Efficient Computation of Option Price Sensitivities for Options of American Style

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Abstract: No front-office software can survive without providing derivatives of option prices with respect to underlying market or model parameters, the so called Greeks. If a closed form solution for an option exists, Greeks can be computed analytically and they are numerically stable. However, for American style options, there is no closed-form solution. The price is computed by binomial trees, finite difference methods or an analytic approximation. Taking derivatives of these prices leads to instable numerics or misleading results, specially for Greeks of higher order. We compare the computation of the Greeks in various pricing methods and conclude with the recommendation to use *Leisen-Reimer trees*.

Keywords: American Options, Greeks, Leisen-Reimer trees.
JEL classification: C63, F31

1. Introduction

We examine which is a suitable method to compute Greeks for American style call and put options in the Black-Scholes model. We choose an exchange rate for the underlying following a geometric Brownian motion,

$$dS_t = S_t[(r_d - r_f)\,dt + \sigma\,dW_t], \tag{1}$$

under the risk-neutral measure. As usual r_d denotes the domestic interest rate, r_f the foreign interest rate, σ the volatility. The analysis we do is also applicable to equity options, but we take the foreign exchange market as an example. For contract parameters maturity in years T, strike K and put/call indicator ϕ, which is $+1$ for a call and -1 for a put, the payoff of the option is

$$[\phi(S_T - K)]^+ = \max[0, \phi(S_T - K)]. \tag{2}$$

We denote by $V(t, x)$ the value of an American style put or call at time t if the spot S_t takes the value x. It is well known (see e.g. Karatzas and Shreve 1998) that in this model the value at time zero is given by

$$V(0, S_0) = \sup_{\tau \in \mathcal{T}} \mathbb{E}[e^{-r_d\tau}[\phi(S_\tau - K)]^+], \tag{3}$$

TABLE 1: COMMONLY USED GREEKS, t IS RUNNING TIME AND $x = S_0$

Delta	Δ	V_x
Gamma	Γ	V_{xx}
Theta	Θ	V_t
Rho (domestic)	ρ_d	V_{r_d}
Rho (foreign)	ρ_f	V_{r_f}
Vega		V_σ
Volga		$V_{\sigma\sigma}$
Vanna		$V_{x\sigma}$

TABLE 3: APPROXIMATION FOR $\frac{\partial^2 f}{\partial x^2}$

Order	Difference Quotient
2	$\frac{f(x-h)-2f(x)+f(x+h)}{h^2}$
2	$\frac{2f(x-2h)-f(x-h)-2f(x)-f(x+h)+2f(x+2h)}{14h^2}$
4	$\frac{-f(x-2h)+16f(x-h)-30f(x)+16f(x+h)-f(x+2h)}{12h^2}$

where $\mathcal{T}$ is the set of all stopping times taking values in $[0, T]$. A closed-form solution for this optimization problem has not yet been found.

1.1 Option Price Sensitivities

Option price sensitivities, the so-called *Greeks* of option values are derivatives with respect to market variables or model parameters. The most commonly used Greeks are listed in Table 1. Numerous relationships and properties of the Greeks for European style options are presented in Reiss and Wystup (2001). Other relevant publications include the work by Carr (2001), Broadie and Glasserman (1996) in the case of Monte Carlo simulations, Pelsser and Vorst (1994) in the case of binomial trees, the work by Eric Benhamou (2003) and (2004), who uses Malliavin calculus, and the contribution by Rogers and Stapleton (1998) using binomial trees with a random number of steps. Joubert and Rogers (1997) use a lookup table for a fast, accurate and inelegant valuation of American options. Formulae for Greeks of many exotic foreign exchange options are published in Hakala and Wystup (2002).

1.2 Approximation by Finite Difference Quotients

We summarize the common methods of numerical differentiation in Tables 2 and 3. For vanna one can use

$$\frac{f(x_i+h, x_k+h) - f(x_i-h, x_k+h) - f(x_i+h, x_k-h) + f(x_i-h, x_k-h)}{4h^2}. \tag{4}$$

TABLE 2: APPROXIMATION FOR $\frac{\partial f}{\partial x}$

Order	Difference Quotient
1	$\frac{f(x+h)-f(x)}{h}$
1	$\frac{f(x)-f(x-h)}{h}$
2	$\frac{f(x+h)-f(x-h)}{2h}$

2. Computation Methods

2.1 Binomial Trees

The computation of option values with binomial trees was introduced by Cox, Ross and Rubinstein (CRR) (1979), where the assumption is used that the log-returns are binomially distributed. It is known that in the limiting case this converges to the continuous Black-Scholes model. Some of the enhancements include Jarrow and Rudd (1983), who developed a moment matching method for the parameters. Tian (1993) constructed binomial and trinomial trees and showed how to compute the model parameters to obtain weak convergence to the Black-Scholes model in the Lindeberg sense. Hull and White (1993) enhanced the precision of the binomial model using a control variate technique, as it is common in Monte Carlo simulations. Leisen and Reimer (1996) modify the parameters of the binomial tree to minimize the oscillating behavior of the value function. We review this technique in the following section.

2.1.1 The Method of Leisen and Reimer

As the convergence of the binomial tree based value to the limit is not monotone but rather oscillatory (see Figure 1), the goal here is to achieve maximum precision with a minimum number of time steps N. However, one can not expect that decreasing the step size $\Delta T = T/N$ will yield a more precise value when using the methods by Cox-Ross-Rubinstein, Tian or Jarrow-Rudd. Leisen and Reimer (1996) developed a method in which the parameters u, d and p of the binomial tree can be altered in order to get better convergence behavior.

Instead of choosing the parameters p, u and d to get convergence to the normal distribution Leisen-Reimer suggest to use inversion formulae reverting the standard method—they use normal approximations to determine the binomial distribution $B(n, p)$. In particular, they suggest the following three inversion formulae to replace p (probability of an up move) by $p(d_-)$.

