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Centre for Practical Quantitative Finance

No. 16

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October 2008

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Publisher:

Frankfurt School of Finance & Management
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Closed Formula for Options with Discrete Dividends and its Derivatives

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*The author wishes to thank Millennium bcp investimento, S.A. for the financial support being provided during the course of his PhD. studies.

Abstract

We present a closed pricing formula for European options under the Black–Scholes model and formulas for its partial derivatives. The formulas are developed making use of Taylor series expansions and by expressing the spatial derivatives as expectations under special measures, as in Carr, together with an unusual change of measure technique that relies on the replacement of the initial condition. The closed formulas are attained for the case where no dividend payment policy is considered. Despite its small practical relevance, a digital dividend policy case is also considered which yields approximation formulas. The results are readily extensible to time dependent volatility models but no so for local-vol type models. For completeness, we reproduce the numerical results in Vellekoop and Nieuwenhuis using the formulas here obtained. The closed formulas presented here allow a fast calculation of prices or implied volatilities when compared with other valuation procedures that rely on numerical methods.

Key words: Equity option, discrete dividend, hedging, analytic formula

Contents

1 Introduction

1.1 Motivation

The motivation to return to this issue is the fact that whenever a new product, model or valuation procedure is developed, the problem that arises with discrete dividends is dismissed or overlooked by applying the usual approximation that transforms the discrete dividend into a continuous stream of dividend payments proportional to the stock price. After all that has been said about the way to handle discrete dividends, there are still strong reasons to justify such an approach.

We recall here the reasons that underlie the use of this method by the majority of market participants and pricing tools currently available. We choose the word *method* to refer to this procedure because we believe it to be more suitable than *model*. The reason for this is that, if one considers two options with different maturities, written on the same underlying stock, this method implies two different diffusion price processes for the same underlying stock under the same measure. This is naturally an unreasonable choice to model the underlying stock price, especially because it would admit arbitrage opportunities.

The drivers behind the huge popularity of this method are mostly due to the (i) tractability of the valuation formulas, (ii) applicability to any given model for the underlying stock, and (iii) the preserved continuity of the option price when crossing each dividend date.

However, it has some significant drawbacks. First and foremost the inaccurate pricing it produces has to be mentioned. Furthermore, the error grows larger as the dividend date is farther away from the valuation date. This is exactly the opposite behavior of what one would expect from an approximation – a larger period of time between the valuation date and the dividend date means that the option valuation functions are smoother, and thus should be easier to approximate. The other side of the inaccurate pricing coin is the fact that this method does not provide a hedging strategy that will guarantee the replication of the option payoff at maturity. To sum up, no numerical procedure based on this method returns (or converges to) the true value of the option, whatever the model.

It still seems like the advantages outweigh the drawbacks since it is the most widely used method.

An example may help to demonstrate this. Consider a stochastic volatility model with jumps. Now consider the valuation problem of an American style option under this model. The complexity of this problem is such that a rigorous treatment of discrete dividends, i.e. a modification of the underlying's diffusion to account for that fact, would render the model intractable.

1.2 Description of the Problem

The problem arises due to the fact that the diffusion models, like the Black-Scholes (BS) model, are no longer an acceptable description of the stock price dynamics when the stock pays discrete dividends.

The risks that occur in this context are mainly the potential losses arising from incorrect valuation and ineffective hedging strategy. We address both of these issues in this paper.

The most natural extension to the a diffusion model to account for the existence of discrete dividends is to consider the same diffusion, for example,

$$dS_t = S_t \left(r dt + \sigma dW_t \right), \tag{1}$$

and add a negative jump with the same size as the dividend, on the dividend payment date as

$$S_{t_D} = S_{t_D^-} - D, \tag{2}$$

where S is the stock price, r is the constant interest rate, σ is the volatility and W is a standard Brownian motion. $S_{t_D^-}$ refers to the time immediately before the dividend-payment moment, t_D, and S_{t_D} to the moment immediately after.

