

FX Column: Interview with Hans-Peter Deutsch

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Hans-Peter is a founder of d-fine, a leading quantitative and technology consulting firm in Europe. Prior to that, he was a partner at Arthur Andersen Financial Risk Consulting. He holds a ‘summa cum laude’ Ph.D. in theoretical physics and is author of many international publications in physics and mathematical finance, including his classical textbook *Derivatives and Internal Models*, now in its 5th Edition. For many years Dr Deutsch was Guest Lecturer for the Mathematical Finance Programme at the University of Oxford, UK, and Chairman of the Advisory Board of the MathFinance Institute Goethe-Universität in Frankfurt, Germany. In 2010, he stepped out of the business world and became a yoga teacher and travel photographer. In recent years, he has engaged in various mathematical endeavors, ranging from the efficiency of bitcoin mining to the mathematically best intonations for musical instruments. That latest project led to the groundbreaking work *Musical Tonality*, published in 2023.

You have a long history in building financial risk models. How did you get into music and what was your motivation to attack the question of tonality?

I got into music long before I worked in finance, around the same time when I got into physics and mathematics: at school as a young teenager. And indeed, it is very basic physics of overtones and resonances already taught in high school and summarized in Figure 1 which clearly and unmistakably tells us what sounds good and why. Since that time, I have been constantly baffled by how completely these basic physical facts are ignored by the intonation we use in Western music for centuries. Now, if you stubbornly ignore physics, you have to come up with other (usually psycho logical or cultural) “reasons” for why something sounds good (to us). A widespread misconception is that what we perceive as euphonious depends on what we are accustomed to in our lives. However (and quite similar to ideologies and religions, by the way),

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this is only true in the following sense: as long as your only choices are wrong intonations (explanations) of the world, the one that will sound best to you will be the one hammered into your brain by the society you coincidentally happened to grow up in. But if you are able to look beyond the programming of your upbringing and open your mind, unhindered by tradition and unbiased by opinion, to see (or in our case, hear) what is truly out there, you end up with physics — and mathematics as the only language with enough logical depth to describe and make sense of what you find. So, I always thought, if I start from the hard, physical facts, and apply strictly logical thinking, I should find the tone system which adheres optimally (in a logically sound sense) to the physics of euphony — without ever needing to resort to psychology, culture, or tradition. And during the pandemic, I finally embarked on this endeavor, resulting in the almost 600 pages of Musical Tonality [1].

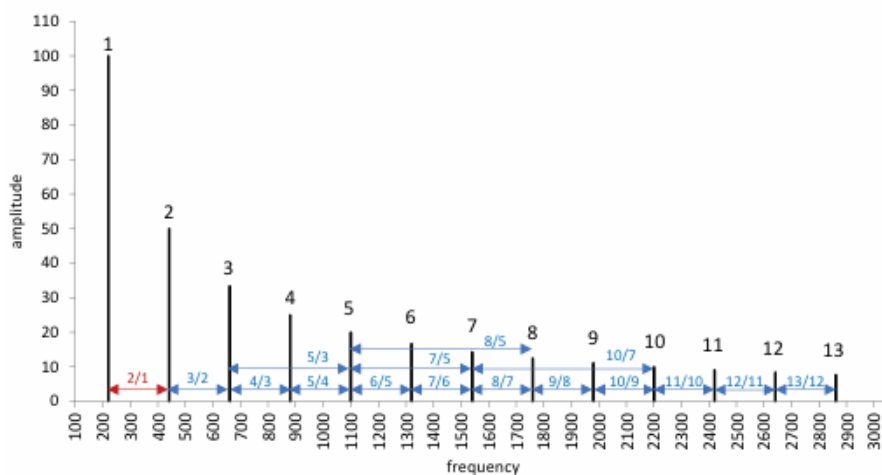


Figure 1: Harmonic spectrum of a sound with pitch of 220 Hz. Black numbers above the bars denote harmonics. Frequency ratios between harmonics (some denoted by blue arrows) are ratios of integers. Perfect resonance in red. From [1]

What you discovered has been a question for humanity for centuries. Why did it take so long for us figure it out?

