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Return distributions of equity-linked retirement plans under jump and interest rate risk

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Abstract We consider a savings plan, where the paid capital is guaranteed at time of retirement, in the German market available as *Riester-Rente* and supported by federal cash payments and tax benefits. We generalize several capital guarantee mechanisms to payment plans and compare their distribution: the return distribution of a classical insurance strategy with investments in the actuarial reserve fund, a CPPI strategy, and a Stop loss strategy, in optimistic, standard and pessimistic market scenarios. To model the distribution we use a jump diffusion process parameterized to resemble the MSCI World index for the stock investment and a Hull-White Extended Vasicek process, calibrated to the euro zero-bond curve, for the risk free investment. We also analyze how fee structures and gap risk affect the performance of these savings plans. Additionally, we present a very simple parameter estimation method for this kind of simulation studies.

Keywords CPPI · Stop loss · Capital guarantee mechanisms · Jump diffusion model · Short rate process · Retirement provision plan · Hull-White Extended Vasicek

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1 Introduction

In recent years an increasing demand for capital guaranteed equity-linked life insurance products and retirement plans has emerged. In Germany a retirement plan called *Riester-Rente* is supported by the state with cash payments and tax benefits. The retirement plans have to preserve invested capital. The company offering a *Riester-Rente* has to ensure that at the end of the savings period at least all cash inflows are available. Due to investors' demand for high returns, banks and insurance companies are not only offering saving plans investing in risk free bonds but also in products with high equity proportion. Companies offering equity-linked guaranteed retirement plans face a big challenge. Due to long maturities of the contracts of more than 30 years it is not possible to just buy a protective put as is usually done when hedging the short term risk of investments. Many different concepts are used by banks and insurance companies to generate this guarantee or to reduce the remaining risk. They vary from simple Stop loss strategies to complex dynamic hedging strategies. In our work we analyze the return distribution generated by some of these strategies and determine their risk. We consider several examples:

- A classical insurance strategy with investments in the actuarial reserve fund. In this strategy a large proportion of the invested capital is held in the actuarial reserve fund to fully generate the guarantee. Only the remaining capital is invested in products with a higher equity proportion. The actuarial reserve fund is considered risk free. It usually guarantees a minimum yearly interest rate.
- A capped constant proportion portfolio insurance strategy (CPPI), which is similar to the traditional reserve fund in that it ensures not to fall below a certain floor in order to generate the guarantee. In contrast to the traditional strategy the amount necessary to generate the guarantee is not fully invested in risk free products. The amount invested in more risky equity products is leveraged for a higher equity exposure. Continuous monitoring ensures that the guarantee is not at risk, since the equity proportion is reduced with the portfolio value approaching the floor.
- A Stop loss strategy where all the money is invested into pure equity until the floor is reached. If this happens all the invested capital is shifted into the risk free products in order to provide the guarantee at the end.

There are also equity-linked life insurance guarantees where the insurance company promises to pay out the maximum of the invested amount and an investment in an equity fund reduced by a guarantee cost (usually yearly as a percentage of the fund value). The return distribution of these products highly depends on the guarantee cost. Due to the long maturities of the contracts, the pricing of this guarantee cost is not straightforward and the price is strongly model-dependent. For this reason they are not included in this comparative study. An introduction to equity-linked guarantees and their pricing can be found in [14].

After generalizing the strategies for a stream of payments, we simulate their return distribution and analyze how fee structures often used by insurance companies affect the return distribution. Since relevant for the German market we also study the impact of the federal cash payments by analyzing an investment plan that maximizes them.

In order to model the distribution of the stock market we extend the jump diffusion model by Kou [15] to allow for displaced jumps. Therefore, we go beyond the classical Black-Scholes model [7] and explicitly allow for jumps in the market as we could observe them within the last two years. One reason for using a jump diffusion model instead of a simple geometric Brownian motion is that it better represents reality by accounting for the heavy tail distribution of stock index returns. Therefore not only the simulation of the underlying process is more realistic but also the rebalancing between equity and fixed income funds in the strategies under consideration. The second reason is that Stop loss and CPPI both only generate a safe guarantee in a market without jumps. We analyze how often a CPPI and a Stop loss strategy fail if we allow for jumps. We present a very simple procedure for estimating parameters for jump processes that avoids the stability problems one usually experiences when estimating jump diffusion parameters by means of moment matching or other techniques.

In most of the literature the floor for CPPI and Stop loss is assumed to change only deterministically over time. This is not a realistic assumption since the floor is usually determined by the zero bond price with the same maturity as the structure itself and zero bond prices are far from being deterministic. Also the performance of the actuarial reserve fund depends on the current level of interest rates. For this reason we model rates by a Hull-White Extended Vasicek process, calibrated to the euro zero bond curve as of November 2009.

Past research on simulating the actuarial reserve fund has mainly focused on the purpose of asset liability management (ALM), see [1, 9, 13] and the references therein.

Extensive research is available on the general theory of constant proportion portfolio insurance. Black and Perold [6] derive general properties of uncapped constant proportion portfolio insurance under log normally distributed stock returns. The impact of discrete trading in this model is analyzed in [2]. Several authors discuss the relation between CPPI and option based portfolio insurance (OBPI) [4, 5]. The distribution properties in the continuous case for CPPI with constraints on borrowing is done in [3] where it is shown that the log normality of the cushion process is lost in this case. An analysis of uncapped CPPI structures under jump risk is done in [11]. We present an analysis for the capped CPPI with jump risk and interest rate risk. Balder and Mahayni also focus on Stop loss in [3], which can be seen as a special limiting case of capped CPPI.

Despite these articles dealing with different strategies there is no simulation study that compares all those when they are used in the context of pension plans. A simulation of several different strategies under a continuous model is done in [10] but lacks the actuarial approach and considers only a single payment instead of payment streams.

We provide a comprehensive study comparing guarantee structures used for pension funds. This study is applicable to the *Riester-Rente* in Germany, but also to general guaranteed pensions plans. We base our analysis on a model that is parametrized to a well known global stock index, the MSCI World, and fits the distribution of its returns very well. Moreover we take into account the additional risk of stochastic rates. Instead of assuming a single payment at the beginning, as is usually the case when assessing CPPI and Stop loss structures, we analyze a plan with monthly payments, as it is the natural case in pension business, for two reasons. Firstly, this has an impact on the average stock exposure since in the case of a de-leveraging (also called Cash-Lock) due to a large market drop shortly after the start of the saving period, the following monthly payments are able to build up a new equity investment over time. Secondly, this is also important for a realistic analysis of the gap risk in CPPI and Stop loss. Our detailed analysis of the impact of path dependent fees charged by the insurance companies is of large practical interest from a client perspective. We also analyze to what extent these large fees can be compensated by payments from the state, which the insurant receives in case of the *Riester-Rente* in Germany. Our in-depth explanation of the simulation technique and parameter estimation should also provide a guideline for similar simulation studies. A less detailed analysis under simplified assumptions can be found in [12].

Our conclusion is that one of the driving factors of the return distribution of the *Riester-Rente* product is the concept that is used to generate the guarantee. With equity exposure ranging from 36% to 100% the return distributions vary significantly. For a monthly saving plan and a small leverage factor of 3 or 4 the risk of failing to generate the guarantee in the CPPI structure turns out to be rather small even when rates are stochastic due to the continuous exposure reduction in periods of negative equity returns. Another very important factor of the return distribution is the fee structure of the contract. For some contracts only 85 % of the capital is actually invested in the strategy. The remaining 15% is taken by the insurance or bank as sales and maintenance fees. With yearly management fees of the underlying funds the return distribution is additionally weakened.

This paper is organized as follows. In Sect. 2 we describe the equity and interest rate model used for simulation and parameter estimation. In Sect. 3 we provide a description of the capital guarantee mechanisms being compared and their generalization to payment streams. Section 4 describes the simulation horizon and the payments. In Sect. 5 we exhibit and analyze the results not considering costs, Sect. 6 introduces the cost structures we analyze and presents the results including costs. The risk of failing to generate the guarantee in case of jumps is studied in Sect. 7. Section 8 concludes.