Camp-Paulson-Inversion formula (for arbitrary n)

$$p(z) = \left(\frac{b}{a}\right)^2 \sqrt[3]{\frac{(9a-1)(9b-1)+3z\sqrt{a(9b-1)^2+b(9a-1)^2-9abz^2}}{(9b-1)^2-9bz^2}} \tag{5}$$

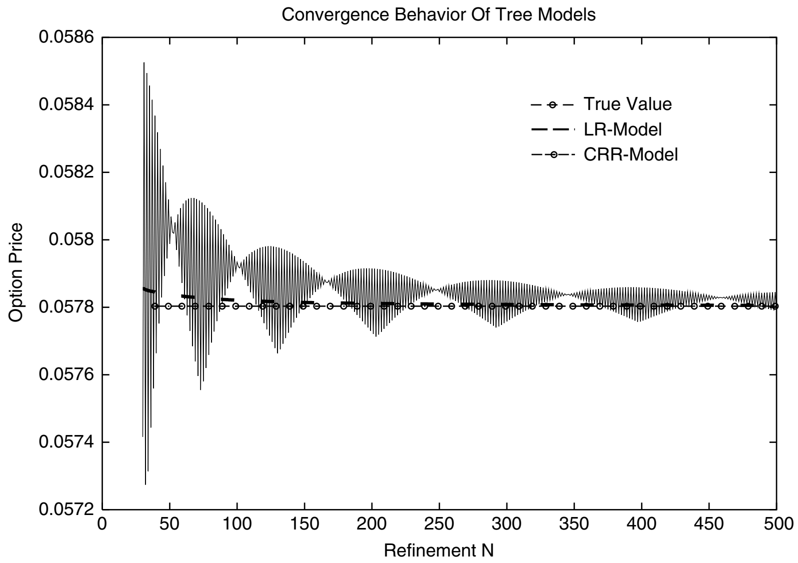

Figure 1: Convergence Behavior of Tree Models with parameters $S = .81$, $K = .9$, $T = 1$, $r_d = .02$, $r_f = .035$, $\sigma = .3$

with $a = n - j$, $b = j + 1$ and z as input values for the standard normal distribution one uses in the Black-Scholes formula.

Peizer-Pratt-Inversion formula 1 $(n = 2j + 1)$

$$p(z) = \frac{1}{2} + \operatorname{sign}(z)\frac{1}{2}\sqrt{1 - \exp\left[-\left(\frac{z}{n + \frac{1}{3}}\right)^2\left(n + \frac{1}{6}\right)\right]} \tag{6}$$

Peizer-Pratt-Inversion formula 2 $(n = 2j + 1)$

$$p(z) = \frac{1}{2} + \operatorname{sign}(z)\frac{1}{2}\sqrt{1 - \exp\left[-\left(\frac{z}{n + \frac{1}{3} + \frac{1}{10\cdot(n+1)}}\right)^2\left(n + \frac{1}{6}\right)\right]} \tag{7}$$

Then the model parameters are defined by

$$u = e^{(r_d - r_f)\Delta t}\frac{p(d_+)}{p(d_-)}, \tag{8}$$

$$d = \frac{e^{(r_d - r_f)\Delta t} - p(d_-)u}{1 - p(d_-)}, \tag{9}$$

$$d_\pm = \frac{\ln\frac{S_0}{K} + (r_d - r_f \pm \frac{1}{2}\sigma)T}{\sigma\sqrt{T}}. \tag{10}$$

Using this method, Leisen and Reimer observe much better convergence behavior.

To compute the Greeks, one can easily use approximations for delta, gamma and theta directly from the tree if the tree satisfies $u = 1/d$, as for example in the CRR model. Let $\Delta T = T/N$ be the step size of an option with maturity T and

$$V_n^i, \quad i = 0, \ldots, n,$$

be the value of the option at time $n\Delta T$, $n \leq N$, if the underlying is $S_n^i = Su^i d^{n-i}$. Then the approximations are given by

$$\Delta \approx \frac{V_1^1 - V_1^0}{S(u - d)} \tag{11}$$

$$\Gamma \approx \frac{\frac{V_2^2 - V_2^1}{S(u^2 - 1)} - \frac{V_2^1 - V_2^0}{S(1 - d^2)}}{S(u^2 - d^2)} \tag{12}$$

$$\Theta \approx \frac{V_0^0 - V_2^1}{2\Delta T}. \tag{13}$$

Vega, Volga and the Rhos can be computed using the difference quotients in Tables 2 and 3, Vanna based on Equation (4). Leisen-Reimer trees do not satisfy $u = 1/d$. Nevertheless, delta and gamma can be computed as described above. Theta needs to be determined numerically since at $2\Delta T$ we have $S_2^1 = Sud \neq S$ for the value.

2.2 Finite Differences

The implementation we use for finite differences is essentially based on the *PREMIA2 project* or Andersen and Brotherton-Ratcliff (1998).

The value $u(t, X_t = \log(S_t))$ of a European style option in the Black-Scholes model obeys the PDE

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) + \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}(t,x) + \left(r_d - r_f - \frac{\sigma^2}{2}\right)\frac{\partial u}{\partial x}(t,x) \\ \quad -r_d u(t,x) = 0 \text{ in } [0,T) \times \mathbb{R}, \\ u(T,x) = \psi(\exp(x)), \forall x \in \mathbb{R}. \end{cases}$$

where ψ is the payoff at maturity T.

Let $x = \log(S_0)$. Then we let the log spot range in $D \triangleq [x - l, x + l]$ with a suitably chosen l, usually about 3 to 4 standard deviations. We discretize the range using the grid $\{x_i\}$ defined by

$$x_i \triangleq x - l + \frac{2il}{M}, \quad \text{for } 1 \leq i \leq M - 1.$$

We approximate the differential operator

$$A\phi \triangleq \frac{1}{2}\sigma^2\frac{\partial^2\phi}{\partial x^2} + \left(r_d - r_f - \frac{\sigma^2}{2}\right)\frac{\partial\phi}{\partial x} - r_d\phi$$

by a discrete operator A_h

$$A_h u_h(t,x_i) = \frac{\sigma^2}{2}\frac{\partial^2 u_h}{\partial x^2}(t,x_i) + \left(r_d - r_f - \frac{\sigma^2}{2}\right)\frac{\partial u_h}{\partial x}(t,x_i) - r_d u_h(t,x_i),$$

where the functions $u_h(t,\cdot)$ are defined by

$$\frac{\partial^2 u_h}{\partial x^2}(t,x_i) = \frac{1}{h^2}(u_h(t,x_{i+1}) - 2u_h(t,x_i) + u_h(t,x_{i-1})),$$
$$\frac{\partial u_h}{\partial x}(t,x_i) = \frac{1}{2h}(u_h(t,x_{i+1}) - u_h(t,x_{i-1})).$$