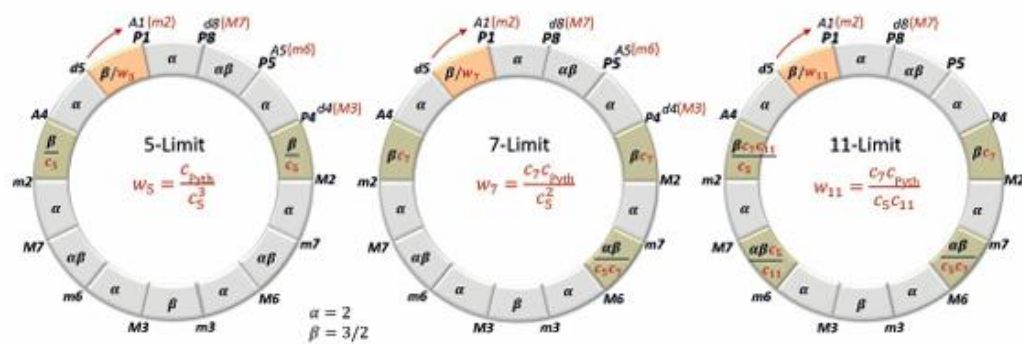
For millennia, humans always ran into the following conflict: if they made tone systems which adhered to the physical laws of euphony as in historic just intonations, then identical intervals had different frequency ratios when played over different tones, making playing, composing, and especially transposing music a nightmare of inconsistencies. After discovering the structural and tonal rings as the fundamental principles of tone systems in Chapter 5 of Musical Tonality [1], I was able to categorize such inconsistencies based on how they broke these rings, see Figure 2.

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On the other hand, if they made a tone system suitable for consistently composing and playing music then it would very badly adhere to the physics of euphony. The search for tone systems which facilitate consistent music and adhere to the physics of euphony has been ‘a battleground for the great minds of Western civilization’, as Stuart Isacoff puts it in his book with this very title [2].

Figure 2: How historic 5-, 7- and 11-limit just intonations break their tonal rings. Broken orange segments encapsulate wolfishness, quantified by the wolf-factors expressed as comma-combinations in the middle of the rings. Broken beige segments generate all other inconsistencies. Non-diatonic intervals (their historically intended but wrong names in red) are shown as far as the broken rings can generate their historically intended but wrong frequency ratios. From [1]



There is an abundance of literature documenting this struggle. But almost none of them was of any use to me. To expose the pure essence and lay bare the logical structure of everything, a fresh start from scratch was necessary, unencumbered by the back and forth of history. This way of thinking is quite established in some scientific fields like, for instance, physics or mathematics, where people routinely wrestle with extremely hard problems, at the very edge, and in many cases well beyond, the capabilities of the human brain. Such fields usually have separate branches called ‘History of Science’, which are taught separately from the respective ‘core science’ itself. This is possible because the scientific objects and facts exist independent of how (or whether at all) they were discovered by humans. In addition, they can be explained and connected by logic alone, without ever having to resort to the (hi)story of who discovered what and when. As an example, consider the earth orbiting around the sun. The earth has been doing that regard less of any humans ever discovering it, and even before humans existed. The history of its (human) discovery, from Nicolaus Copernicus to Galileo Galilei and all the way to Johannes Kepler, is surely interesting (all the disputes with the Catholic church and what not), and we humans, being human(-centric), inherently like stories about humans. But such stories are not always helpful or necessary to solve a problem.

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A clear look at the bare logic of a problem (like how the inverse square law of gravitation leads to elliptical trajectories), free from any historic embellishments, is often more helpful in finding its solution. Nowhere is this desire for clarity and simplicity better summarized than in Albert Einstein's über-famous quote: Everything should be (explained) as simple as possible — but not any simpler [3].

Figure 3: The sets which two positive numbers x, y are mapped into by arithmetic operations, depending on x, y belonging to the sets specified in the first column. Closure highlighted in green. The last line is for Tonal Numbers. The last two columns show the sets to which the prime exponents, and numerators and denominators of the irreducible ratios belong. From [1]

x, y	$x + y$	$x - y$	$x * y$	x / y	x^y	Prime Exponents	Irreducible Ratio
\mathbb{Z}_+	\mathbb{Z}_+	\mathbb{Z}	\mathbb{Z}_+	\mathbb{Q}_+	\mathbb{Z}_+	\mathbb{N}	$\mathbb{N} / \mathbb{1}$
\mathbb{Q}_+	\mathbb{Q}_+	\mathbb{Q}	\mathbb{Q}_+	\mathbb{Q}_+	$\mathbb{Q}_+^{\mathbb{Q}}$	\mathbb{Z}	\mathbb{N} / \mathbb{N}
$\mathbb{Q}_+^{\mathbb{Q}}$	\mathbb{A}_+	\mathbb{A}	$\mathbb{Q}_+^{\mathbb{Q}}$	$\mathbb{Q}_+^{\mathbb{Q}}$	\mathbb{A}_+	\mathbb{Q}	$\mathbb{Q}_+^{\mathbb{Q}} / \mathbb{Q}_+^{\mathbb{Q}}$

