FX Column: Can Volga of a Long Vanilla Option be Negative?

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Volga is an abbreviation for volatility-gamma, strictly non-related to the river in Russia. It is sometimes also called *vomma* or *d-vega-d-vol*. Mathematically, it is defined as the second derivative of the value of a derivative contract with respect to volatility σ and can consequently be interpreted as the first derivative of vega with respect to volatility, because vega is already the first derivative of the value with respect to volatility. For vanilla options in the Black-Scholes model the analytic formula for volga is

$$Se^{-r_f T}\sqrt{T}n(d_+)\frac{d_+ d_-}{\sigma},$$

where as usual S denotes the spot reference, r_f the foreign continuous interest rate (or dividend rate) in a currency pair FOR-DOM, so in EUR-USD with spot 1.1500, it would be the EUR interest rate, T the time to maturity (in years), σ the volatility, $d_\pm = \frac{\ln\frac{F}{K}\pm\frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$ the usual auxiliary terms in the Black-Scholes formula, which some folks denote by d_1 and d_2 if they haven't been exposed to a symmetry missionary like myself. For completeness, F denotes the Forward price and K the strike price. Note that the formula does not depend on whether the option is a call or a put. And of course, for non-vanilla options, there are also formulas, but they often turn out to be quite lengthy. Let's focus on vanilla volga in this column.

Figure 1 illustrates the volga profile of a vanilla call (or put, doesn't matter) on the spot and time space.

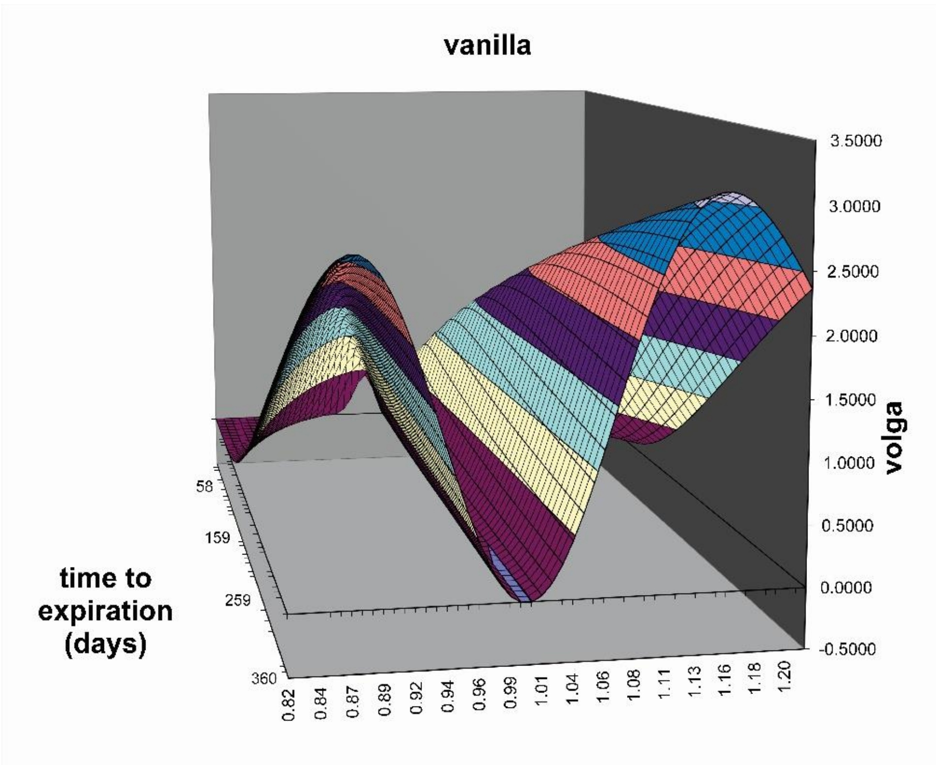

Figure 1: volga profile of a long vanilla option

Volga being a type of gamma insinuates that it must be always non-negative, and the graph in Figure 1 seems to confirm this. However, when we look closely in the middle, which is at-the-money for the option, the situation appears to be a bit ambiguous, and one might ask whether volga is *really* always non-negative. A negative *gamma* would be equivalent to a negative probability density and consequently indicate a butterfly arbitrage. But what about a negative volga? This would mean that more volatility would cause less volatility risk, which may feel counter-intuitive at first glance. In fact, this question came up in one of my training courses on FX options (Figure 2), which have been running for 15 years all over the world, and there hasn't been a single course without a puzzle like this.