Now we determine $u_h(t,x_i)$, $(0 \le i \le M)$ such that for $0 \le t \le T$, $1 \le i \le M-1$ the conditions

$$\begin{cases} \frac{d}{dt}u_h(t,x_i) + A_h u_h(t,x_i) = 0, \\ u_h(T,x_i) = \psi(x_i), \\ u_h(t,x-l) = \psi(x-l), \\ u_h(t,x+l) = \psi(x+l) \end{cases} \tag{14}$$

hold. We let $u_h(t) \triangleq (u_h(t,x_1),\dots,u_h(t,x_{M-1}))^T$ and

$$\alpha \triangleq \frac{\sigma^2}{2h^2} - \frac{1}{2h}\left(r_d - r_f - \frac{\sigma^2}{2}\right),$$
$$\beta \triangleq -\frac{\sigma^2}{h^2} - r_d,$$
$$\gamma \triangleq \frac{\sigma^2}{2h^2} + \frac{1}{2h}\left(r_d - r_f - \frac{\sigma^2}{2}\right).$$

Then we can write the operator A_h applied to $u_h(t,\cdot)$ as $A_h u_h(t,\cdot) = M^h u_h(t) + v^h$, where

$$M^h = \begin{bmatrix} \beta & \gamma & 0 & \cdots & 0 & 0 \\ \alpha & \beta & \gamma & 0 & \cdots & 0 \\ 0 & \alpha & \beta & \gamma & \cdots & 0 \\ 0 & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha & \beta & \gamma \\ 0 & 0 & 0 & \cdots & \alpha & \beta \end{bmatrix}, \quad v^h = \begin{bmatrix} \psi(x-l)\alpha \\ 0 \\ \vdots \\ 0 \\ \psi(x+l)\gamma \end{bmatrix}. \tag{15}$$

For the time-discretization we use the standard-θ-scheme ($\theta \in [0,1]$). We choose the step size k such that $T = Nk$ and construct the approximation

$$u_{h,k}(t,x) = \sum_{n=0}^{N} u_h^n(x)\mathbf{1}_{[nk,(n+1)k]}(t),$$

where $u_h^0,\dots,u_h^N$ satisfy the equations

$$\begin{cases} u_h^N = \psi_h, \\ u_h^n(x-l) = \psi(x-l) \quad \text{for } 0 \le n \le N-1, \\ u_h^n(x+l) = \psi(x+l) \quad \text{for } 0 \le n \le N-1, \\ \frac{u_h^{n+1} - u_h^n}{k} + A_h(u_h^{n+1} + \theta(u_h^n - u_h^{n+1})) = 0 \quad \text{for } 0 \le n \le N-1. \end{cases} \tag{16}$$

For $\theta = 0$ we obtain the fully explicit scheme, for $\theta = 1$ the fully implicit scheme and for $\theta = \frac{1}{2}$ the so-called Crank-Nicholson scheme.

In the explicit case $\theta = 0$ the definition of A_h reduces the approximation scheme (16) to

$$\left| \begin{array}{l} u_h^N = \psi \\ \text{for } 1 \le n \le M-1: \\ u_h^n(x_i) = p_1 u_h^{n+1}(x_{i-1}) + p_2 u_h^{n+1}(x_i) + p_3 u_h^{n+1}(x_{i+1}), \end{array} \right.$$

where

$$p_1 = k\left(\frac{\sigma^2}{2h^2} - \frac{b}{2h}\right),\ p_2 = 1 - k\left(r_d + \frac{\sigma^2}{h^2}\right),\ p_3 = k\left(\frac{\sigma^2}{2h^2} + \frac{b}{2h}\right), \tag{17}$$

and $b = r_d - r_f - \frac{1}{2}\sigma^2$. The scheme is stable if $k \le \frac{h^2}{\sigma^2 + (r_d - r_f)h^2}$.

In all other cases $1 \ge \theta > 0$ we need to solve a system of linear equations at each time step

$$\mathbf{M}u_{k,h}(jk,\cdot) = \mathbf{N}u_{k,h}((j+1)k,\cdot)$$

where the tri-diagonal matrices $\mathbf{M}$ and $\mathbf{N}$ take the form

$$\begin{pmatrix} b_1 & c_1 & 0 & \cdots & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & \cdots & 0 \\ 0 & a_3 & b_3 & c_3 & \cdots & 0 \\ 0 & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{M-1} & b_{M-1} & c_{M-1} \\ 0 & 0 & 0 & \cdots & a_M & b_M \end{pmatrix}.$$

$\mathbf{M}$ is given by

$$a_i = \theta k\left(\frac{b}{2h} - \frac{\sigma^2}{2h^2}\right), \quad b_i = 1 + \theta k\left(r + \frac{\sigma^2}{h^2}\right), \quad c_i = -\theta k\left(\frac{b}{2h} + \frac{\sigma^2}{2h^2}\right),$$

and $\mathbf{N}$ is given by

$$a_i = (1-\theta)k\left(\frac{\sigma^2}{2h^2} - \frac{b}{2h}\right), \quad b_i = 1 - (1-\theta)k\left(r + \frac{\sigma^2}{h^2}\right),$$
$$c_i = (1-\theta)k\left(\frac{b}{2h} + \frac{\sigma^2}{2h^2}\right).$$

Solving a system of equations of the kind $\mathbf{M}u = v$, where u and v are M-dimensional vektors can be carried out with the Gauss-Seidel-Factorisation, which is based on the fact that a regular matrix can be decomposed into the product $\mathbf{M} = \mathbf{LU}$ with a lower triangular matrix $\mathbf{L}$ and an upper triangular matrix $\mathbf{U}$ whose diagonal entries are all equal to 1. The solution of a system of the form $\mathbf{LU}z = v$ will be done in two steps $\mathbf{L}y = v$, $\mathbf{U}z = y$.

One realizes that if $\mathbf{M}$ is triangular, then $\mathbf{L}$ and $\mathbf{U}$ are triangular as well and hence we only need to find the upper diagonal of $\mathbf{U}$ and the two diagonals of $\mathbf{L}$. The computation of $\mathbf{L}$, $\mathbf{U}$ and v happens in the same step

$$\left| \begin{array}{l} b'_M \triangleq b_M, \qquad y_M \triangleq v_M \\ \text{For } 1 \le i \le M-1, \ i \text{ increasing:} \\ \qquad b'_i = b_i - c_i a_{i+1}/b'_{i+1}, \\ \qquad y_i = v_i - c_i y_{i+1}/b'_{i+1}. \end{array} \right.$$

$$\left| \begin{array}{l} u_1 = y_1/b'_1 \\ \text{For } 2 \le i \le M, \ i \text{ decreasing:} \\ \qquad z_i = (y_i - a_i u_{i-1})/b'_i. \end{array} \right.$$

Remark. We require the pivot-elements b_i to be non-zero.