There are some common objections to this formulation though. A first objection may be the assumption that the stock price will fall by the amount of the dividend size. This objection is mainly driven by the effects taxes have on the behavior of financial agents and thus market prices. We will not consider this objection in this paper and thus assume model (2) to be valid. A second objection may be that the dividend payment date and amount are not precisely known until a few months before their payment. We also believe this to be the case, but to consider it in a proper fashion, the model would grow significantly in complexity. Our goal is rather to devise a simple variation that can be applied to a wide class of models, that does not worsen the tractability of the model and that produces accurate results.

Finally, one can argue that the model admits negative prices for the stock price S. This is in fact true and can easily be seen if one takes the stock price $S_{t_D^-}$ to be smaller than D at time t_D^-. A simple solution to this problem is to add an extra condition, for example replace (2),

$$S_{t_D} = S_{t_D^-} - D \quad \text{if } S_{t_D^-} > D. \tag{3}$$

However, in most practical applications, this is not of great importance since the vast majority of the companies that pay dividends, have dividend amounts that amount to a small fraction of the stock price, i.e. less than 10% of the stock price, rendering the probability assigned to negative prices very small. For this reason we may drop this condition whenever it would add significant complexity.

In the next section we review the existing literature on this subject and the reasons that underlie the use of the method most popular among practitioners. We then turn to develop the formulas in section 2 and in section 3 we reproduce the numerical results in [19]. Section 4 concludes.

1.3 Literature Review

Here we shortly review the literature on modifications of stock price models to cope with the discrete dividend payments. Merton [12] (1973) analyzed the effect of discrete dividends in American calls and states that the only reason for early exercise is the existence of unprotected dividends. Roll [15](1977), Geske [9](1979), and Barone-Adesi and Whaley [1](1986) worked on the problem of finding analytic approximations for American options. John Hull [11] in the first edition of his book, 1989, establishes what was to be the most used method to cope with discrete dividends. The method works by subtracting from the current asset price, the net present value of all dividends to occur during the life of the option. The reasons for its popularity and acceptance were the facts that it would preserve the continuity of the option price across the dividend payment date and that it could cope with multiple dividends. One can show that this formulation is exact only if the dividend is paid immediately after the valuation date. On the other end of the spectrum, Musiela and Rutkowsky [13] (1997) propose a model that adds the future value at maturity of all dividends paid during the lifetime of the option to the strike price. Again one can show that this formulation is exact only if the dividend payment happens just before the option matures. To balance these two last methods, Bos and Vandermark [5] (2002) devise a method that divides the dividends in "near" and "far" and subtracts the "near" dividends from the stock price and adds the "far" dividends to the strike price. This method performs better than the previous but it is not exact, especially in the case of options not at-the-money. A method that considers a continuous geometric Brownian motion with jumps at the dividend payment dates is analyzed in detail by Wilmott [20] (1998) by means of numerical methods. Berger and Klein [2](1998) propose a non-recombining binomial tree method to evaluate options under the jump model. Bos *et al.* [4](2003) devise a method that adjusts the volatility parameter to correct the subtraction method stated above. Haug *et al.* (2003) review existing methods' performance and pay special attention to the problem of negative prices that arise within the context of the jump model and propose a numerical quadrature scheme. Björk [3] (1998) has one of the clearest descriptions of the discrete dividends problem for European options and provides a formula for proportional dividends. Shreve [16] (2004) also states the result for proportional dividends. Recently, Vellekoop and Nieuwenhuis [19] (2006) describe a modification to the binomial tree method to account for discrete dividends preserving the crucial recombining property.

2 Closed Formula

We start by developing an arbitrage argument that will enable us to address the existence of discrete dividends in a unified approach, irrespective of the option type and model. Here we restrict ourselves to the large set of models that display the Markov property. Our arguments are equivalent to what can be found in the literature, see Björk [3] for a clear and concise description.

Consider an option with a maturity T on the underlying S, whose value at time t_0 is $V(S_0, t_0)$.