2 Model setup

As mentioned in Sect. 1, we would like to go beyond the classical Black-Scholes model and allow for exponentially distributed jumps in the market. We extend the model by Kou [15, 16] by allowing the jumps to be displaced. This allows us to distinguish between the small movements arising from the discretization of the Brownian motion, which in theory has no jumps, and the larger movements (jumps) arising from the Poisson component of the process. It is therefore assumed that the logarithm of the jumps can take values only outside the interval $(-\kappa, +\kappa)$. This is different from [15] in which, instead, jumps can take any value in the set of non-zero real numbers.

The governing equation for the displaced double exponential jump diffusion model (DDE) is

$$\frac{\mathrm{d}S_t}{S_{t-}} = \gamma \mathrm{d}t + \sigma_1 \mathrm{d}W_t^1 + \mathrm{d}\left(\sum_{j=1}^{N_t}(V_j - 1)\right)$$

which is solved by

$$S_t = S_0 \exp\left[\left(\gamma - \frac{\sigma_1^2}{2}\right)t + \sigma_1 W_t^1\right] \prod_{j=1}^{N_t} V_j \tag{1}$$

where

- (W_t^1) is a standard Brownian motion,
- (N_t) is a Poisson process with intensity $\lambda > 0$ and
- V_j are independent identically distributed random variables $V_j \sim e^Y$, where Y represents the relative jump size with a minimal jump of κ, therefore leading to jumps of Y in the range $(-\infty, -\kappa] \cup [\kappa, +\infty)$,

with parameters

- γ denoting the drift,
- σ_1 denoting the volatility,
- λ denoting the expected number of jumps per year.

The processes (W_t^1), (N_t), and the random variables V_j are all independent. We choose the jumps Y to be double-exponentially distributed assuming only values outside the interval $(-\kappa, +\kappa)$.

We illustrate sample paths drawn from this distribution for a period of 35 years in Fig. 1.

Except for the drift, the parameters are estimated to resemble the daily log returns of the MSCI World index[1] for the last thirty years. For the drift we choose different scenarios. The details of the parameter estimation are given in Appendix 1. We show the main parameters in Table 1. We do not intend to simulate actively managed funds.

To calculate the current value of the future liability (floor) and the performance of the fixed income investments we use the zero bond curve as of October 1 2009. The curve $D_{0,t}$, $0 \leq t \leq T$ is extracted from money market and swap rate quotes from Reuters by bootstrapping. Interpolation is done linear in the rates. See Table 2 for the calculated discount factors and the market quotes as provided by Reuters. We simulate twice, once with the zero bond curve kept deterministic. In this case market risk is only due to the equity process and the zero bond price at time $t > 0$ is given by

[1] MSCI Daily TR (Total Return) Gross (gross dividends reinvested) in USD.

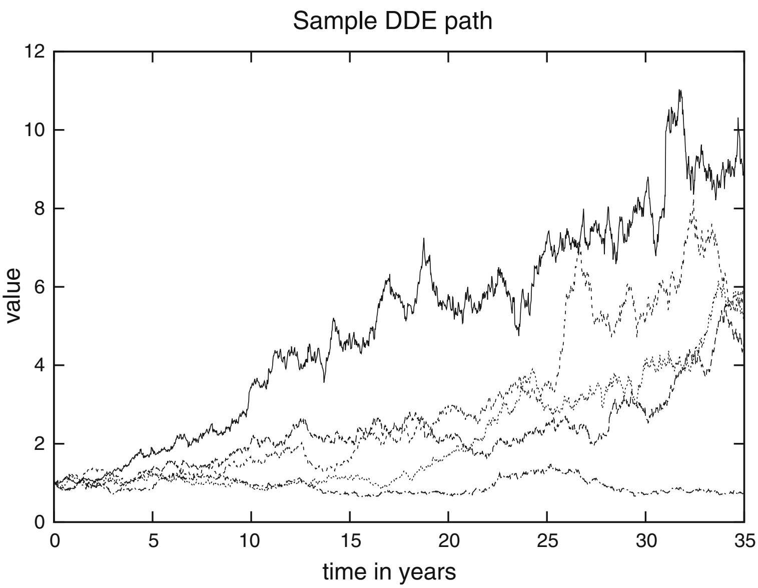

Fig. 1 Displaced Double-Exponential jump process: simulated paths with parameters $\mu := \gamma + \lambda \mathrm{E}[e^{Y} - 1] = 6\ \%$, $\sigma_1 = 14.3\ \%$, $\lambda = 5.209$, $\kappa = 2.31\ \%$

Table 1 Estimated parameters for the DDE-process

Parameter	Value
Volatility of the diffusion part $\hat{\sigma_1}$	11.69%
Jump intensity λ	5.209
Minimum jump size κ	2.31%
Expected jump size above minimum jump size	1.121%

$$D_{t,T} = \frac{D_{0,T}}{D_{0,t}}.$$

In a second run we also simulate the short rate. The stochastic short rate increases the risk for these capital guarantee structures because the guarantee level of the future liabilities is stochastic as well. For modeling the short rate we use a Hull-White Extended Vasicek model as described for example in [8].

The process has the governing equation

$$\mathrm{d}r_t = [\vartheta_t - a r_t]\mathrm{d}t + \sigma_2 \mathrm{d}W_t^2 \tag{2}$$

with constants a and σ_2 and time dependent ϑ_t chosen to exactly fit the term structure of interest rates. The Brownian motions W^1 and W^2 are usually correlated by ρ. For simplicity we use $\rho = 0$ in this study. The advantage of this process is its perfect fit to the current yield curve of interest rates, which ensures that at the beginning of the simulation the allocation decision is really as it would be on

Table 2 Extracted discount factors from money market quotes and swap rates

Date	Instrument	Rate (%)	Discount factor $D_{0,T}$ (%)
11/3/2009	Money market	0.43	99.96
12/3/2009	Money market	0.59	99.89
1/4/2010	Money market	0.75	99.80
2/3/2010	Money market	0.84	99.71
3/3/2010	Money market	0.92	99.61
4/5/2010	Money market	1.01	99.48
5/3/2010	Money market	1.06	99.37
6/3/2010	Money market	1.10	99.26
7/5/2010	Money market	1.14	99.13
8/3/2010	Money market	1.18	99.01
9/3/2010	Money market	1.20	98.88
10/4/2010	Swap	1.20	98.80
10/3/2011	Swap	1.71	96.64
10/3/2012	Swap	2.15	93.76
10/3/2013	Swap	2.46	90.64
10/3/2014	Swap	2.71	87.30
10/5/2015	Swap	2.92	83.85
10/3/2016	Swap	3.15	80.11
10/3/2017	Swap	3.24	77.01
10/3/2018	Swap	3.36	73.71
10/3/2019	Swap	3.46	70.47
10/5/2020	Swap	3.55	67.25
10/4/2021	Swap	3.64	64.09
10/3/2022	Swap	3.72	61.07
10/3/2023	Swap	3.79	58.18
10/3/2024	Swap	3.84	55.43
10/3/2029	Swap	3.99	44.76

Reuters page "EURIRS". The dates are quoted in American convention as Month/Day/Year

October 1 2009. We therefore calibrate the parameters in (2) to the observed initial term structure of interest rates $D_{0,t}$, $0 \leq t \leq T$ and simulate the future rates. We assume that the price for a zero-coupon bond with maturity T at time t, given the short rate r_t, is calculated by

$$D_{t,T} = A_{t,T} \exp\left(-R_{t,T} r_t\right)$$

with

$$R_{t,T} = \frac{1 - \exp(-a(T-t))}{a}$$

and

$$A_{t,T} = \frac{D_{0,T}}{D_{0,t}} \exp\left(-R_{t,T} \frac{\partial \ln D_{0,t}}{\partial t} - \frac{\sigma_2^2 (1 - \exp(-2at)) R_{t,T}^2}{4a} \right).$$

To have an analytic expression is of advantage concerning the computational time of the simulations, because the calculation of the zero-coupon bond price is required in every time step to compute the current value of the future liabilities.