Einstein found it necessary to say that because the theoretical models constructed by scientists to make sense of nature, as mathematically complicated as they might look, often are still crude simplifications of reality. So, in physics, his famous quote is used as a dire warning against oversimplification — of which physicists are in constant danger, since the processes they try to describe (particularly anything with the word ‘quantum’ in it) are just so darn difficult. In many other fields of human activities however (politics, for instance), the danger lies in the opposite direction: people speak in terms more complicated and cumbersome than necessary, making their words more difficult to understand than the actual concepts they describe. Now, the peculiarity in the field of musical intonations is that Einstein's principle is violated on both sides: things are oversimplified by using not enough mathematics to really analyze, understand, and solve the problems at hand. At the same time, things are made unnecessarily complicated by inventing names, preferably taken from ancient languages like Latin or Greek, sounding mighty impressive like Septimal Kleisma, Major Diesis, or some such. However, giving a problem a name, even if taken from an ancient language, is not science, if it doesn't help in understanding or solving that problem. In short (and a bit over-simplified): the history of musical intonations contains too much Latin and too little math, violating both sides of Einstein's principle. It is like burning both ends of a candle: You won't make more light; you'll just burn your hands. And that's why it took so long to figure it out.

What role does mathematics play? Do you have examples?

I'm glad you asked, and I will take the liberty to answer this on a more general, maybe even philosophical level. The way I see it, there are two fundamentally different kinds of problems in this + world. The kind that can be solved by influencing people's opinions, and the other kind — for which you need math. The first kind arises from, and is solved by human interactions, from lovesickness (if you convince your desired partner to love you back, your problem is solved) over

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court proceedings (if you convince the judge that you are innocent, your problem is solved) to politics and wars (if the warring counterparts can be convinced that their dispute is not worth fighting over, the problem is solved). Natural languages are designed for this type of problem — what I call “opinion management”: expressing one’s own opinion and trying to influence the opinions of others (after all, this is what discussions are all about.) However, this is just humanity as a whole being occupied with itself. And just like a company occupied only with internal politics (with nobody producing any products and nobody going out to sell them) will not survive, humanity as a whole would not have survived (let alone evolved) if we had not solved some problems of the other kind — the ones that don’t arise from human interactions and have nothing to do with human opinions, but involve understanding facts and discovering truth by logical deduction. Here, the logical arguments are often so convoluted and involve so many steps that mathematics, the language of logic, is the only way to precisely formulate such arguments and to make sure that each step of the way stays on solid ground.

Human opinions are absolutely irrelevant here, and natural languages just won’t do. If humanity had relied on natural languages and opinion-management only, without mathematics, we would have had neither electricity nor medicine, no engines at all (neither steam nor combustion nor electric or any other kind), no computers, no internet and no understanding of our world whatsoever. Only opinions and mysticism. So, of course, mathematics is the only way for my specific endeavor here, too. To make sure that I made only strictly logical deductions from undisputed physical facts, without getting influenced by any opinions (of which the history of music is chock-full). Here is one example: As mentioned earlier, I had to start over from scratch and build everything completely new. This goes right down to the very numbers I worked with: I had to identify a new number system, namely, the largest set of numbers which could ever occur in a tone system based on the harmonic (overtone) series. It turns out that this is the set of rational powers of positive rational numbers which I call Tonal Numbers. I have shown that they are closed under multiplication and division only, but under no other operation (not even under addition!), which is exactly what’s needed for musical intonations, see Figure 3. They are also the largest set of numbers for which prime factorizations and irreducible ratios can be defined — even though most of them are irrational. If the mathematician in you now wonders how the heck could an irrational(!) number ever be prime-factorized, go ahead and read Appendix E of Musical Tonality [1]. I’m sure you will enjoy it, even if you are not into music.

Can you describe the essence of your findings?