We now assume that an unexpected dividend, of size D, is paid at the date t_D, with $t_0 < t_D < T$. The value of the option at the moment t_D is changed because the underlying value has decreased by the size of the dividend.

More precisely, the option value change corresponds to the following difference

$$\Delta V = V(S_{t_D^-}, t_D) - V(S_{t_D}, t_D). \tag{4}$$

That is, the difference between the value of the option taking the underlying price just before and after the dividend payment. This difference measures the impact of the change in the option price with respect to a discrete change (neither accounted for nor predicted) in the value of the underlying.

From the point of view of the buyer/seller of such an option, this change in value may come as an unexpected profit or loss, depending on the specific option one considers. This profit or loss was not predicted by the hedging strategy, nor could it be so because any profits or losses from the hedging strategy should, by construction, be reflected on the initial price of the option. This inclusion, in turn, would eliminate their occurrence.

We now consider the difference ΔV as a payoff profile of a virtual derivative contract. This new contract, together with the option V, for which the existence of dividends was ignored, form a special kind of portfolio whose aggregate value is the value of the option when the existence of the discrete dividend is considered. The special property of this portfolio is that none of its components can exist by themselves. For European style options, this special property is not a constraint since every claim in the portfolio has a lifespan from t_0 to T. The same is not true for American style or barrier options that can cease to exist before time T.

We are now ready to state the following lemma.

Lemma 2.1. *Let V be random variables defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)$. Let V be the arbitrage-free price of an option on a stock that pays a discrete dividend D at time t_D according to the policy $\mathbb{I}_A$, where $A = \{\omega : \text{the dividend is paid}\}$.*

Then, for each $\omega \in \Omega$,

$$V(S_{t_D}, t_D)(\omega) = V(S_{t_D^-}, t_D^-)(\omega) + \left[V(S_{t_D^-} - D, t_D^-)(\omega) - V(S_{t_D^-}, t_D^-)(\omega)\right]\mathbb{I}_A. \tag{5}$$

Proof. If $\omega \in A$, then the dividend policy determines the actual payment of the dividend, and we have

$$\begin{aligned} V(S_{t_D}, t_D)(\omega) &= V(S_{t_D^-} - D, t_D^-)(\omega) \\ &= V(S_{t_D}, t_D)(\omega). \end{aligned} \tag{6}$$

The opposite case, if $\omega \notin A$, i.e., when the dividend policy determines that the dividend is not paid, we have

$$\begin{aligned} V(S_{t_D}, t_D)(\omega) &= V(S_{t_D^-}, t_D^-)(\omega) \\ &= V(S_{t_D}, t_D)(\omega). \end{aligned} \tag{7}$$

Thus we have verified that (5) holds for each $\omega \in \Omega$. □

With lemma 2.1 we have established the value of the derivative V at time t_D as a function of S at time t_D^-, i.e., just before the dividend is paid. We can now consider $V(S_{t_D}, t_D)$ as a payoff of a claim maturing at t_D^-. Furthermore, the underlying asset process S_t, with $t_0 < t < t_D$ is just the Black-Scholes model as in (1) since there are no dividend payments during this time interval.

In this setting, the valuation of the option at time t_0, $V(S_{t_0}, t_0)$, is just the discounted expectation under the risk-neutral measure with the process S_t given by (1).

We can thus state the following corollaries:

Corollary 2.1. *For European style options we have*

$$\begin{aligned} &V_D(S_{t_0}, t_0) = \\ &\quad e^{-r(t_D-t_0)}\mathbb{E}^Q_{t_0}\left[V\left(S_{t_D^-}, t_D\right) + \left(V\left(S_{t_D^-} - D, t_D\right) - V\left(S_{t_D^-}, t_D\right)\right)\cdot \mathbb{I}_A\right] = \\ &\quad V\left(S_{t_0}, t_0\right) + e^{-r(t_D-t_0)}\mathbb{E}^Q_{t_0}\left[\left(V\left(S_{t_D^-} - D, t_D\right) - V\left(S_{t_D^-}, t_D\right)\right)\cdot \mathbb{I}_A\right], \end{aligned}$$

where $\mathbb{E}^Q_{t_0}[X]$ stands for the expectation under the risk-neutral measure Q, with respect to the σ-algebra $\mathcal{F}_{t_0}$, of the random variable X.