Figure 2: An FX Options instructor in Warsaw in June 2021

There are many ways to answer the questions. The mathematician would most likely perform her standard procedure, calculate the derivative of volga with respect to the strike and set the result equal to zero, i.e., applying high-school math to a non-high-school formula so it feels intellectual. Here we go:

The first order condition can be written as $-2l + d_+ d_+ d_- s = 0$, where we use the abbreviations $l = \ln\frac{F}{K}$ and $s = \sigma\sqrt{T}$ or equivalently $8l^3 + 4s^2l^2 - (16s^2 + 2s^4)l - s^6 = 0$. This cubic equation in l generally has three real roots (Cardano formula), which meets our expectation, as when we look at Figure 1, we would expect two maxima in the wings and one minimum in the center. There will be one root minimizing volga, which is near the forward price, and two more roots maximizing volga. Corresponding optimizing strikes are obtained via $K = Fe^{-l}$.

Example

If we take a forward price $F = 1.0000$, volatility σ = 10%, maturity $T = 1$ year, then $s = 0.1$, and the three optimal strikes are listed in Table 1. The center strike, which is near the forward price, minimizes volga and does in fact take a negative value; so mathematically, we are done.

strike	strike	forward delta	volga
K_+	1.15492	8.22%	314.46%
K_0	1.00001	51.99%	-0.996%
K_-	0.87020	92.51%	272.96%

Table 1: Strike prices leading to maximum and minimum values for volga

Intuition

An experienced options instructor would probably look at the question from another point of view. The graph in Figure 3 shows the value of a vanilla call options as a function of volatility on the y-axis. We know that for contracts with convex payoff-functions like a call or a put option the value increases with volatility: **"the higher the vol, the higher the value"**. But how does the value function grow as volatility increases. I typically let the audience put high values of volatility into their option pricing tool and let them observe. Some of them conclude that the option value must be bounded – but not all of them! Some like to argue for linear or log-like growth. Well, it must clearly be bounded, because the call option entitles the holder to buy 1 EUR at maturity, which cannot be worth more today than a discounted EUR. Therefore, the function must have an inflection point and must be concave down on the right-hand side, which in turn means a negative second derivative as high-school math has taught us.

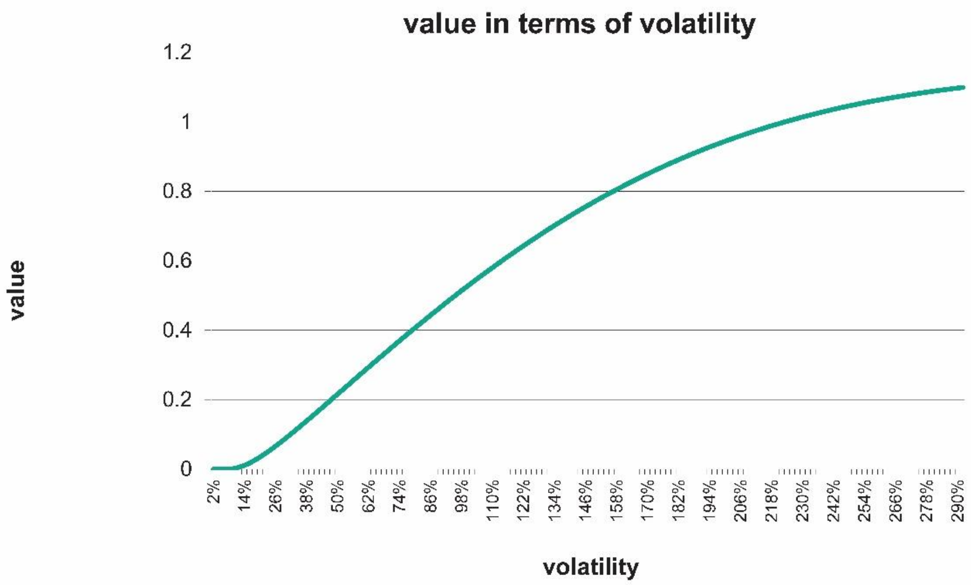

Figure 3: Black-Scholes value of an at-the-money vanilla call option as a function of volatility

The second derivative is called volga and is hence negative for an at-the-money option when volatilities are high. Since volatilities in FX markets are usually around 10% or even less for G10 pairs, a negative volga is hardly observed in practice. For commodity and crypto markets volatilities are higher though.