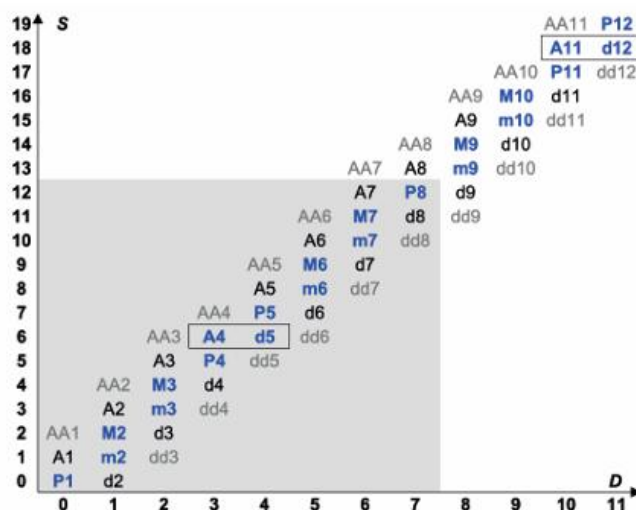
The essence lies in the deep insights (like e.g. the tonal rings and tonal numbers mentioned above) gained along the path to optimal intonation. In fact, one could argue that this optimal intonation is merely an application of the profound and novel findings presented in Musical Tonality [1] — which clarify all aspects of tuning systems based on harmonic overtones and answer all questions, some of which have been debated for centuries. Here are a few examples: Tone structures are sets of intervals between discrete tones.

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One important (not new but often deliberately ignored) insight is that such interval sets are two-dimensional and you can add/subtract them like vectors in a coordinate system if you use the right coordinates — namely, their chromatic and diatonic steps (not tones!) as in **Figure 4**.

Figure 4: Intervals in the coordinate system of diatonic steps D and chromatic steps (semitones) S . Besides the diatonic intervals (blue) only singly (black) and doubly (grey) diminished/augmented intervals are written out. Tritones are framed. The grey area is the first octave with intervals having coordinates in the range $0 \leq D \leq 7$ and $0 \leq S \leq 12$. From [1]



Intonations, on the other hand, are mappings assigning frequency ratios to intervals. These mappings must be functions with one unique frequency ratio per interval, an aspect heavily violated by historic just intonations which lead to all kinds of inconsistencies. Now, a new and very important insight is that the inverse mapping, in the opposite direction from rational numbers (frequency ratios) to intervals, is always the same unique function — for all (even inconsistent) just intonations of a given tone structure! I call this the Inverse of Just Intonations, or IJI for short. The parameters of this function, which I call seed coefficients, are structural, intonation-independent constants which I explicitly determined for the Western tone structure. Obviously, this puts all discussions (which have been raging for centuries) about what interval name to give to what rational number to rest. For instance, Huygens' tritone $7/5$ is a diminished fifth, whereas Euler's tritone $10/7$ is an augmented fourth, without any doubt, as determined by IJI.

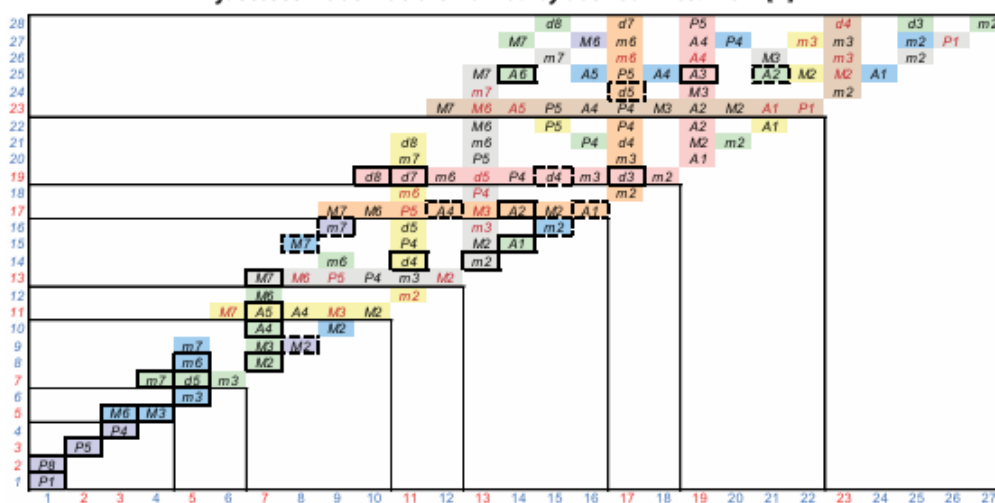
But the implications go much further. For instance, I derived what I call a comma factorization, which factorizes any given rational number into a product of the Pythagorean frequency ratio (whose prime factorization contains only primes 2 and 3) for the interval associated via IJI with that given number, and so-called commas. These commas are all automatically generated by the seed coefficient of IJI, and equal exactly the commas found historically (by tedious trial and error) like the syntonic comma, the septimal comma, etc. This implies the deep insight that all rational frequency ratios for the same interval have the 2,000-year-old Pythagorean intonation (which is consistent, by the way) at their core and can only ever differ by commas. What's even better, this shows a way forward toward optimization: to find the best frequency ratio for a given interval, start with its Pythagorean ratio and attach commas to it until you find the comma combination which (through cancellations, etc.) results in the irreducible ratio involving the smallest integers. This