Corollary 2.2. *For American style options we have*

$$V_D(S_{t_0}, t_0) = \max_{\tau \in \mathcal{T}_{t_0, t_D}} \mathbb{E}^Q_{t_0}\left[e^{-r(\tau - t_0)}\Phi_D\left(S_\tau\right)\right],$$

with

$$\Phi_D\left(S_\tau\right) = \begin{cases} \Phi\left(S_\tau\right) & \text{if } \tau < t_D \\ V\left(S_{t_D^-}, t_D\right) + \left(V\left(S_{t_D^-} - D, t_D\right) - V\left(S_{t_D^-}, t_D\right)\right)\cdot \mathbb{I}_A & \text{if } \tau = t_D, \end{cases} \tag{8}$$

and

$$V\left(S_{t_D^-}, t_D\right) = \max_{\tau \in \mathcal{T}_{t_D, T}} \mathbb{E}^Q_{t_D}\left[e^{-r(\tau - t_D)}\Phi\left(S_\tau\right)\right], \tag{9}$$

and $\Phi(x)$ is the payoff of the option if the stock price is at x, $\mathcal{T}_{t_D,T}$ is the set of all stopping-times taking values in the interval (t_D, T) and $\mathcal{T}_{t_0,t_D}$ the corresponding set for the interval (t_0, t_D).

Since Lemma 2.1 does not impose any conditions on the form of $V_D(S_{t_D}, t_D)$, we are allowed to use it several times to account for several dividend payments, iterating between the lemma and the relevant corollary for the option at hand. This procedure will be used in the following sections to obtain the pricing formula.

2.1 General Derivation

In this section we will focus on European style options. We will develop a general approach and identify the necessary conditions for its validity. In the next section, we provide an example of a call option under the Black-Scholes model which does satisfy the validity conditions. The corresponding proofs are given in appendix A.

As induced by the lemma above, the approach to the dividend problem should start by targeting the last dividend before maturity, and then proceed by moving backwards in time until the first dividend.

Therefore, let there be n dividends D_i and payment dates t_i, with $i = 1, \cdots, n$ and $t_0 < t_1 < \cdots < t_n < T$.

We will keep the notation $S_{t_i^-}$ to refer to the stock price at the time just before the i^{th} dividend payment but we will identify t_i^- with t_i to lighten notation, since they are equal in the limit.

Last Dividend Before Maturity

From corollary 2.1 we can state that

$$C_n(S_{t_{n-1}}, t_{n-1}) = C\left(S_{t_{n-1}}, t_{n-1}\right) + \\ e^{-r(t_n - t_{n-1})}\mathbb{E}^Q_{t_{n-1}}\left[\left(C\left(S_{t_n^-} - D_n, t_n\right) - C\left(S_{t_n^-}, t_n\right)\right) \cdot \mathbb{I}_{A_n}\right], \tag{10}$$

where C stands for the price of the derivative claim ignoring the existence of the dividend D_n and C_n is the price of the same claim but acknowledging the existence of the dividend D_n.

In this form, and for most models, it is impossible to find a closed formula solution for the expectation above. We thus choose to replace the difference that it contains by its corresponding Taylor series expansion. The necessary condition for the exchange is the following:

Condition 2.1. *Let $C\left(S_{t_n^-}, t_n\right)$ be a function infinitely differentiable with respect to $S_{t_n^-}$, and let the corresponding infinite Taylor series expansion be convergent.*

We thus have,

$$\mathbb{E}^Q_{t_{n-1}}\left[\left(C\left(S_{t_n^-} - D_n, t_n\right) - C\left(S_{t_n^-}, t_n\right)\right) \cdot \mathbb{I}_{A_n}\right] \\ = \mathbb{E}^Q_{t_{n-1}}\left[\sum_{i=1}^{\infty} \frac{(-D_n)^i}{i!} C^i\left(S_{t_n^-}, t_n\right) \cdot \mathbb{I}_{A_n}\right], \tag{11}$$

with C^i referring to the i^{th} derivative of the claim price with respect to $S_{t_n^-}$[1].