To determine the Greeks it appears advantageous to use the information contained in the grid rather than the formulae in Tables 2 and 3 to compute approximations for delta, gamma and theta through

$$\Delta_h = \frac{u_h^0(\exp(x+h)) - u_h^0(\exp(x-h))}{S(e^h - e^{-h})}, \tag{18}$$

$$\Gamma_h = \frac{\frac{u_h^0(\exp(x+h)) - u_h^0(\exp(x))}{S(e^h-1)} - \frac{u_h^0(\exp(x)) - u_h^0(\exp(x-h))}{S(1-e^{-h})}}{S(e^h - e^{-h})}, \tag{19}$$

$$\Theta_h = \frac{u_h^k(e^x) - u_h^0(e^x)}{k}. \tag{20}$$

2.3 Analytic Approximations

Since there is no closed form solution available for American style call or put options and the need for fast computation is eminent, several analytic approximations have been developed. However, one needs to be careful using these for the computation of derivatives, as it is well-known that approximating a function does not necessarily imply that the approximation is also a good approximation of the function's derivatives.

2.3.1 Approximation by Barone-Adesi and Whaley

We outline the method to compute the value function for American style options proposed by MacMillan (1986) and Barone-Adesi and Whaley (1987).

We introduce the notation $v_S = \frac{\partial v}{\partial S}$, $v_{SS} = \frac{\partial^2 v}{\partial S^2}$ and $v_t = \frac{\partial v}{\partial t}$ and X for the strike. Furthermore, we let $V(S,T)$ be the value of an American style options and $v(S,T)$ be the value of a European style option. The values of calls will be denoted by $C(S,T)$ and $c(S,T)$, the values of puts by $P(S,T)$ and $p(S,T)$ respectively. The key idea for the approximation rests on the fact that since the Black-Scholes PDE holds for both the European and the American style option, the early-exercise-premium

$$\varepsilon_C(S,T) = C(S,T) - c(S,T) \tag{21}$$

must satisfy

$$\tfrac{1}{2}\sigma^2 S^2 \varepsilon_{SS} - r_d \varepsilon + (r_d - r_f) S \varepsilon_S + \varepsilon_t = 0. \tag{22}$$

Using the abbreviations $\tau \triangleq T - t$, $K(\tau) \triangleq 1 - e^{-r_d \tau}$, $M \triangleq \frac{2r_d}{\sigma^2}$, $N \triangleq \frac{2(r_d - r_f)}{\sigma^2}$ and $\varepsilon_C(S,K) \triangleq K(\tau) f(S,K)$ Equation (22) implies

$$S^2 f_{SS} + NSf_S - \frac{M}{K} f - (1-K) M f_K = 0. \tag{23}$$

The authors now argue that the term $(1-K)Mf_K = 0$ is neglegible for small and large τ[1].

The resulting ordinary differential equation

$$S^2 f_{SS} + NSf_S - \frac{M}{K} f = 0 \tag{24}$$

has the general solution

$$f(S) = a_1 S^{q_1} + a_2 S^{q_2}, \tag{25}$$

where the roots of the characteristic polynomial are given by

$$q_{1,2} = \frac{-(N-1) \mp \sqrt{(N-1)^2 + 4\frac{M}{K}}}{2}$$

with $q_1 < 0$ and $q_2 > 0$ since $\frac{M}{K} > 0$. Since $q_1 < 0$, $a_1 \neq 0$ would imply $\lim_{S \to 0} f(S) = \infty$, whence we must have $a_1 = 0$.

Using equation

$$C(S,\tau) = c(S,\tau) + K a_2 S^{q_2} \tag{26}$$

we can derive restrictions on a_2, namely

1. for $S = 0$ Equation (26) implies $C(S,T) = 0$.
2. $C(S,\tau)$ must be increasing in S. Therefore, $a_2 > 0$.
3. the r.h. side of (26) must not intersect the line $S - X$, but only touch it in at the optimal exercise level S^*. For $S \le S^*$ the value of the American call is given by Equation (26). For $S > S^*$ its value is $S - X$.

In order to find S^* we differentiate

$$S^* - X = c(S^*, \tau) + K a_2 (S^*)^{q_2}. \tag{27}$$

with respect to S^* and obtain

$$1 = e^{(b-r_d)\tau} \mathcal{N}(d_1(S^*)) + K q_2 a_2 (S^*)^{q_2}, \tag{28}$$

where $d_1(S^*) = \frac{\ln \frac{S^*}{X} + \left(b + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}$ and $b = r_d - r_f$.

Then one solves Equation (28) for a_2 and plugs the result into Equation (27) to reach

$$S^* - X = c(S^*, \tau) + \frac{1 - e^{(b-r_d)\tau}\mathcal{N}[d_1(S^*)]}{q_2}, \tag{29}$$

where S^* is the only unknown and can be easily found numerically. As a result we get

$$C(S,\tau) = \begin{cases} c(S,\tau) + A_2 \left(\frac{S}{S^*}\right)^{q_2}, & \text{if } S < S^* \\ S - X, & \text{otherwise}, \end{cases} \tag{30}$$

where $A_2 = \frac{(1-e^{(b-r_d)\tau}\mathcal{N}[d_1(S^*)])}{q_2}$.

Remark: Note that A_2 is only positive if $b < r_d$, i.e. $r_f > 0$, which is usually satisfied. Similarly for puts Equation (21) holds in the form

$$\varepsilon_P(S,T) = P(S,T) - p(S,T). \tag{31}$$

Here in Equation (25) we need $a_2 = 0$ and hence

$$P(S,\tau) = p(S,\tau) + Ka_1 S^{q_1}. \tag{32}$$

To get a_1 we employ the optimal exercise level S^{**} defined by

$$X - S^{**} = p(S^{**},\tau) - \frac{1 - e^{(b-r_d)\tau}\mathcal{N}[-d_1(S^{**})]}{q_1}. \tag{33}$$

The American style put is then approximated by

$$P(S,\tau) = \begin{cases} p(S,\tau) + A_1 \left(\frac{S}{S^{**}}\right)^{q_1}, & \text{if } S > S^{**} \\ X - S, & \text{otherwise}, \end{cases} \tag{34}$$

where $A_1 = -\frac{(1-e^{(b-r_d)\tau}\mathcal{N}[-d_1(S^{**})])}{q_1}$.

3. Comparison of the Methods

Now we compare the efficiency of the different valuation procedures outlined before. We consider a Euro call USD put option with a strike of 0.9000, 3 months maturity. Market data are assumed to be 10% volatility, 3.5% Euro interest rate, 2% USD interest rate. In this scenario, the value of the European and American put are identical, so the European put can be taken as a benchmark for the American style value and Greeks. The value of the American call will be strictly larger than the value of the European call.