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could never have been done before, because no one knew about comma factorizations — and because of this little problem: before you can even think about finding the best of all frequency ratios for a given interval, you must first know all frequency ratios belonging to that interval in the first place. Something nobody could ever agree on but is now completely settled by IJI as shown explicitly in **Figure 5**.

Figure 5: Unique intervals associated by IJI to irreducible just ratios $1 \leq m/k \leq 2$ with m =line and k =column index up to $m, k < 29$. Colors according to prime-limits. Bold intervals have smooth ratios (i.e., not far away from equal temperament). Intervals having their absolute justest frequency ratios are framed by solid lines. If such an absolute justest ratio is not smooth then the justest smooth ratio is framed by dashed lines. From [1]



These examples are just two of many insights gained in Musical Tonality. They are neither the deepest nor the most profound. I have chosen them because they are the easiest to describe here, and because they demonstrate how such insights can lead to optimization ideas. But they enable optimizing frequency ratios for individual intervals only, not for intonations as a whole — which requires finding the “best possible” set of frequency ratios for all intervals of a tone structure. That’s a whole different can of worms and needs a lot more (and more complicated) insights and tools, like, e.g., the tonal rings and tonal numbers mentioned above, generator logarithms, characteristic and natural determinants, natural bases, diatonic quanta, and mathematical measures for sound quality. It is not even remotely possible to explain all of this in the limited space available here. You will need to read Musical Tonality. It has 600 pages for a reason. How is it related to Equal Temperament. which we have all used for centuries?

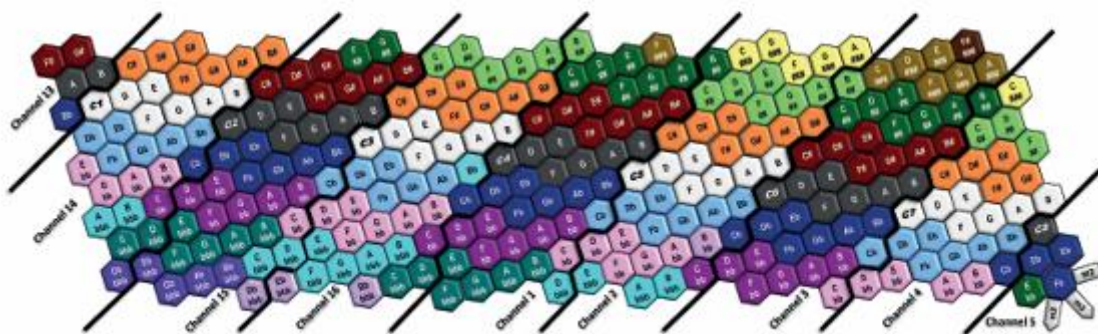
As mentioned above, the major flaw of historical just intonations is that they are inconsistent, producing many different frequency ratios for the same interval. The major flaw of Equal Temperament is the exact opposite: it is degenerated, having many different intervals with the same frequency ratio. People call such intervals “enharmonic equivalents”, but they are structurally different. Denying this difference makes the two dimensions of the tone structure collapse into one — which is the reason why so many people stubbornly deny this two-dimensionality (which is the most basic fact you need to understand before you can understand anything else), even up to the point that they keep building instruments like pianos with one-dimensional rows of keys instead of two-dimensional arrays of buttons. But the Circle of Fifths

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never closes (in stark contrast to the tonal rings in Musical Tonality [1], by the way, which actually, truly close in all consistent intonations) and an augmented seventh is not a perfect eighth, even if Equal Temperament makes them sound the same; and a B# is not a C, even if the piano has only one key for both of them.