3 Products

In this section we describe the capital guaranteeing investment strategies as well as the assumptions we made for implementing them. All strategies rely on rebalancing invested money between a risk free and a risky investment. In practice, there is usually no totally risk free investment available. In the following we refer to "fixed income fund" as the risk free fund and assume that it invests in high rated government bonds with maturity according to the strategy under consideration. We refer to "equity fund" as the risky fund, which is assumed to invest in the MSCI World index. According to the standard version of a pension plan we consider in our study a stream of payments $p_0, \ldots, p_n$ at time points $t_0, \ldots, t_n$ before maturity, denoted by T, instead of a single payment p_0 as it is usually done when comparing dynamic capital structures. At each payment time t_i the amount guaranteed by the insurance company increases by p_i[2]. With

$$\begin{aligned} G_0 &= p_0 \\ G_i &:= G_{i-1} + p_i, \ i = 1, \ldots, n \end{aligned}$$

we define

$$\begin{aligned} B_t &:= G_i \ for \ t \in [t_i, t_{i+1}), \ i = 0, \ldots, n-1 \\ B_t &:= G_n \ for \ t \in [t_n, T) \end{aligned} \tag{3}$$

as being the guarantee level in place at time point t.

3.1 Classical insurance strategy with investment in the actuarial reserve fund

The current value of the future liability is calculated and a sufficient amount to meet this liability in the future is invested in the actuarial reserve fund. The actuarial reserve fund is assumed to be a fixed income fund and accrues interest implied by the current zero bond curve with a minimum guaranteed rate of 2.25 %[3]. The excess amount that is not required for the guarantee is invested in the risky asset. We assume that the calculation of the amount needed to meet future liability is based on the guaranteed interest rate of 2.25 %. So, at each payment date t_i the amount

[2] Note that in the following it is not assumed that the insurance company knows about all payments at the beginning.

[3] This applies to life insurance contracts in Germany until the end of 2011. For contracts signed after December 2011 the guaranteed interest is 1.75%.

$p_i(1+2.25\,\%)^{-(T-t_i)}$ is invested in the actuarial reserve fund and $p_i - p_i(1+2.25\,\%)^{-(T-t_i)}$ in the equity fund. See Fig. 2 for a picture of the exposure distribution in the classical insurance case. The excess return of the actuarial fund above the guaranteed interest is assumed to be added to the equity fund once a year.

3.2 Constant proportion portfolio insurance (CPPI)

The constant proportion portfolio insurance (CPPI) also invests some proportion into a fixed income fund and the residual part into the more risky equity fund. The difference to the actuarial fund is that instead of investing only the excess amount of the protection level (floor) F_t into the risky asset, the excess amount is leveraged in order to allow for a higher equity participation. The investment is monitored on a continuous basis to guarantee that the investment does not fall below the floor. The original CPPI is based on a constant value B which has to be protected at a certain time T, different to the situation we described for the payment stream where B_t can increase over time. Let F_t be the present value of the future liability B, so

$$F_t := D_{t,T}B$$

with $D_{t,T}$ being the price at time point t of a zero bond maturing at T. Investing the amount F_t risk free could guarantee B in T. Let further P_t denote the total value of the fund and m a positive leverage factor. The amount that can be invested in the risky asset at time t, E_t (also called exposure), is determined by the equation

$$E_t = \min(m(P_t - F_t), P_t).$$

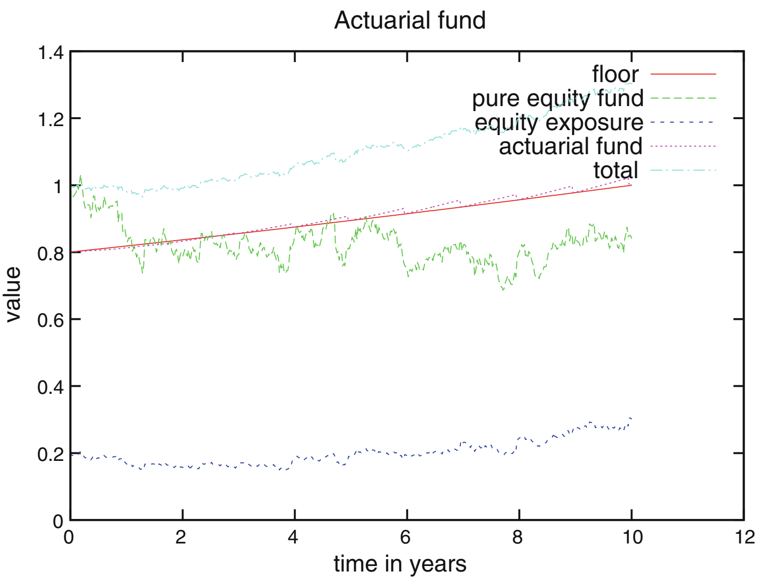

Fig. 2 Simulated path for a classical insurance strategy and 10 year investment horizon

The fixed income investment then is $P_t - E_t$. The excess of P_t over F_t is often called cushion in the CPPI literature. The leverage factor determines the participation in the equity returns. The higher the leverage factor m, the higher the participation in returns but also the risk to reach the floor. The values commonly used for m are between 2 and 4 for these saving plans, but banks also sell more aggressive CPPI structures with leverage factors of 5 and higher. For a sample path of a CPPI structure with leverage factor 3 and its equity and fixed income distribution see Fig. 3. For a leverage factor of 1, the CPPI becomes static in the sense that no rebalancing is necessary. This strategy guarantees 100 % capital protection in a continuous model. In a model with jumps this is not the case. The strategy is subject to gap risk, which is the risk that due to a jump in the market rebalancing is not possible and the fund value drops below the floor. We neglect liquidity issues here, which would cause an additional risk. However, due to continuous reduction of equity exposure, this risk is rather small compared to the risk of a Stop loss. In the general theory of constant proportion portfolio insurance the exposure is often not assumed to be capped by the portfolio value, but for the investment plans considered here borrowing is not allowed. See for example [6] for the general theory of constant proportion portfolio insurance. In the usual Black Scholes model with constant volatility where the stock S_t is modeled by

$$\frac{\mathrm{d}S_t}{S_t} = \mu \mathrm{d}t + \sigma \mathrm{d}W_t$$

and interest rates are kept deterministic, the final value of the uncapped CPPI can be shown to be

$$P_t = BD_{t,T} + \frac{P_0 - BD_{0,T}}{S_0^m} \exp\left\{ (1-m)\left(-\ln D_{0,t}\right) + \left(m - m^2\right) \frac{\sigma^2}{2} t \right\} S_t^m. \tag{4}$$

A very interesting property is that the uncapped CPPI is not path dependent. It is just the sum of a fixed income investment and a part proportional to S_t^m. This representation can be found in [2] for the case of constant rates. The generalization for the case of deterministic but not necessarily constant rates is straightforward. By the same arguments as used in [2] one can show that the expectation of P_t can be increased to any value by increasing the leverage factor as long as the equity drift μt is greater than the integrated forward rate $\int_0^t f_{0,s} \mathrm{d}s = -\ln D_{0,t}$. For the sake of completeness we derive (4) and $\mathbb{E}[P_t]$ in Appendix 2. These properties are lost in the capped version as shown in [3].