Furthermore, it would be helpful to interchange the expectation and summation in equation (11). For this interchange, the following conditions have to hold:

Condition 2.2.
Let

$$\sum_{i=1}^{\infty} \frac{(-D_n)^i}{i!} C^i\left(S_{t_n^-}, t_n\right) \cdot \mathbb{I}_{A_n} \cdot \phi\left(S_{t_n^-}\right)$$

[1]To be coherent with the notation used below, we should write $C^{(i)}$ instead of C^i. The abbreviated notation was chosen to lighten the formulas below that make repeated use of these functions.

converge uniformly on any interval $(a,b) \in [S_{t_n^-}, S_{t_n^-} - D_n]$, *where* $\phi\left(S_{t_n^-}\right)$ *denotes the distribution function of* $S_{t_n^-}$ *given* $\mathcal{F}_{t_{n-1}}$, *and may either*

$$\mathbb{E}^Q_{t_{n-1}}\left[\left|\sum_{i=1}^{\infty}\frac{(-D_n)^i}{i!}C^i\left(S_{t_n^-},t_n\right)\cdot \mathbb{I}_{A_n}\right|\right]$$
$$or \ \sum_{i=1}^{\infty}\mathbb{E}^Q_{t_{n-1}}\left[\left|\frac{(-D_n)^i}{i!}C^i\left(S_{t_n^-},t_n\right)\cdot \mathbb{I}_{A_n}\right|\right]$$

be finite.

This is enough to state the general expression for a call with only one dividend payment

$$\begin{aligned}C_n(S_{t_{n-1}},t_{n-1}) =& C\left(S_{t_{n-1}},t_{n-1}\right)+\\ & e^{-r(t_n-t_{n-1})}\sum_{i=1}^{\infty}\frac{(-D_n)^i}{i!}\mathbb{E}^Q_{t_{n-1}}\left[C^i\left(S_{t_n^-},t_n\right)\cdot \mathbb{I}_{A_n}\right].\end{aligned} \tag{12}$$

Having the convergence property fulfilled for the series, one can now truncate the series to a given η_n that is going to depend on the specific problem at hand. Thus, we proceed with a truncated series

$$\begin{aligned}C_n(S_{t_{n-1}},t_{n-1}) =& C\left(S_{t_{n-1}},t_{n-1}\right)+\\ & e^{-r(t_n-t_{n-1})}\sum_{i=1}^{\eta_n}\frac{(-D_n)^i}{i!}\mathbb{E}^Q_{t_{n-1}}\left[C^i\left(S_{t_n^-},t_n\right)\cdot \mathbb{I}_{A_n}\right],\end{aligned} \tag{13}$$

with η_n determined in light of the specific problem at hand, based on a criteria of added contribution of the terms beyond the η_n term.

From the Last Dividend to the First

From corollary 2.1 and lemma 2.1 we can state that

$$\begin{aligned}C_{n-1}(S_{t_{n-2}},t_{n-2}) = C_n\left(S_{t_{n-2}},t_{n-2}\right) + e^{-r(t_{n-1}-t_{n-2})}.\\ \mathbb{E}^Q_{t_{n-2}}\left[\left(C_n\left(S_{t_{n-1}^-}-D_{n-1},t_{n-1}\right)-C_n\left(S_{t_{n-1}^-},t_{n-1}\right)\right)\cdot \mathbb{I}_{A_{n-1}}\right],\end{aligned} \tag{14}$$

where C_n stands for the price of the same derivative claim ignoring the existence of the dividend D_{n-1}, and C_{n-1}, its price, acknowledging that fact.

Again, we would like to replace the difference by its corresponding Taylor series expansion. Consequently, we have to impose that C_n also fulfills condition 2.1.