Figure 6: Two-dimensional isomorphic keyboard layout for a hexagonal MIDI controller. To facilitate orientation, different colors mark different accidentals, and alternating darker and brighter shades mark different octaves. Thick black lines are MIDI channel borders. The Main MIDI-channel containing C4 (middle C) is Channel 1. From [4]



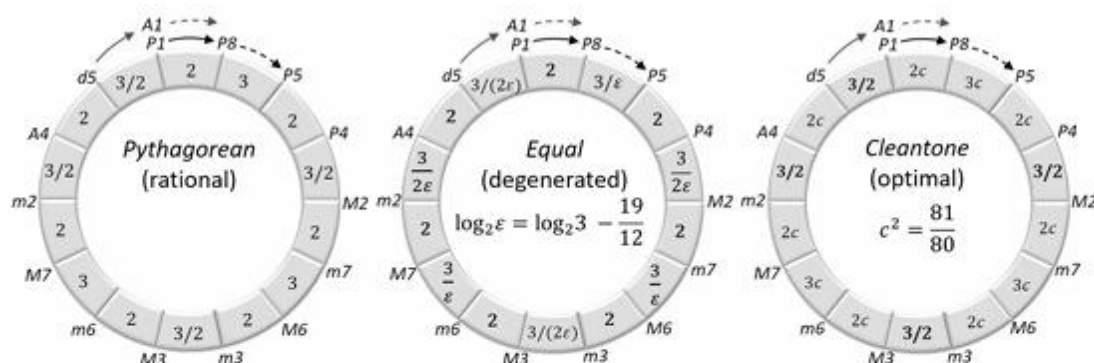
A two-dimensional keyboard as in **Figure 6** from my ‘Implementation Paper’ [4] has no such problems and is also iso morphic, meaning that any musical entity, whether scale, chord or melody, has only one shape that never changes regardless of the key it is played in. And yes, Equal Temperament also sounds bad, since (except for the octave 2/1) all its frequency “ratios” are irrational, i.e., not ratios of integers at all — let alone of small integers. This completely ignores the physics of euphony. The reason why it still sounds okay(ish) is that it comes very close to the first two overtones in the harmonic spectrum, which are factors 2, respectively 3 away from the base tone. But this is not the accomplishment of the intonation, but the sole result of the tone structure having 12 steps per octave. Any other number of steps per octave, for example, 10, would sound awful, making it immediately obvious that Equal Temperament as an intonation does nothing for euphony.

You see, the tone structure itself is already the result of an optimization — before intonations even enter the stage. Twelve steps per octave being optimal follows from the fact that $19/12$ is a continued fraction approximation [5] for $\log_2 3$. If you interpret $\log_2 3$ as the “distance” between a base tone and its 2nd overtone, and likewise $\log_2 2 = 1 = 12/12$ as the distance between the base tone and its 1st overtone (the octave), then this implies that if you cover the distance of an octave with 12 steps, then you hit the first overtone exactly while 19 such steps will come very close to the 2nd overtone. This is a simple structural optimization involving discrete, integer numbers (of steps per octave), without using any of the continuous degrees of freedom intonations bring to the table to optimize further. And Equal Temperament does not do this further optimization. It just freerides on the optimization already achieved by the tone structure and slaps “equality” onto it. What a waste! Imagine what can be achieved if you use an intonation’s continuous degrees of freedom to optimize further for euphony! Well, out comes Cleantone Temperament [1] — the truly

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optimal intonation for the Western tone structure, where half of all intervals have exactly the desired ratios of small integers as in (inconsistent!) just intonation. In particular, the major and minor thirds have their ideal ratios $5/4$ and $6/5$. So, all stacks of thirds like fifths, sevenths, and most chords(!) are perfect. But Cleantone Temperament is still consistent, retaining all benefits of equal temperament like free transpositions through all musical keys. As to how this relates to equal temperament: This is best understood via the tonal rings in Figure 7: All consistent intonations have unbroken tonal rings and can be characterized by how their ring-segments differ from the Pythagorean intonation (which is the only consistent intonation which hits the first two overtones exactly). As it turns out, the error of the continued-fraction approximation $19/12$ for $\log_2 3$ controls the deviation of Equal Temperament from the Pythagorean intonation. By contrast, for Cleantone Temperament this deviation is controlled by the square root of the syntonic comma $81/80$. These are very deep insights which cannot be fully explained here. You would need to read Musical Tonality [1] to fully appreciate the implications of **Figure 7**.