In our study we consider a stream of payments $p_0, \ldots, p_n$ at time points $t_0, \ldots, t_n$ instead of a single payment p_0 as in the situation described above. One possibility to treat this situation could be to set up n independent CPPI structures P_t^i starting at time points $t_0, \ldots, t_n$ each with its own rebalancing equation $E_t^i = \min\left(m\left(P_t^i - F_t^i\right), P_t^i\right)$ where $F_t^i = D_{t,T}\, p_i$. The total portfolio value P_t would then be

$$P_t = \sum_{i | t_i \le t} P_t^i.$$

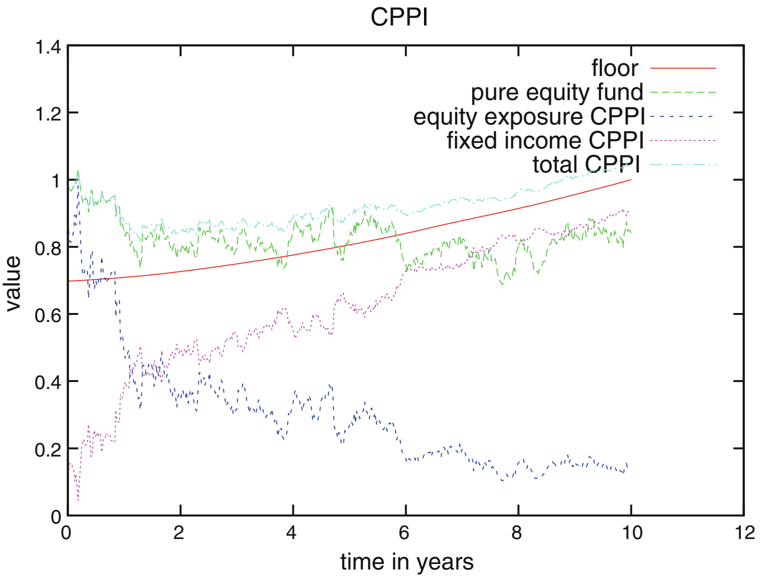

Fig. 3 Simulated CPPI path with leverage factor 3 and 10 years investment horizon

This setup seems unreasonable from a computational and administrative point of view. Therefore we assume that the insurance company only holds one CPPI portfolio for each insurant and handles the payments such that they increase the total guarantee of the CPPI. The floor to be protected is then $F_t = D_{t,T}\,B_t$ with B_t as defined in Eq. (3).

It can easily be seen from the definition above that in the uncapped case there is no difference between setting up n individual CPPI funds at time points $t_0, \ldots, t_n$ or only increasing the guarantee level of a single CPPI structure at each time point t_i. In the capped CPPI, instead, this procedure can lead to a higher equity proportion since

$$\begin{aligned} &\min[m(P_{t_i} - F_{t_i-}), P_{t_i}] + \min\left[m\left(p_i - D_{t,T}p_i\right), p_i\right] \\ &\quad \le \min\left[m\left((P_{t_i} + p_i) - (F_{t_i-} + D_{t,T}p_i)\right), P_{t_i} + p_i\right], \end{aligned}$$

where a strict inequality arises if $m(P_{t_i} - F_{t_i}) < P_{t_i}$ and $m\left(p_i - D_{t,T}p_i\right) > p_i$, or viceversa.

3.3 Stop loss strategy

In the Stop loss strategy 100 % of the invested amount is held in the risky fund as long as its value is larger than the floor. When this value reaches the floor, or goes below, all the investment is moved to the fixed income fund to generate the guarantee at maturity. See Fig. 4 for a path where the Stop loss barrier is reached and all the investment is shifted into the fixed income fund. With P_t being the fund value at time t we define the stopping time

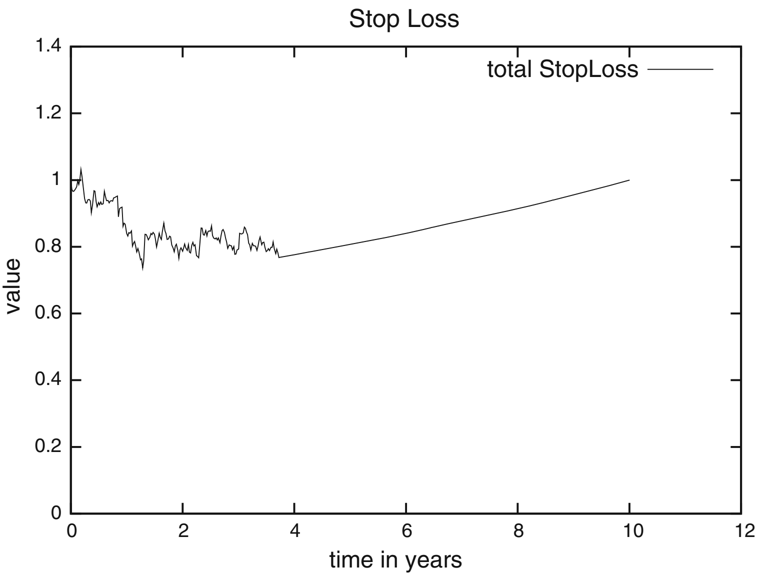

Fig. 4 Simulated Stop loss path with 10 years investment horizon

$$\tau := \inf\{t \geq 0 : P_t \leq F_t\}$$

with $F_t = D_{t,T}\,B_t$. The strategy invests for $t < \tau$ in the equity fund and for $t \geq \tau$ in the fixed income fund. Just like the CPPI strategy, in a continuous model, it is risk free in the sense that P_t can reach the floor but there is no risk that it goes below. However, in a model with jumps we are subject to gap risk. We neglect liquidity issues here, which actually force the insurer to liquidate the risky asset before it reaches the floor level. This issue becomes especially important for large funds, which is often the case for retirement plans or life insurance products since the amount to liquidate is so large that its trading actually influences the market.

The question is how to generalize the Stop loss strategy to payment streams. In case there was no investment in the fixed income fund before time point t_i the situation is clear. The new cash flow p_i is invested in equity and the guarantee level increases from B_{t_i-} to B_{t_i}. If at a time point later than t_i the fund falls below $F_t = D_{t,T}\,B_t$ all investment is moved to the fixed income fund. However, it is not clear what to do after the Stop loss was triggered. In case there is no huge gap loss due to a large market jump, the new cash flow could actually bring the total account value P_t back above the floor level F_{t_i}, at least if the discount factor $D_{t,T}$ is below 1, in which case $D_{t,T}\ p_i \leq p_i$. Therefore, it would be possible to move all the investment back into the equity fund until F_t is reached again for $t \geq t_i$. However, in our situation of a payment stream with many small payments, this procedure doesn't seem reasonable because the difference $P_{t_i} - F_{t_i}$ can be so small that it is very likely that all the investment has to be moved back to the fixed income fund very soon. Therefore, we assume that the fixed income fund has only positive cash flows during the lifetime of the contract. Once money is invested in the fixed income fund, it stays there until maturity. New cash inflow from the client is always invested in equity and as soon as the total fund value (fixed income and equity fund) reaches F_t all money is invested in the fixed income fund. This is a practical approach and generalizes the Stop loss concept to payment streams.

4 Payments to the contract and simulation horizon

Since capital guaranteed life insurance products are especially popular in Germany under the master agreement of the *Riester-Rente*, we consider a typical payment plan with an horizon of 20 years. To be eligible for the maximal amount of cash payments and tax benefits the insured has to spend at least 4 % of his yearly gross income for the insurance product, including the federal cash payments, but no more than 2,100 Euro. In this case he receives federal cash payments of 154 Euro per year, and additional 185 Euro for each child born before January 1 2008 and 300 Euro for each child born on or after this date. Even though this is not the focus of this paper, we assume a savings plan that allows for these benefits in order to compare typical cost charges against the state benefits. We consider the situation of a person being 45 years old[4] when entering the contract and earning 30,000 Euro a year. We further assume that he has one child born after January 1 2008 but before entering the contract. In this case the insured receives 454 Euro from the state, so he actually only has to pay 746 Euro per year to reach 1,200 per year (4 % of his income). This is a very high support rate of 37.8 %. For comparison, if we take an investor without children, earning 52,500 Euro per year, the support rate would be only 7.3 %. We assume a monthly payment of 100 Euro and do not distinguish between payments made by the state and by the insured. The total nominal amount is $20 \times 1,200 = 24{,}000$ Euro if we assume that he retires in 20 years. This is the amount the issuer of the plan needs to guarantee at retirement. There is no guarantee during the lifetime of the contract. Especially in the case that the insured dies before retirement, the current payments to the contract are not guaranteed and only the current account value can be transferred to another contract or paid out. In case the contract is terminated early, payments from the state will be claimed back.

5 Results without costs

We present the simulation results for different scenarios. Since the drift is hard to estimate from past realizations, we consider three different drift assumptions. In Fig. 5 it can be seen how the distribution varies between the different strategies under the DDE-model with stochastic rates in the standard scenario with $\mu = \gamma + \lambda\, \mathbb{E}\,[e^Y - 1\,] = 6\ \%$.

In Tables 3, 4 and 5 we list the mean and the median of the accumulated capital available at retirement for all three scenarios. Also the average equity exposure is

4 The age of the insured is actually not important but it determines the lifetime of the contract as the time to retirement.