The parameters for Leisen-Reimer binomial trees are $N^{bin} = 2000$ time steps, the parameters for the finite differences are $N_S^{fd} = 1130$ spot steps, $N_T^{fd} = 1130$ time steps and $\theta = 0.5$ (Crank-Nicholson).

We compare these methods with the approximation by Barone-Adesi and Whaley (BAW) and the Black-Scholes method.

3.1 Value Function

Figure 2 shows the value functions of a call as a function of the current spot. One of the weaknesses of BAW is that the precision can't be improved by changing a parameter.

We observe furthermore in Table 4 that in the BAW method the exercise boundary will be reached too early as compared to the binomial trees and finite differences. The computation of the Greeks will inherit this feature, whence we can't expect high accuracy for the Greeks using BAW near the optimal exercise boundary.

The advantage of BAW is its speed, the method is superior if you need to price a vanilla far away from the optimal exercise boundary.

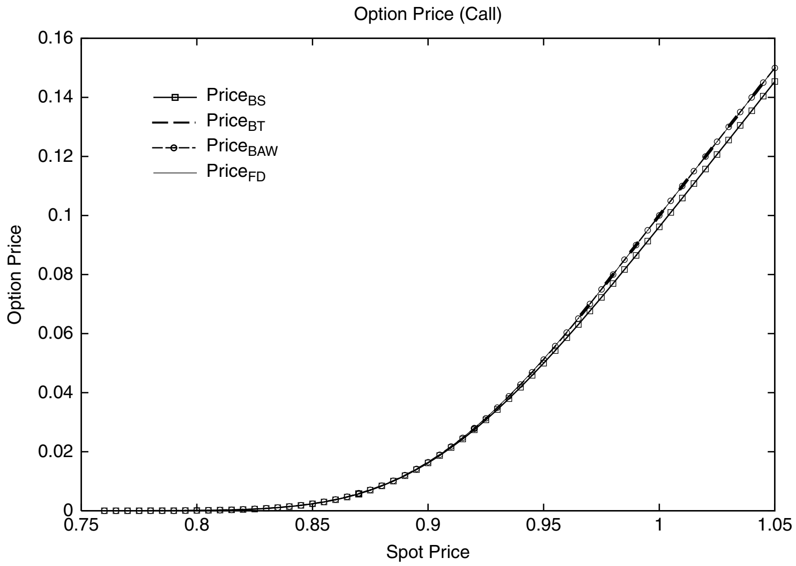

Figure 2: American call with $K = .9$, $T = 3m$, $r_d = .02$, $r_f = .035$, $\sigma = .1$

TABLE 4: COMPARISON OF THE VALUES IN USD USING THE METHODS NEAR THE OPTIMAL EXERCISE BOUNDARY

Spot	BS	BT	BAW	FD
0.97	0.06765478	0.07007488	0.0700066	0.07007446
0.971	0.06857433	0.0710532	0.07100095	0.07105275
0.972	0.06949664	0.0720353	0.072	0.07203482
0.973	0.07042164	0.07302112	0.073	0.07302061
0.974	0.07134923	0.07401058	0.074	0.07401004
0.975	0.07227935	0.07500358	0.075	0.07500305
0.976	0.0732119	0.07600002	0.076	0.076
0.977	0.07414681	0.077	0.077	0.077
0.978	0.075084	0.078	0.078	0.078

3.2 Delta and Gamma

We take these Greeks directly from the PDE grid or the tree as the information comes at no extra computational cost. Figures 3 and 4 show the expected behavior for BAW: delta approaches 1 to quickly and gamma

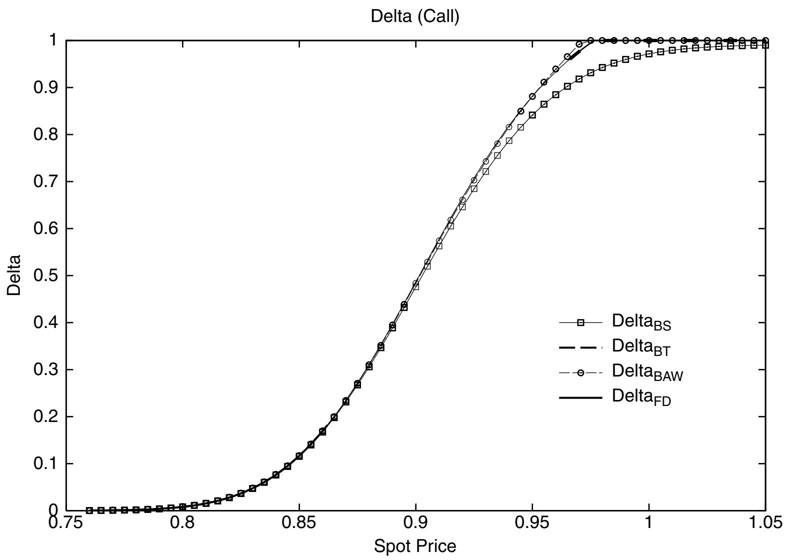

Figure 3: Delta of an American call with $K = .9$, $T = 3m$, $r_d = .02$, $r_f = .035$, $\sigma = .1$

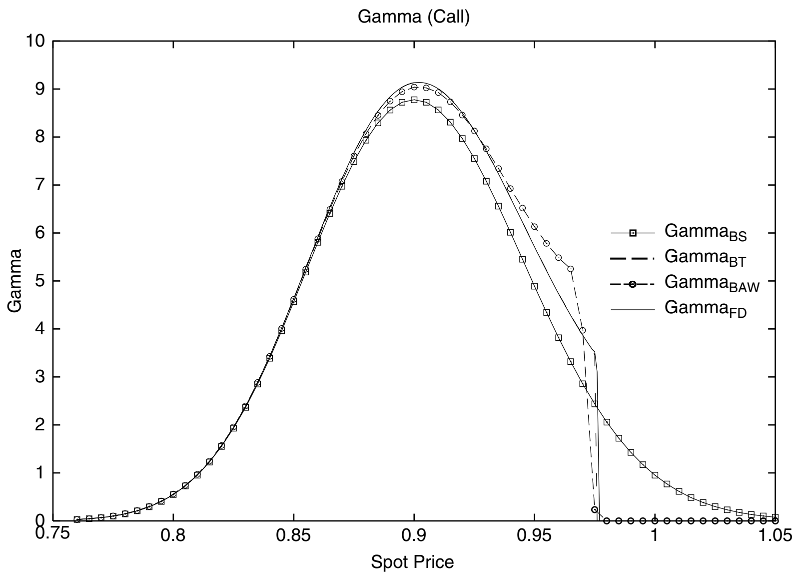

Figure 4: Gamma of an American call with $K = .9$, $T = 3m$, $r_d = .02$, $r_f = .035$, $\sigma = .1$

approaches 0 to quickly. The Leisen-Reimer trees and finite differences yield equally good values for delta and gamma. However, we observed that the values of gamma near the optimal exercise boundary tend to be more stable using Leisen-Reimer trees.