A risk manager would probably explain why higher volatility leads to lower volatility risk, by arguing that for a higher volatility the probability density becomes flatter and wider. This density is the same as gamma when we ignore discounting. Gamma in turn is a multiple of vega[1], so when vega becomes flatter, the maximum vega decreases (and the wing vega increases). Vega is shown in Figure 4.

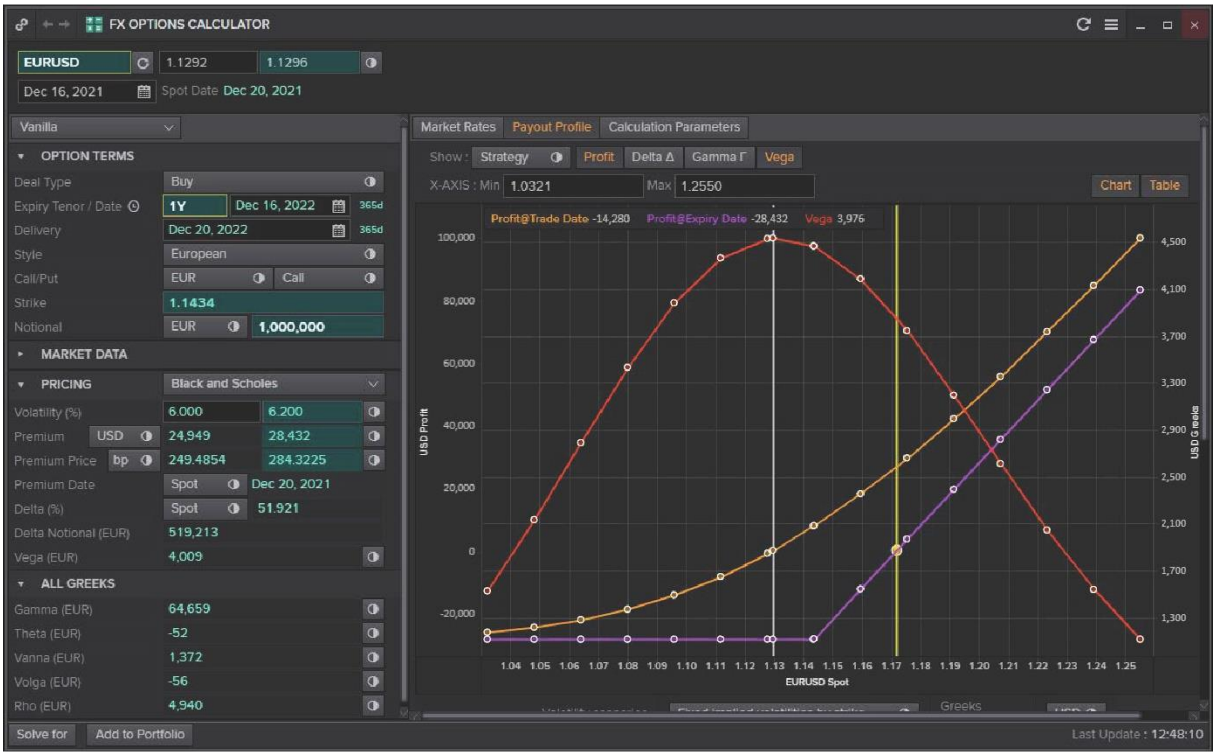

Figure 4: Example vanilla option with negative volga, source: Eikon

Summary

Vanilla volga can be negative and usually is negative near the at-the-money point, especially when volatilities are high. An example in EUR-USD is shown in Figure 4. And as usual, high-school math can take you a long way!

[1] Reiss, O. and Wystup, U. Efficient computation of option price sensitivities using homogeneity and other tricks, *The Journal of Derivatives* Vol. 9 No. 2, Winter 2001.