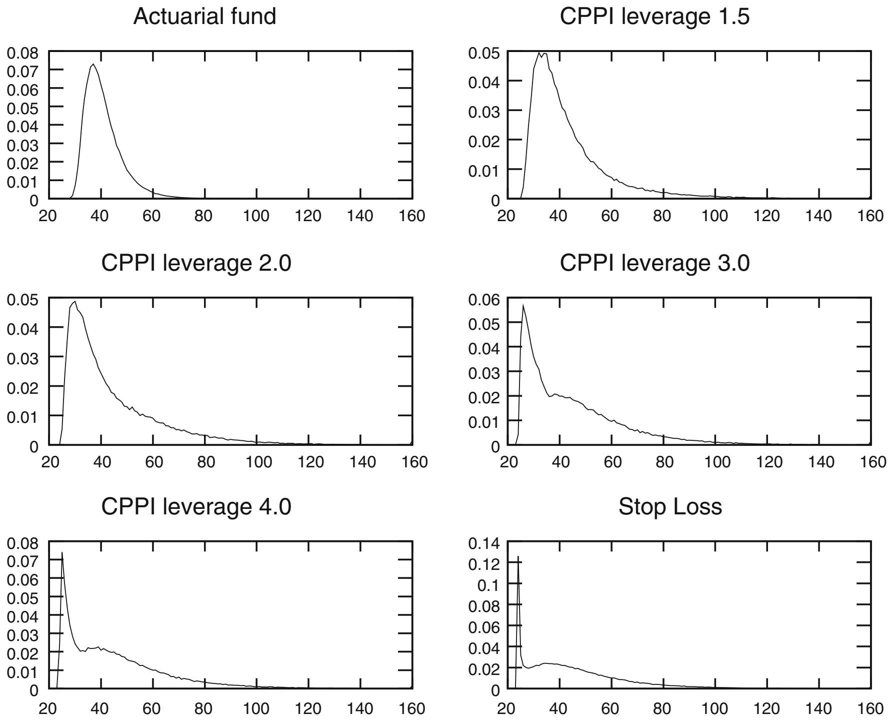

Fig. 5 Return distribution of the different strategies. We list the capital available at retirement (in units of 1,000 EUR) on the *x*-axes

shown for all strategies. It is calculated as an average over time and over simulations. The Stop loss strategy has an expected distribution very close to the pure equity investment since it has the highest equity participation. Therefore for the bullish investor this might be the optimal investment for his *Riester-Rente* or for a general investment plan with capital guarantee. A similar return profile is provided by a CPPI structure with a high leverage factor. The advantage of the CPPI product in practice is that due to the continuous reduction of the equity exposure if the market performs badly, the liquidity issue is smaller than for the Stop loss strategy. However, as can be seen in Fig. 5, the risk of returns close to zero is rather high for both, the Stop loss and the CPPI with a high leverage factor. For the bearish investor a classical product with an investment mainly in the actuarial fund or a CPPI product with a small leverage factor could be the better choice. The effect of stochastic rates varies between the strategies. The actuarial fund gains from stochastic rates due to its guaranteed level of interest of 2.25%, which has the effect that the portfolio gains from increasing rates while being protected against rates falling below 2.25%. The other structures suffer from the higher risk to end in a Cash-Lock, a situation where all the money is invested in the fixed income fund and only new cash inflow is partly invested in the equity fund.

Table 3 Results in the standard scenario (drift $\mu = 6\%$)

Strategy	Deterministic rates			Stochastic rates		
	Mean	Median	Exposure (%)	Mean	Median	Exposure (%)
Actuarial	40,933	39,406	36.77	41,635	40,082	38.30
CPPI leverage 1.5	43,570	38,820	73.35	43,165	38,492	72.95
CPPI leverage 2	44,657	38,126	87.68	44,252	37,760	86.82
CPPI leverage 3	45,160	39,785	93.94	44,937	39,488	93.40
CPPI leverage 4	45,279	40,525	95.48	45,109	40,321	95.04
Stop loss	45,392	40,903	96.56	45,269	40,733	96.10

Table 4 Results in the optimistic scenario (drift $\mu = 8\%$)

Strategy	Deterministic rates			Stochastic rates		
	Mean	Median	Exposure (%)	Mean	Median	Exposure (%)
Actuarial	45,100	43,077	39.17	46,002	43,911	40.74
CPPI leverage 1.5	53,069	46,039	78.94	52,435	45,522	78.40
CPPI leverage 2	55,864	48,571	92.57	55,353	47,948	91.74
CPPI leverage 3	56,793	51,018	97.08	56,581	50,768	96.70
CPPI leverage 4	56,970	51,302	98.01	56,826	51,155	97.73
Stop loss	57,084	51,433	98.63	56,975	51,324	98.34

Table 5 Results in the pessimistic scenario (drift $\mu = 4\%$)

Strategy	Deterministic rates			Stochastic rates		
	Mean	Median	Exposure (%)	Mean	Median	Exposure (%)
Actuarial	37,693	36,535	34.48	38,243	37,099	35.96
CPPI leverage 1.5	37,153	34,069	67.72	36,908	33,874	67.44
CPPI leverage 2	36,889	32,365	81.62	36,601	32,148	80.82
CPPI leverage 3	36,713	30,885	89.09	36,516	30,698	88.46
CPPI leverage 4	36,668	30,848	91.22	36,503	30,602	90.64
Stop loss	36,722	32,200	92.84	36,592	31,959	92.19

6 Impact of costs

We study the impact of different cost structures observed in insurance products. Often the fee structure is rather complex and consists of a combination of various fees.

– Sales and Distribution cost: These costs are usually charged to pay a sales fee for the agent who closed the deal with the insured. These fees are usually dependent on the total cash contracted to pay into the product until maturity. However they are charged uniformly distributed over the first 5 years of the contract. In insurance business they are called α-cost.

– Administration cost: These costs are charged on the cash payments to the contract during the entire lifetime of the contract. They are charged to cover administrative costs of the contract. In insurance business they are called β-cost.
– Capital management cost (Cost on accumulated payments): These costs are charged based on the sum of the payments up to the effective date. They are usually charged for capital management. Effective date is every date at which a payment takes place.

To compare the impact of different cost structures we analyze costs that are equivalent in terms of the current value. We assume a total fee of 4% of all payments to the contract, i.e. 960 Euro, charged on the day of payment. This is exactly the β-cost. The current value of these fees with the applied zero-coupon bond curve is 681 Euro, where each fee payment is discounted with the discount value corresponding to its payment date. Now we adjust the α-cost and Capital management cost such that they have the same current value although charged differently over time. This results in α-cost of 3.01 % and Capital management cost of 0.038 %. The reason that the Capital management cost is so small as a percentage is that it is not applied to the single payments but to the sum of all payments which becomes rather high at the end of the contract. Since the typical costs in insurance products are usually a combination of all these fees we also simulate the impact of these three fees together which is a commonly used cost charge (current value $3 \times 681 = 2{,}043$).

The simulation results with fees are shown in Tables 6, 7 and 8. It can be seen that fees have a high impact on the return distribution. Even if the fees have actually the same current value, the impact on the return distribution is different. The α-cost weakens the expected return most since it decreases the exposure at the beginning of the saving period. This impact is very high for the actuarial reserve product, which even without fees only has an average equity participation of 36.77 %. The α-fee reduces this further to 31.30 % as can be seen in Table 6. Fees on the accumulated payments have the least impact since these are mainly charged at the end of the saving period and therefore the impact on the equity exposure is smaller. The insured has to carefully study whether the negative impact of the fee structure is actually fully compensated by the federal cash payments. This highly depends on the cost structure, which varies massively between the different products and on the income and family situation of the insured, which in turn determines the amount of cash benefits from the state. For the situation above with a person with one child for whom the payment stream is optimal in the sense that it maximizes his federal cash payments, the payments add up to 9,080 euro. For people with a greater income or without children the support is considerably lower. Also the fee structures we consider here are only examples, usually the contracts sold in Germany include many different fees whose impact can be much higher. In many cases it may be advisable to choose a product outside the class of *Riester-Rente* that has a smaller cost ratio. In this case the investor can buy less costly products and can freely choose one without capital protection, which has a higher expected return.