Condition 2.3. *Let* $C_n\left(S_{t_{n-1}^-},t_{n-1}\right)$ *be a function infinitely differentiable with respect to* $S_{t_{n-1}^-}$ *and let the corresponding infinite Taylor series expansion be convergent.*

We then get

$$\begin{aligned}\mathbb{E}^Q_{t_{n-2}}\left[\left(C_n\left(S_{t_{n-1}^-}-D_{n-1},t_{n-1}\right)-C_n\left(S_{t_{n-1}^-},t_{n-1}\right)\right)\cdot \mathbb{I}_{A_{n-1}}\right]=\\ \mathbb{E}^Q_{t_{n-2}}\left[\sum_{j=1}^{\infty}\frac{(-D_{n-1})^j}{j!}C_n^j\left(S_{t_{n-1}^-},t_{n-1}\right)\cdot \mathbb{I}_{A_{n-1}}\right].\end{aligned} \tag{15}$$

Rewriting (14) with (13) and (15), we get

$$
\begin{aligned}
&C_{n-1}(S_{t_{n-2}}, t_{n-2}) = C\left(S_{t_{n-2}}, t_{n-2}\right) + \\
&e^{-r(t_n - t_{n-2})}\mathbb{E}^Q_{t_{n-2}}\left[\sum_{i=1}^{\eta_n} \frac{(-D_n)^i}{i!} C^i\left(S_{t_n^-}, t_n\right)\cdot \mathbb{I}_{A_n}\right] + \\
&e^{-r(t_{n-1} - t_{n-2})}\mathbb{E}^Q_{t_{n-2}}\left[\sum_{j=1}^{\infty} \frac{(-D_{n-1})^j}{j!} \frac{\partial^j}{\partial S^j_{t_{n-1}^-}}\left(C\left(S_{t_{n-1}^-}, t_{n-1}\right) + \right.\right. \\
&\left.\left. e^{-r(t_n - t_{n-1})}\sum_{i=1}^{\eta_n} \frac{(-D_n)^i}{i!}\mathbb{E}^Q_{t_{n-1}}\left[C^i\left(S_{t_n^-}, t_n\right)\cdot \mathbb{I}_{A_n}\right]\right)\cdot \mathbb{I}_{A_{n-1}}\right].
\end{aligned}
$$

We resolve the derivative of $C\left(S_{t_{n-1}^-}, t_{n-1}\right)$ to establish the term for the direct effect of the dividend D_{n-1}, and the derivative with respect to $\mathbb{E}^Q_{t_{n-1}}\left[C^i\left(S_{t_n^-}, t_n\right)\cdot \mathbb{I}_{A_n}\right]$ for the combined effect of both dividends,

$$
\begin{aligned}
&C_{n-1}(S_{t_{n-2}}, t_{n-2}) = C\left(S_{t_{n-2}}, t_{n-2}\right) + \\
&e^{-r(t_n - t_{n-2})}\sum_{i=1}^{\eta_n} \frac{(-D_n)^i}{i!}\mathbb{E}^Q_{t_{n-2}}\left[C^i\left(S_{t_n^-}, t_n\right)\cdot \mathbb{I}_{A_n}\right] + \\
&e^{-r(t_{n-1} - t_{n-2})}\sum_{j=1}^{\infty} \frac{(-D_{n-1})^j}{j!}\mathbb{E}^Q_{t_{n-2}}\left[C^j\left(S_{t_{n-1}^-}, t_{n-1}\right)\cdot \mathbb{I}_{A_{n-1}}\right] + \\
&e^{-r(t_n - t_{n-2})}\sum_{j=1}^{\infty} \frac{(-D_{n-1})^j}{j!}\sum_{i=1}^{\eta_n} \frac{(-D_n)^i}{i!}\cdot \\
&\mathbb{E}^Q_{t_{n-2}}\left[\frac{\partial^j\left(\mathbb{E}^Q_{t_{n-1}}\left[C^i\left(S_{t_n^-}, t_n\right)\cdot \mathbb{I}_{A_n}\right]\right)}{\partial S^j_{t_{n-1}^-}}\cdot \mathbb{I}_{A_{n-1}}\right].
\end{aligned} \tag{16}
$$