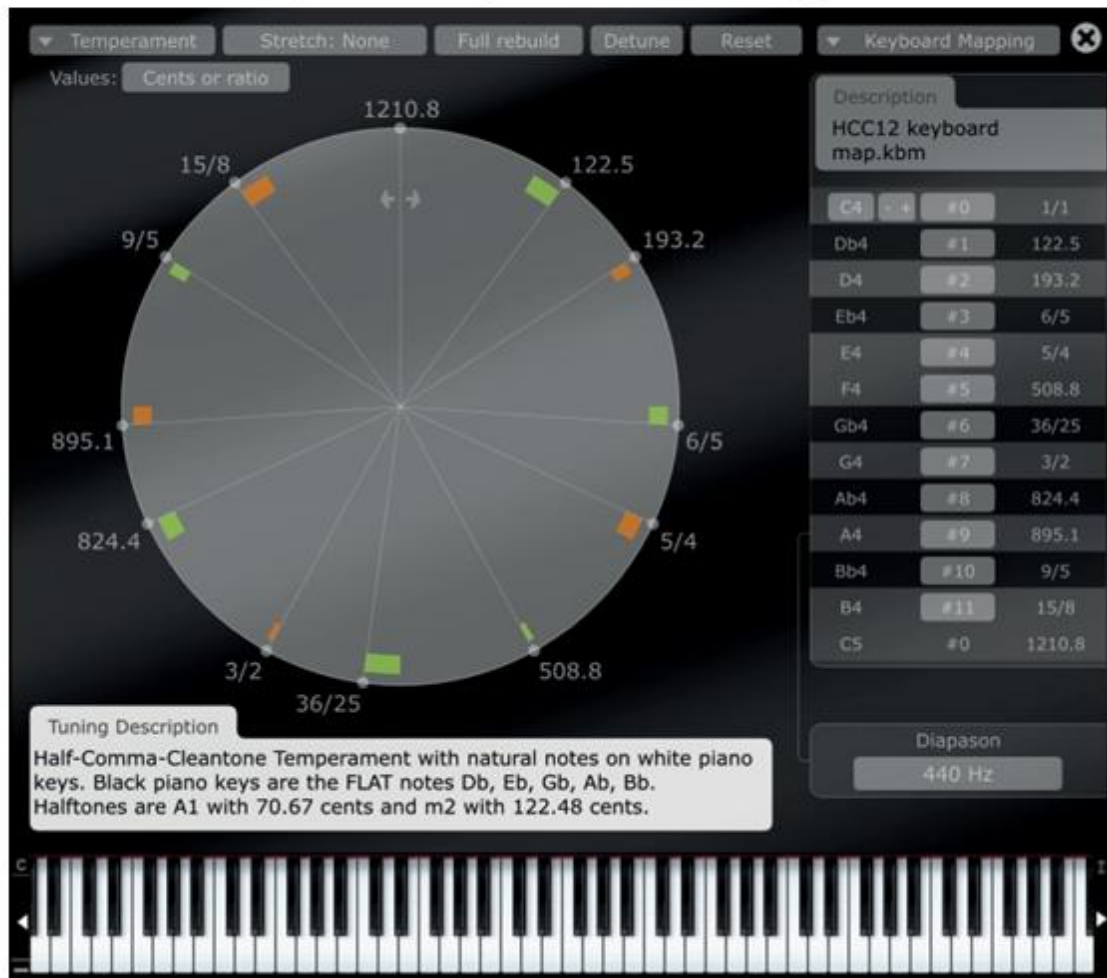
Can I tune my piano myself based on your findings? What tools do I need?

Yes, you can — as long as your piano is digital and uses a sound generating software which allows for re-tuning. I have just published a paper [4] explaining how to practically implement Cleantone Temperament in this way, using virtual instruments (e.g., Pianoteq [6]) and digital audio workstations — with detailed, explicit implementation guides for traditional one-dimensional keyboards like pianos and synthesizers, as well as MIDI Controllers as in Figure 6 with two-dimensional hexagonal keyboards like the Lumatone [7]. You need your sound generating software to be able to read so-called Scale Files (.scl files) and Keyboard Maps (.kbm files) [8]. It is all explained in the paper. **Figure 8** shows how this looks in practice for a piano tuned via Pianoteq to Cleantone Temperament. Of course, with one-dimensional piano keyboards (made for Equal Temperament), you must decide which of the many enharmonic equivalents (which all sound identical in Equal Temperament, but different in Cleantone Temperament) will get the honor of being assigned to piano keys.