Table 6 Results in the actuarial reserve product with costs in the standard scenario

Cost	Deterministic rates			Stochastic rates		
	Mean	Median	Exposure (%)	Mean	Median	Exposure (%)
No cost	40,933	39,406	36.77	41,635	40,082	38.30
α-cost	38,922	37,685	31.30	39,623	38,366	33.00
β-cost	39,111	37,756	34.08	39,813	38,442	35.67
Cost on accum. payments	39,169	37,744	35.39	39,871	38,424	36.98
All fees applied	35,239	34,283	26.26	35,942	34,975	28.13

Table 7 Results in the CPPI strategy with leverage factor 3 with costs in the standard scenario

Cost	Deterministic rates			Stochastic rates		
	Mean	Median	Exposure (%)	Mean	Median	Exposure (%)
No cost	45,160	39,785	93.94	44,937	39,488	93.40
α-cost	43,049	37,461	92.26	42,731	37,026	91.35
β-cost	43,249	37,535	92.54	42,980	37,177	91.86
Cost on accum. payments	43,312	37,557	92.71	43,064	37,221	92.10
All fees applied	39,118	32,581	88.31	38,695	32,090	86.97

Table 8 Results in the Stop loss strategy with costs in the standard scenario

Cost	Deterministic rates			Stochastic rates		
	Mean	Median	Exposure (%)	Mean	Median	Exposure (%)
No cost	45,392	40,903	96.56	45,269	40,733	96.10
α-cost	43,327	38,959	95.52	43,129	38,698	94.74
β-cost	43,522	39,078	95.57	43,350	38,856	94.94
Cost on accum. payments	43,595	39,114	95.61	43,447	38,907	95.06
All fees applied	39,547	35,004	92.55	39,277	34,598	91.33

7 Impact of jumps

The CPPI and the Stop loss strategy are risk free in a continuous equity model. However, in a model with jumps, we are exposed to *gap risk*, which means that the value of the portfolio of risky assets can fall below the floor. In this case the strategy fails to generate the guarantee. In practice the leverage factor is chosen such that even a very big jump in the market still maintains the guarantee. For example, a 20 % jump in the market does not bring the portfolio value below the floor for a portfolio that has been rebalanced before the jump if the leverage factor is below 5. In this case all the equity exposure would be lost, but the guarantee could exactly be generated. So, with a moderate leverage factor the

Table 9 Results in the Stop loss strategy with costs in the standard scenario

Product	Deterministic rates		Stochastic rates	
	Number of paths with gap	Average realized gap	Number of paths with gap	Average realized gap
Actuarial	0	0	0	0
CPPI leverage 1.5	0	0	535	1,060.99
CPPI leverage 2	0	0	967	921.04
CPPI leverage 3	0	0	2,596	746.68
CPPI leverage 4	0	0	4,720	689.25
CPPI leverage 8	44	55.57	12,312	710.49
CPPI leverage 9	136	60.58	13,461	744.24
CPPI leverage 10	329	64.41	14,495	775.07
Stop loss	19,620	109.98	20,516	543.40

remaining gap risk is negligible. This is also reflected in our simulation results. For the Stop loss strategy the situation is different since the equity exposure does not decrease when approaching the floor level, which yields a much greater gap risk.

We calculate how often the strategy fails on average due to a jump. The expected number of guarantee shortfalls is shown in Table 9 for both, constant rates and stochastic rates.

After the Stop loss level is reached and all money is invested in the fixed income fund it stays there until maturity unless there is new cash flow. Therefore, it might happen that due to the new cash flow there are several guarantee shortfalls in one path. We state in the column **Average realized gap** the average sum of injections to the fund that have to be done to generate the guarantee at maturity. For the CPPI strategy we also look at very high leverage factors of 8, 9 and 10 for comparison with existing literature although they are not used in insurance business.

In case of constant rates even with a leverage factor of 10 the CPPI strategy very rarely fails. This is due to the cap in the CPPI, which has the effect that actually increasing the leverage factor above 3 does not increase the equity exposure much more. The reason for this is the long maturity of the products and the corresponding small zero bond price at the beginning, which yields a high cushion value as a percentage of the portfolio value. The high factor can only have an affect for payments close to maturity. Then the cushion is smaller if the past equity fund performance was very weak and $m(P_i - F_t) \leq P_t$ can hold also for values of m greater than 3. Our results are in line with the findings in [11] where there is an almost zero probability of reaching the floor for multipliers up to 4. However, for an uncapped CPPI a higher leverage does increase the risk considerably.

In the case of stochastic rates we have a higher number of paths with shortfalls. In the CPPI with small leverage factor, they mainly happen in the case of negative interest rates, which is allowed by the model, together with a very low performing equity market period[5]. The current value of the future cash flow is higher than its actual amount in this case. Also the gap is often not realized at once but distributed over several time points in a path due to the fact that the current value of the incoming cash flows is close to or above 1 for a longer time interval. For a higher leverage factor an unfavorable movement of rates and stock prices can lead to a gap more easily and is not limited to the situation where rates are negative. This shows that for a small leverage factor the dynamic reduction of equity exposure when approaching the floor works rather well even when rates are stochastic. When interest rates drop, the floor rises, but in turn, the fixed income investment in zero-coupon bonds performs better. Due to continuous trading most of the fund value is already invested in bonds with the correct maturity when the total portfolio value is close to its floor.

In Appendix 3 we tested further whether our findings hold for a jump model where jumps are not displaced, which turns out to be the case.

8 Summary

We have compared the performance of savings plans within the class of different capital guarantee mechanisms: from the Stop loss to classic investments in the actuarial reserve fund. CPPI strategies with different leverage factors can be viewed as a compromise between these two extremes. In bullish markets savings plans with a high equity ratio perform the best, in bearish markets the classic insurance concept shows better returns. A Stop loss strategy suffers from gap risk, whence a CPPI strategy combines the strength of both gap risk minimization and equity ratio maximization in both, a constant rate scenario as well as in the stochastic rate scenario. Even for a high leverage factor the gap risk of the capped CPPI turns out to be relatively small. The effect of fees on the savings plans dominates the performance, especially in typical fee structures found in the German *Riester-Rente*. The private investor is advised to check carefully if the federal cash payments can compensate the fees taking into account his own salary and tax situation. We propose a jump diffusion model, which is easy to implement and where parameters can be easily estimated. The robustness checks suggest that the assumption of displaced jumps does not lead to very different results than the usual assumption of jump sizes in the entire real line. However, one has to check carefully in each situation if the model is appropriate.

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[5] The possibility of negative rates is often seen as a drawback of Gaussian models and sometimes numerical adjustments are used to avoid them. We decided for two reasons to allow them. Firstly, negative rates could be observed in the past over short time periods. Secondly, a correction would not be in line with the pricing of zero coupon bonds under the risk neutral measure.

Appendix 1

In the following we derive several properties of the displaced double exponential jump diffusion model and the parameter estimation.

Jump distribution

The jump part of the process has the density

$$f_Y(y) = \begin{cases} p\eta_1 e^{-(y-\kappa)\eta_1} & if\, y \geq \kappa, \\ 0 & if\, |y| < \kappa, \\ (1-p)\eta_2 e^{(y+\kappa)\eta_2} & if\, y \leq -\kappa, \end{cases}$$

with $\eta_1 > 1$, $\eta_2 > 0$ and $0 \leq p \leq 1$. We show a graph of the density function in Fig. 6.