3.3 Theta

In BAW and LR, theta has to be computed with difference quotients, in the finite differences we can take it from the grid. This implies about twice or trice the computational cost for trees depending on whether we use $\frac{1}{h}(f(x) - f(x-h))$ or $\frac{1}{2h}(f(x+h) - f(x-h))$. The accuracy of LR and finite differences appears identical (see Figure 5), so we would recommend finite differences to save on the computational cost.

The following Greeks can only be determined using the difference quotients in Tables 2 and 3. The choice of the parameter h is crucial. If we choose it too small, then the lack of precision in the value function will lead to a possibly larger error in the hedge parameter.

3.4 Rho (domestic) and Rho (foreign)

We compute these Greeks with the difference formulae of first order to keep the computation cost under control. In particular, we only need one more computation of $r_d - h$ or $r_f + h$ to compute the approximations

$$\rho_d \approx \frac{1}{h}(f(r_d) - f(r_d - h)) \quad \text{and} \tag{35}$$

$$\rho_f \approx \frac{1}{h}(f(r_f + h) - f(r_f)). \tag{36}$$

We take $h = 0.01$. Figures 7 and 8 show the approximations for ρ_d and ρ_f. Even for step size of $N^{bin} = 100$ in the binomial tree we obtain good approximations for these Greeks. Finite differences tend to oscillate for this grid size as illustrated in Figure 6, so that we would recommend LR trees for the rhos.

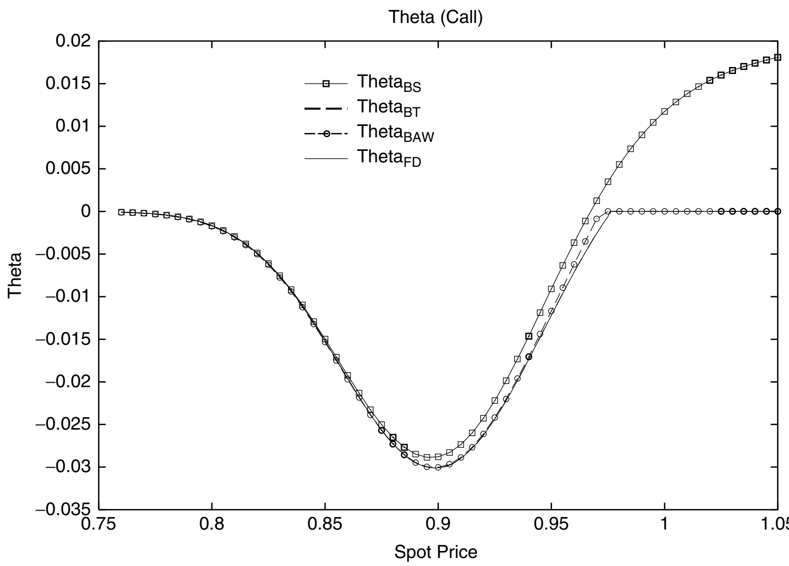

Figure 5: Theta of an American call with $K = .9$, $T = 3m$, $r_d = .02$, $r_f = .035$, $\sigma = .1$

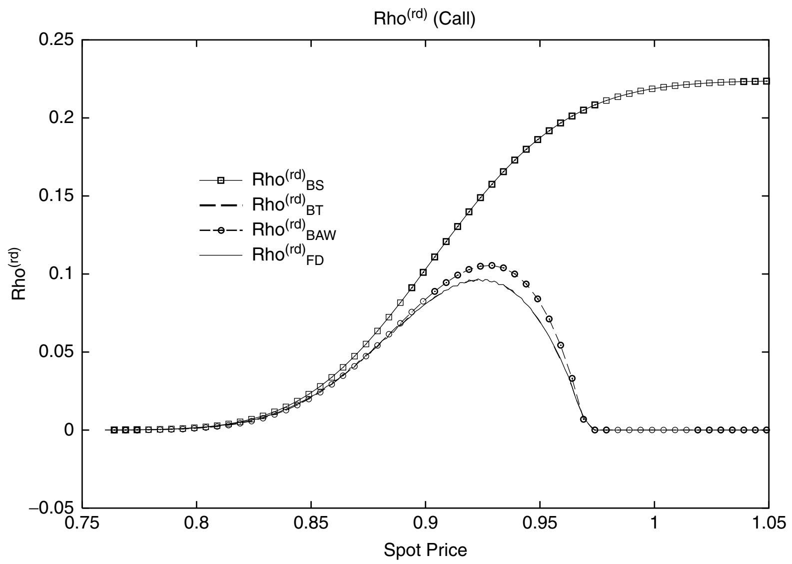

Figure 6: Behavior for the domestic rho of an American call with $N^{bin} = N_t^{fd} = N_S^{fd} = 100$

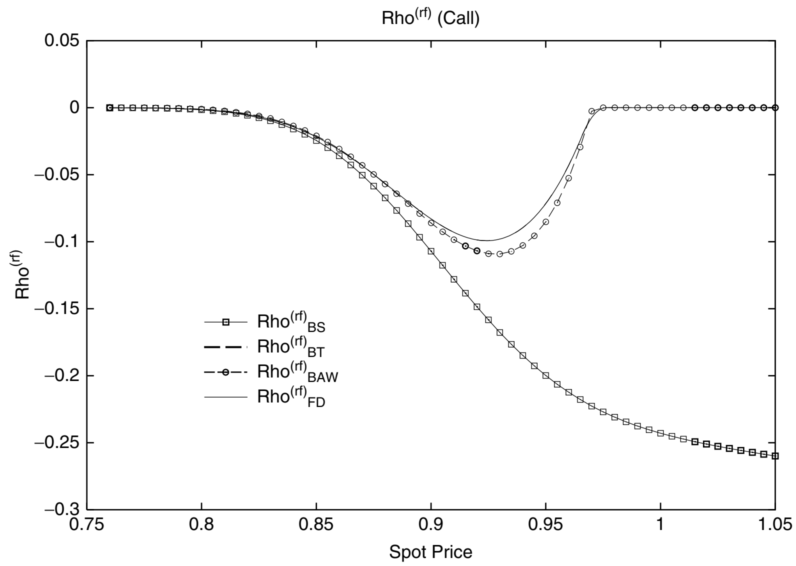

Figure 8: Rho (foreign) of an American call with $K = .9$, $T = 3m$, $r_d = .02$, $r_f = .035$, $\sigma = .1$