We can now interpret the contribution of each term to the final result. The first term sets the starting point as a call assuming no dividends. The first two summations account for the impact of the existence of the dividends D_n and D_{n-1}. The third measures the combined effect of both dividends. One can also notice that the number of terms doubles with each new dividend that is added. One dividend yielded two terms and two dividends yielded four terms. The discount factor applied to each term corresponds to the date of the last dividend payment considered by that term. For further dividends, the same procedure applies.

2.2 Call under the Black–Scholes Model

In this section we will apply the general approach to the case of a European call under the Black–Scholes model.

Before we start deriving the formula we consider a simplification of the problems (13) and (16). If the sets A_i have a form different from Ω, i.e., the dividends are paid in all states of the world, equation (13) becomes much more complex and equation (16) is certainly too complex to solve in closed form. Keeping formulas as simple as possible is then an extra argument in favor of assuming $A_i = \Omega$, adding to those discussed in section 1.2. Unfortunately, this assumption raises a further extra problem unlike those discussed in section 1.2. Even though the Black–Scholes model does indeed satisfy the conditions determined in section 2.1 if $A_i = \left\{\omega : S_{t_i^-} > D_i\right\}$, it fails to do so if $A_i = \Omega$. The proofs are lengthy and can be found in appendix A. All proofs rely on the results by Estrella [7], who showed that a Taylor series expansion of the Black–Scholes formula, with respect to the stock price S_t, converges for a radius of S_t itself.

Still, given the fact that the set $\left\{\omega : S_{t_i^-} \leq D_i\right\}$ has a very small probability in almost all realistic scenarios, we believe that the mentioned simplification is an acceptable compromise.

As a less radical simplification, we consider also the possibility of keeping only $A_n = \left\{\omega : S_{t_n^-} > D_n\right\}$ with regard to the dividend payment closest to maturity. This modification can be applied by replacing formula (20) below by the formulas (57) and (58) in appendix C. In fact, in the case of only one dividend payment, this alternative keeps the problem (13) unchanged and thus is not subject to any of the arguments developed above.

With the referred simplification, $A_i = \Omega$, the problem (16) is reduced to

$$\begin{aligned}
&C_{n-1}(S_t, t) = C\left(S_t, t\right) + \\
&e^{-r(t_n - t)} \sum_{i=1}^{\eta_n} \frac{(-D_n)^i}{i!} \mathbb{E}_t^Q \left[C^i\left(S_{t_n^-}, t_n\right)\right] + \\
&e^{-r(t_{n-1} - t)} \sum_{j=1}^{\eta_{n-1}} \frac{(-D_{n-1})^j}{j!} \mathbb{E}_t^Q \left[C^j\left(S_{t_{n-1}^-}, t_{n-1}\right)\right] + \\
&e^{-r(t_n - t)} \sum_{j=1}^{\eta_{n-1}} \frac{(-D_{n-1})^j}{j!} \sum_{i=1}^{\eta_n} \frac{(-D_n)^i}{i!} \mathbb{E}_t^Q \left[\frac{\partial^j \left(\mathbb{E}_{t_{n-1}}^Q \left[C^i\left(S_{t_n^-}, t_n\right)\right]\right)}{\partial S_{t_{n-1}^-}^j}\right].
\end{aligned} \tag{17}$$

Formulas and Results

We take the Black–Scholes model as in equation (1) and recall the formula for call prices,

$$C(S_t, t) = S_t N(d_+) - K e^{-r(T-t)} N(d_-), \tag{18}$$

$$d_\pm = \frac{\log \frac{S_t}{K} + (r \pm \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}, \tag{19}$$

where S_t is the current price of the underlying stock, t the current time, K the strike price, and T the maturity date of the option. $N(x)$ denotes the cumulative function of the standard normal distribution.