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Figure 8: An implementation of Cleantone Temperament on digital piano using Pianoteq 8, showing explicitly how half of all intervals have their ideal ratios of small integers as in 5-limit just intonation. The other half are all exactly half a syntonic comma (ca. 10.7 cents) away from their ideal ratios – which is STILL closer than Equal Temperament gets the consonant intervals (thirds and sixth) to their ideal ratios. From [4]



For instance, whether the black key above the white F-key should be an F# or a Gb, and whether this white key, which I just so nonchalantly called F, should be an F at all, or rather an E#, etc. These choices, which you save as presets in your instrument, depend on the musical goals; for instance, on the piece to be played, the harmonics and chords to be used, etc. This works surprisingly well in practice. It is truly astonishing how few such choices (presets) are needed to play all existing music on a piano, and how amazingly good it sounds. Of course, you get the full beauty for all notes (without ever needing to change presets) only on real two-dimensional keyboards like in Figure 6. But it's almost more impressive to hear how good an already familiar instrument like the piano can sound when you're finally playing in the best into nation our Western tone structure is capable of.

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Do you think this will cause a revolution in music composition?

I would like to hope that it causes a revolution, not only in music composition but in musical performance as well, since all existing music can be played in Cleantone Temperament. You don't need to specifically compose for it. But, considering the last 2,000 years of music history, I have doubts that this revolution will happen — at least if it depends on humans. For instance, just when I was about finishing Musical Tonicity, I got a call from Springer Publications, since they know me as the author of other books (like Derivatives and Internal Models) published by them. They asked if I would be willing to write a new book they could publish. I said, what a coincidence and perfect timing! I have just finished Musical Tonicity, 600 pages of logic and math, solving everything about musical intonations with harmonic overtones — probably the most interesting and important stuff I have ever done in my life. They said great, send us the manuscript. So I did. You might already guess what happened next. Yep. They didn't send it to a physicist or mathematician, but to a music theorist as a referee, who checked the reference list and complained that the presented arguments are not “anchored in the ongoing discourse”, missing some citations. Well, of course, they aren't! I anchored everything in logic and mathematics. Any ongoing (or past) discourses in music theory weren't of much help — not the least because music theory isn't even the topic of the book in the first place. It's about tuning instruments; not about composing music.

Anyway, it's probably for the better, as I can now simply publish it on the internet for anyone to read, without needing to ask any publisher for permission. On ResearchGate and SSRN, it is quite successful, with thousands of views and hundreds of full-text downloads, which are quite high numbers for these platforms. And, much more importantly, it can be crawled by AI. They will be smart enough to understand it (if they haven't figured everything out by themselves already, anyway). And if they start using the ideas in the book, be it for the music they create by themselves or for the answers they give to humans who ask about these things, we might actually get a musical revolution, at the very least better sounding music. Where can one listen to music tuned by your method? Well, you can listen all day long if you tune your instrument accordingly and play on it (like I've been doing now for many months, without ever looking back). In addition, I will be opening a YouTube channel, where I will play well-known pieces in Cleantone Temperament and also re-play them in Equal Temperament, so that everybody can hear the difference. It's a project for this winter. So, keep searching YouTube for Cleantone Temperament every now and then. At some point, you will get a hit.

REFERENCES

- [1] Deutsch, H-P. 2023. Musical Tonicity. Online. Available at: <https://ssrn.com/abstract=4452394> (Accessed May 2025).
- [2] Isacoff, S. 2009. Temperament: How Music Became a Battleground for the Great Minds of Western Civilization. New York: Knopf Doubleday Publishing Group.
- [3] Einstein, A. 2011. The Ultimate Quotable Einstein. A. Calaprice, Ed. Princeton, New Jersey: Princeton University Press.

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- [4] Deutsch, H-P. 2025. Implementing Cleantone Temperament on MIDI Controllers and Virtual Instruments. Online. Available at: <https://ssrn.com/abstract=5355401>
- [5] Vialar, T. 2016. Handbook of Mathematics. Paris: Books on Demand.
- [6] Modartt. Pianoteq. Online. Available: https://www.modartt.com/pianoteq_overview. (Accessed 2025).
- [7] Lumatone. Lumatone.io. Online. Available: <https://www.lumatone.io/>. (Accessed May 2025).
- [8] huygens-fokker.org. huygens-fokker.org/scala. Online. Available: https://www.huygens-fokker.org/scala/scl_format.html. (Accessed May 2025).