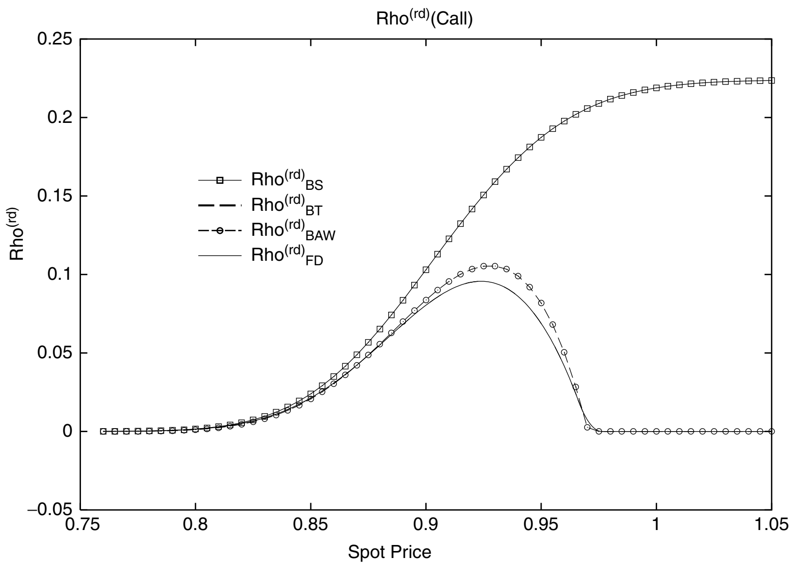

Figure 7: Rho (domestic) of an American call with $K = .9$, $T = 3m$, $r_d = .02$, $r_f = .035$, $\sigma = .1$

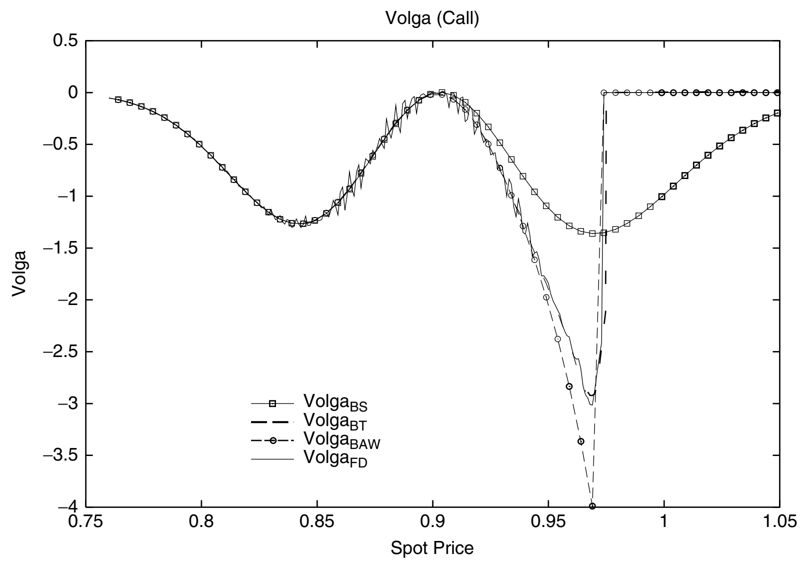

Figure 9: Behavior of an American call volga with $N^{bin} = N_T^{fd} = N_S^{fd} = 100$

3.5 Vega and Volga

For the second derivative we need to choose an approximation from Table 3. Noticeably, the approximations of second order work better than the one of order four. We had specially good experience using

$$\text{Volga} \approx \frac{2f(x-2h) - f(x-h) - 2f(x) - f(x+h) + 2f(x+2h)}{14h^2}. \qquad (37)$$

with $h = 0.05$. A smaller h would require more computational cost for the trees and finite differences, since the resulting value function would have to have higher precision. For small step size we find the precision of the LR trees superior to the finite differences, as illustrated in Figure 9 with $N^{bin} = N_T^{fd} = N_S^{fd} = 100$. Therefore we recommend LR trees for volga. To compute vega it is advisable to use the values already obtained for volga, i.e.

$$\text{Vega} \approx \frac{f(x-2h) - f(x+2h)}{4h}. \tag{38}$$

For small step sizes we find similar behavior for the vega as we found for volga. Therefore, we recommend LR trees to compute vega. The results are displayed in Figures 10 and 11.

3.6 Vanna

We approximate vanna using the second order derivative

$$\frac{f(S+h_S, \sigma+h_v) - f(S-h_S, \sigma+h_v) - f(S+h_S, \sigma-h_v) + f(S-h_S, \sigma-h_v)}{4h_S h_v}, \tag{39}$$

where h_S denotes the step size of the spot S and h_v the step size of the volatility σ. We take $h_S = 0.003$ and $h_v = 0.03$. Compared to the other Greeks, vanna requires very high accuracy in the finite difference based method, to avoid oscillatory behavior as illustrated in Figure 12. LR trees turn out to be the obviously better method here, although we need a fine grid near the optimal exercise boundary.