Similar to the work of Kou [15] we calculate

$$\begin{aligned}\mathbb{E}[Y] &= \int_{-\infty}^{\infty} y f_Y(y)\mathrm{d}y \\ &= \int_{\kappa}^{\infty} y p \eta_1 e^{-(y-\kappa)\eta_1}\mathrm{d}y + \int_{-\infty}^{-\kappa} y(1-p)\eta_2 e^{(y+\kappa)\eta_2}\mathrm{d}y \\ &= p\frac{1}{\eta_1} - (1-p)\frac{1}{\eta_2} + \kappa(2p-1),\end{aligned}$$

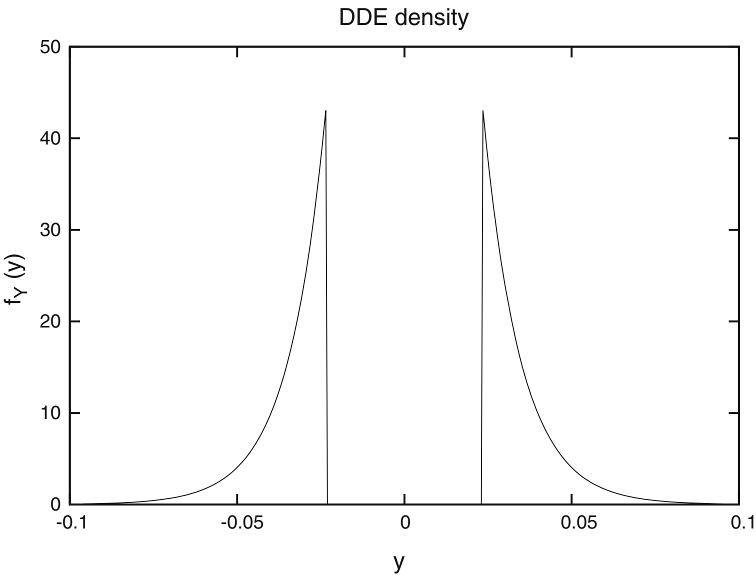

Fig. 6 Displaced Double-Exponential density of Y with parameters $\kappa = 2.31$ %, $\eta_1 = \eta_2 = \eta = 1/1.121$ %, $p = 0.5$

$$\begin{aligned}\mathbb{E}[Y^2] &= \int_{\kappa}^{\infty} y^2 p\eta_1 e^{-(y-\kappa)\eta_1}\mathrm{d}y + \int_{-\infty}^{-\kappa} y^2(1-p)\eta_2 e^{(y+\kappa)\eta_2}\mathrm{d}y \\ &= p\left(\frac{2}{\eta_1^2}+\frac{2\kappa}{\eta_1}\right) + (1-p)\left(\frac{2}{\eta_2^2}+\frac{2\kappa}{\eta_2}\right) + \kappa^2\end{aligned}$$

and

$$\mathbb{E}[e^Y] = p\eta_1 \frac{e^{+\kappa}}{\eta_1 - 1} + (1-p)\eta_2 \frac{e^{-\kappa}}{\eta_2+1}.$$

Using the relation $Var[Y] = \mathbb{E}(Y^2) - \mathbb{E}(Y)^2$ we get

$$\begin{aligned}Var[Y] = (1-p)\frac{(2+\eta_2\kappa(2+\eta_2\kappa))}{\eta_2^2} + p\frac{(2+\eta_1\kappa(2+\eta_1\kappa))}{\eta_1^2} \\ - \left((-1+p)\frac{1+\eta_2\kappa}{\eta_2} + p\frac{1+\eta_1\kappa}{\eta_1}\right)^2.\end{aligned}$$

The drift due to the jump part is given by

$$\begin{aligned}\delta :&= \lambda\mathbb{E}[e^Y - 1] \\ &= \lambda\left(p\eta_1\frac{e^{+\kappa}}{\eta_1-1} + (1-p)\eta_2\frac{e^{-\kappa}}{\eta_2+1} - 1\right).\end{aligned} \tag{5}$$

Variance and volatility of log returns

The variance of the random number $\ln\frac{S_t}{S_0}$ of the process (1) can be written as

$$\begin{aligned}Var\left[\ln\frac{S_t}{S_0}\right] &= \sigma_1^2 t + Var\left[\sum_{j=1}^{N_t} Y_j\right] \\ &= \sigma_1^2 t + \lambda t\mathbb{E}[Y^2]\end{aligned}$$

and the volatility as

$$\sigma_{1tot} = \sqrt{\frac{1}{t}Var\left[\ln\frac{S_t}{S_0}\right]} = \sqrt{\sigma_1^2 + \lambda\mathbb{E}[Y^2]}. \tag{6}$$

Parameter estimation

Here we describe the parameter estimation for the displaced double exponential jump diffusion process. We estimate the parameters based on the historical data of the **MSCI Daily TR (Total Return) Gross (gross dividends reinvested) in USD** for the period between January 1 1980 and October 2 2009. We denote these prices with $x_0, x_1, \ldots, x_N$ and the log-returns by

$$d_i := \ln\frac{x_i}{x_{i-1}}, \quad i = 1, \ldots, N.$$

The estimate for the expected daily log return is

$$\bar{d} = \frac{1}{N}\sum_{i=1}^{N} d_i.$$

The estimate for the total volatility $\hat{\sigma_1}_{tot}$ is

$$\hat{\sigma_1}^2_{tot} = \frac{N_a}{N-1}\left(\sum_{i=1}^{N} d_i^2 - N\bar{d}^2\right),$$

where N_a is the number of observations per year.

To determine the parameters for the jump process we have to define a level κ such that d_i with $\|d_i\| \geq \kappa$ is considered to be a jump. To determine this κ we calculate for a given level $u \in [0, 1]$ the u-quantile a and the $(1-u)$-quantile b of the empirical distribution. For the analyzed MSCI World Index it turns out that a and b are almost symmetric and κ is taken such that $\kappa \simeq b$ and $-\kappa \simeq a$. The level u should be chosen such that the resulting returns are intuitively considered as jumps. If u is chosen too high, even small log-returns are considered jumps, and if u is too low, almost no jumps occur. Of course, this level is subjective. We have chosen $u = 0.01$. In this case the minimal jump size is estimated $\kappa = (b-a)/2 = 2.31\%$ such that only daily changes of more than 2.31 % are considered to be jumps. Smaller changes can be explained with the diffusion part with sufficiently high probability. It also turns out that for the analyzed MSCI World Index, the average up-jump and the average down-jump is almost equal and we use $\eta = \eta_1 = \eta_2$ and $p = 0.5$.

The value η of the single parameter exponential distribution is chosen such that the mean of the distribution fits the mean of the observed jumps. From the financial data and the already fixed parameters we obtain $h = 1/\eta = 1.121\%$. The number of jumps divided by the total number of observations yields an estimate for the jump frequency. Annualizing this frequency we can estimate λ to be 5.21, which intuitively is 2% of the trade days per year.

Finally we have to correct the estimator for the volatility according to (6) since the volatility consists of the jump part and the diffusion part. We define the constant drift parameter γ in (1) by $\gamma := \mu - \delta$ with δ as defined in (5) such that the process S_t has the desired total drift μ, meaning

$$\mathbb{E}[S_t] = S_0 e^{t\mu}.$$

We summarize the estimated parameters in Table 10.

Appendix 2

Here we derive the distributional properties of the simple CPPI without borrowing restrictions under the following model[6]:

$$\mathrm{d}S_t = S_t[\mu \mathrm{d}t + \sigma \mathrm{d}W_t] \tag{7}$$

$$\mathrm{d}D_{t,T} = f_{0,t} D_{t,T}\mathrm{d}t$$

[6] Hence we assume a deterministic instantaneous spot rate r_t coinciding with the corresponding forward rate $f_{0,t}$ observed at time 0.

Table 10 Estimated parameters for the DDE-process

Parameter	Value
Total volatility $\hat{\sigma}_{1\,tot}$	14.3%
Volatility of the diffusion part $\hat{\sigma}_1$	11.69%
Jump intensity λ	5.209
Minimum jump size κ	2.31%
Expected jump size above minimum jump size h	1.121%
Drift adjustment δ	0.339%

The value of the floor is $F_t = D_{t,T}B$. We follow mainly [2] with the difference that the instantaneous interest rate is allowed to change deterministically over time. Let $C_t := P_t - F_t$ be the cushion process. Then

$$\begin{aligned}
\mathrm{d}C_t =& \mathrm{d}P_t - \mathrm{d}F_t \\
=& \left(\frac{mC_t}{S_t}\mathrm{d}S_t + \frac{(P_t - mC_t)}{D_{t,T}}\mathrm{d}D_{t,T}\right) - \frac{F_t}{D_{t,T}}\mathrm{d}D_{t,T} \\
=& C_t\left(\frac{m}{S_t}\mathrm{d}S_t - \frac{m}{D_{t,T}}\mathrm{d}D_{t,T} + \frac{P_t}{C_tD_{t,T}}\mathrm{d}D_{t,T} - \frac{F_t}{C_tD_{t,T}}\mathrm{d}D_{t,T}\right) \\
=& C_t\left(\frac{m}{S_t}\mathrm{d}S_t - \frac{m-1}{D_{t,T}}\mathrm{d}D_{t,T}\right) \\
=& C_t\left(\frac{m}{S_t}\mathrm{d}S_t - (m-1)f_{0,t}\mathrm{d}t\right) \\
=& C_t\left(m\sigma\mathrm{d}W_t + \left[m\mu - (m-1)f_{0,t}\right]\mathrm{d}t\right),
\end{aligned}$$

which is solved by

$$\begin{aligned}
C_t =& C_0 \exp\left(m\sigma W_t + \int_0^t (m\mu - (m-1)f_{0,s})\mathrm{d}s - \frac{1}{2}\sigma^2 m^2 t\right) \\
=& C_0 \exp\left(m\sigma W_t + m\mu t - (m-1)\int_0^t f_{0,s}\mathrm{d}s - \frac{1}{2}\sigma^2 m^2 t\right) \\
=& C_0 \frac{S_t^m}{S_0^m} \exp\left(-(m-1)\int_0^t f_{0,s}\mathrm{d}s + (m - m^2)\frac{1}{2}\sigma^2 t\right),
\end{aligned} \tag{8}$$

since $S_t = S_0 \exp\left(\mu t - \frac{1}{2}\sigma^2 t + \sigma W_t\right)$. Together with $C_0 = P_0 - B\, D_{0,T}$ and $F_t = B\, D_{t,T}$ we obtain

$$P_t = BD_{t,T} + \frac{P_0 - BD_{0,T}}{S_0^m}\exp\left\{(1-m)\int_0^t f_{0,s}\mathrm{d}s + \left(m - m^2\right)\frac{\sigma^2}{2}t\right\}S_t^m.$$

which is the same as (4) since $\int_0^t f_{0,s}\mathrm{d}s = -\ln D_{0,t}$.

To derive $\mathbb{E}[P_t]$, we recall that $\mathbb{E}[X] = \exp\left[\mu_X + 1/2\sigma_X^2\right]$, if $\ln X$ is normally distributed with mean μ_X and variance σ_X^2. Using Eq. (8) we get

$$\begin{aligned}\mathbb{E}[P_t] =& \mathbb{E}[F_t] + \mathbb{E}[C_t] \\ =& BD_{t,T} + C_0 \exp\left(m\mu t - (m-1)\int_0^t f_{0,s}\mathrm{d}s \right) \\ =& BD_{t,T} + (P_0 - BD_{0,T}) \exp\left(m\int_0^t (\mu - f_{0,s})\mathrm{d}s + \int_0^t f_{0,s}\mathrm{d}s \right).\end{aligned}$$

This shows that in the uncapped CPPI the expected value of the total investment can be increased by increasing the multiplier m as long as the integrated forward rate is smaller than the integrated equity fund drift.

Appendix 3

In order to test robustness we also estimate the parameters for the double exponential jump diffusion model without displaced jumps and compare the simulation results. We follow [11] by estimating the parameters based on the empirical characteristic function of the logarithmic returns. Let $\hat{\Psi}(u)$ be the empirical characteristic function and $\Psi_\theta(u)$ the theoretical characteristic function for the model with parameter set θ. The estimation then reduces to minimizing the quadratic distance

$$\int_{-K}^{K} |\hat{\Psi}(u) - \Psi_\theta(u)|^2 w(u)\mathrm{d}u,$$

where $w(u)$ is a weighing function and K the cutoff parameter. We use the weighing function:

$$w(u) := \frac{\exp\left(-\sigma_D^2 u^2\right)}{1 - \exp(\sigma_D^2 u^2)},$$

with σ_D^2 being the variance of the log returns $d_1, \ldots, d_N$. The minimization is done with a simulated annealing algorithm. We summarize the estimated parameters in Table 11. The minimization converges very slowly and different value for the temperature of the annealing algorithm and different starting values lead to rather different estimations for h and λ. This is due to the fact that there are several combinations of the two parameters which lead to almost the same fit to the data. It is hard to distinguish whether there are small jumps with high intensity or larger jumps with low intensity. The parameter p and $\hat{\sigma}_1$ turn out to be much more stable. While p is very close to 0.5 as we found also with our method, the volatility $\hat{\sigma}_1$ is estimated greater without the displaced jumps. This is reasonable since there is a positive probability that also the diffusion part of the process causes movements above the minimum jump size κ when discretized. These movements are neglected

Table 11 Estimated parameters by empirical characteristic function

Parameter	Value
Volatility of the diffusion part $\hat{\sigma_1}$	13.96%
Jump intensity λ	48.63
Probability of a down jump p	0.51
Expected jump size h	0.34%

Table 12 Expected values and shortfalls under double exponential jump diffusion model with constant rates (standard scenario)

Product	Mean	Median	Exposure (%)	Number of paths with gap	Average realized gap
Actuarial	40,967	39,405	36.78	0	0
CPPI leverage 1.5	43,636	38,842	73.37	0	0
CPPI leverage 2	44,742	38,158	87.68	0	0
CPPI leverage 3	45,237	39,799	93.91	0	0
CPPI leverage 4	45,352	40,538	95.47	0	0
Stop loss	45,443	40,918	96.35	19,666	72.91

in our estimation method for the diffusion part since they are all attributed to the jumps. We show the simulation results in Table 12. We find that the results for the mean, the median and the exposure are very similar. Both model describe the return distribution very well. For the normal double exponential jump diffusion model the gap risk is also zero for the capped CPPI. For Stop loss, the average realized gap is smaller than with the displaced jumps due to the lower expected jump size but in turn the higher intensity.

References

1. Bacinello AR (2001) Fair pricing of life insurance participating policies with a minimum interest rate guaranteed. ASTIN Bull 31(2):275–297
2. Balder S, Brandl M, Mahayni A (2009) Effectiveness of CPPI strategies under discrete-time trading. J Econ Dyn Control 33(1):204–220
3. Balder S, Mahayni AB (2010) Cash-lock comparison of portfolio insurance strategies. Working paper, MSM, Duisburg University
4. Bertrand P, Prigent JL (2003) Portfolio insurance strategies: a comparison of standard methods when the volatility of the stock is stochastic. Int J Business 8(4):15–31
5. Bertrand P, Prigent JL (2005) Portfolio insurance strategies: OBPI versus CPPI. Finance 26:5–32
6. Black F, Perold A (1992) Theory of constant proportion portfolio insurance. J Econ Dyn Control 16(3-4):403–426
7. Black F, Scholes M (1973) The pricing of options and corporate liabilities. J Political Econ 81(3):637–654
8. Brigo D, Mercurio F (2001) Interest rate models-theory and practice. Springer, Berlin

9. Briys E, de Varenne F (1997) On the risk of insurance liabilities: debunking some common pitfalls. J Risk Insur 64:673–694
10. Cesari R, Cremonini D (2003) Benchmarking, portfolio insurance and technical analysis: a Monte Carlo comparison of dynamic strategies of asset allocation. J Econ Dyn Control 27(6):987–1011
11. Cont R, Tankov P (2009) Constant Proportion Portfolio Insurance in presence of jumps in asset prices. Math Finance 19:379–401
12. Detering N, Weber A, Wystup U (2011) Return distributions of equity-linked retirement plans, vol 2, 2nd edn. Statistical tools for finance and insurance. Springer, Berlin, pp 393–413
13. Gerstner T, Griebel M, Holtz M (2009) Efficient deterministic numerical simulation of stochastic asset-liability management models in life insurance. Insurance Math Econ 44(3):434–446
14. Hardy M (2003) Investment guarantees: modelling and risk management for equity-linked life insurance. Wiley Finance, London
15. Kou SG (2002) A jump-diffusion model for option pricing. Manage Sci 48(8):1086–1101
16. Kou SG, Wang H (2004) Option pricing under a double exponential jump diffusion model. Manage Sci 50(9):1178–1192