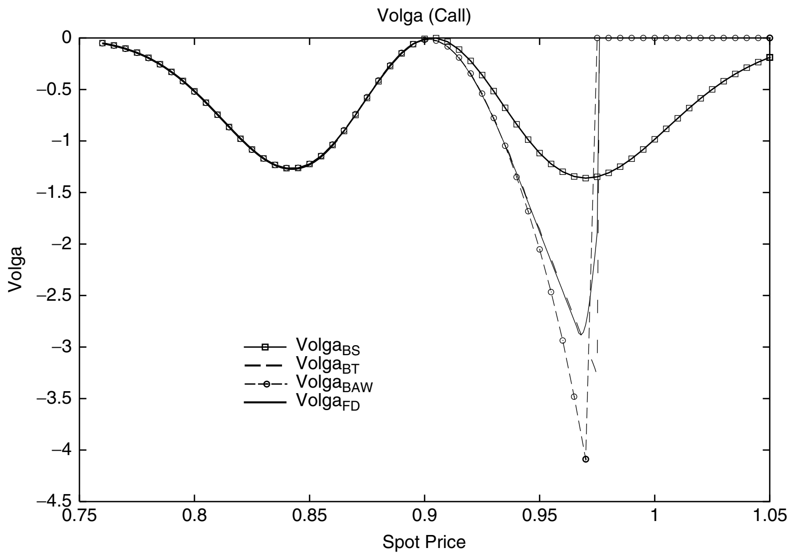

Figure 11: Volga of an American call with $K = .9$, $T = 3m$, $r_d = .02$, $r_f = .035$, $\sigma = .1$

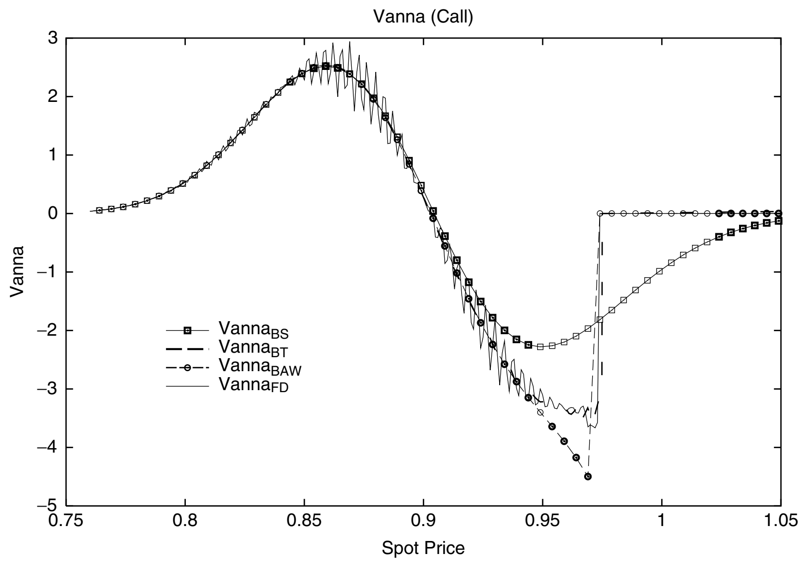

Figure 12: Behavior of an American call vanna with $N^{bin} = N_T^{fd} = N_S^{fd} = 100$

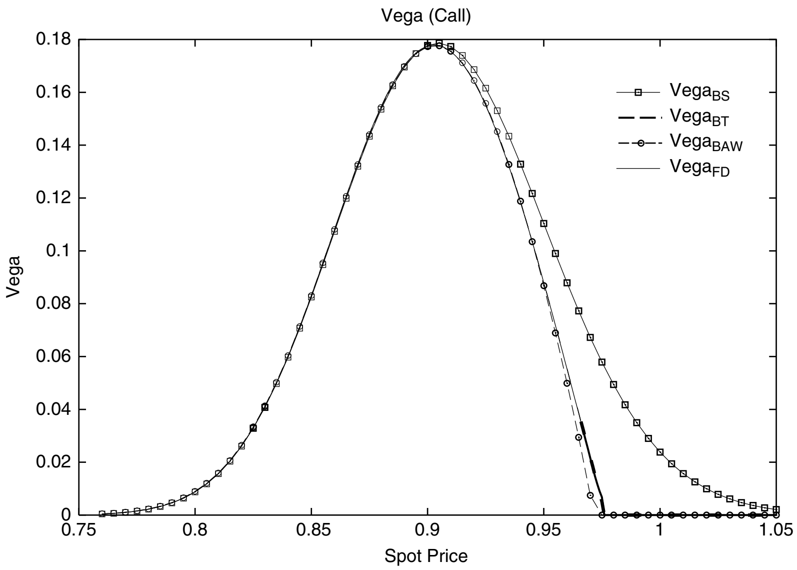

Figure 10: Vega of an American call with $K = .9$, $T = 3m$, $r_d = .02$, $r_f = .035$, $\sigma = .1$

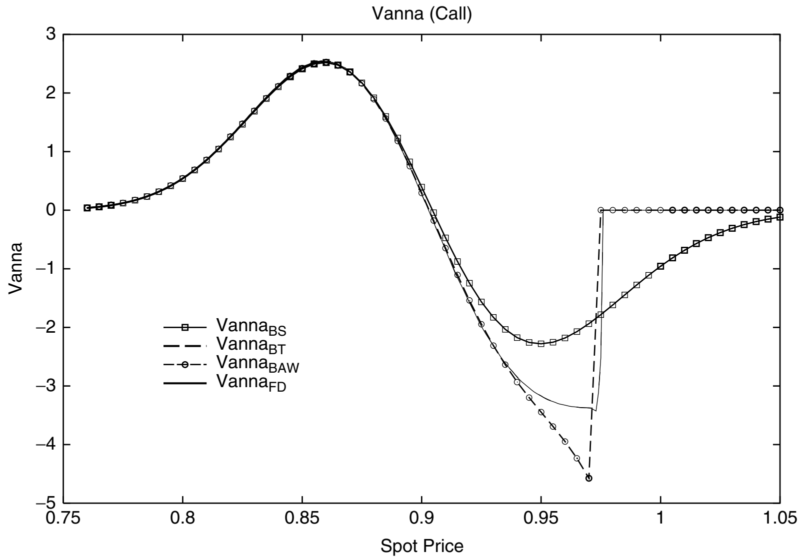

Figure 13: Vanna of an American call with $K = .9$, $T = 3m$, $r_d = .02$, $r_f = .035$, $\sigma = .1$

4. Summary

We analyzed three commonly used methods to determine the value of American style options with regard to their efficiency to compute the hedge parameters (Greeks), in particular: delta, gamma, theta, vega, vanna, volga, domestic and foreign rho. These were the analytic approximation by Barone-Adesi and Whaley, the finite difference method with Crank-Nicholson scheme and the binomial model in the variant of Leisen and Reimer.

The method by Barone-Adesi and Whaley is working with a fixed an non-improvable precision. Moreover, it lacks precision near the optimal exercise boundary. Its only strength lies in its speed.

We confirmed that using finite differences will deliver approximations for delta, gamma and theta directly from the grid without additional computational cost. Except for theta we obtain the same result for the binomial trees. Leisen Reimer trees yield more precise results for delta and gamma.

The remaining Greeks can not be taken from the grid, but have to be computed using finite difference quotients. We observed that Leisen Reimer trees are the superior method.

FOOTNOTES & REFERENCES

1. This hints at a weaker quality of the method for medium length maturities.

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