Since we are going to rely heavily on the derivatives of this function, it is useful to mention here as well, the formula for a i^{th} derivative of the call price following Carr [6]

$$C^i\left(S_t, t\right) = S_t^{-i} \sum_{j=1}^{i} \mathcal{S}_1\left(i, j\right) \delta^j, \tag{20}$$

$$\delta^j = S_t N\left(d_+\right) + K e^{-r(T-t)} \frac{N'\left(d_-\right)}{\sigma\sqrt{T-t}} \sum_{h=0}^{j-2} \frac{H_h\left(d_-\right)}{\left(-\sigma\sqrt{T-t}\right)^h},$$

where $N'\left(x\right)$ denotes the probability density function of the standard normal distribution, $\mathcal{S}_1\left(i, j\right)$ the Stirling number of the first kind and $H_i\left(d\right)$ are Hermite polynomials.

Also from Carr [6] we need the result that allows us to express the derivatives of call prices, in the BS model, with respect to S_t, in the form of expectations.

In general, the spacial derivatives can be calculated by the following expectation

$$C^i\left(S_t,t\right)=e^{\left(r+\frac{1}{2}i\sigma^2\right)(i-1)(T-t)}\mathbb{E}_t^{S^i}\left[f^{(i)}\left(S_T\right)\right] \tag{21}$$

where the operator $\mathbb{E}^{S^i}$ indicates that the expectation is calculated from the diffusion

$$dS_t=S_t\left((r+i\sigma^2)dt+\sigma dW_{i,t}\right), \tag{22}$$

and where $W_{i,t}$ is a standard Brownian motion under the measure S^i, which has S^i as *numeraire.* $f^{(i)}\left(S_T\right)$ is the i^{th} derivative of the payoff function with respect to S_T.

For example, the delta of a call $C^1\left(S_{t_n},t_n\right)$ is the value, a time t_n, of a derivative paying $\mathbb{I}_{S_T>K}$ units of the underlying stock S, expressed also in units of S,

$$C^1\left(S_t,t\right)=\mathbb{E}_t^{S}\left[\mathbb{I}_{S_T>K}\right].$$

We shall also need expectations of the derivatives observed at a given time, such as

$$\mathbb{E}_t^{Q}\left[C^i\left(S_{t_n},t_n\right)\right],\ \text{with } t\le t_n\le T. \tag{23}$$

By (21) and making use of the tower property for conditional expectations, we have

$$\mathbb{E}_t^{S^i}\left[C^i\left(S_{t_n},t_n\right)\right]=e^{\left(r+\frac{1}{2}i\sigma^2\right)(i-1)(T-t)}\mathbb{E}_t^{S^i}\left[f^{(i)}\left(S_T\right)\right]. \tag{24}$$

To compute (23) we still need the following

Proposition 2.1. *Let* $v_i\left(S_t,t\right)=\mathbb{E}_t^{S^i}\left[f^{(i)}\left(S_T\right)\right]$, *then*

$$\mathbb{E}_t^{Q}\left[C^i\left(S_{t_n},t_n\right)\right]=e^{\left(r+\frac{1}{2}i\sigma^2\right)(i-1)(T-t_n)}v_i\left(e^{-i\sigma^2(t_n-t)}S_t,t\right). \tag{25}$$

Proof. From (21),

$$\begin{aligned}\mathbb{E}_t^{Q}\left[C^i\left(S_{t_n},t_n\right)\right]&=\mathbb{E}_t^{Q}\left[e^{\left(r+\frac{1}{2}i\sigma^2\right)(i-1)(T-t_n)}\mathbb{E}_{t_n}^{S^i}\left[f^{(i)}\left(S_T\right)\right]\right]\\&=e^{\left(r+\frac{1}{2}i\sigma^2\right)(i-1)(T-t_n)}\mathbb{E}_t^{Q}\left[\mathbb{E}_{t_n}^{S^i}\left[f^{(i)}\left(S_T\right)\right]\right].\end{aligned} \tag{26}$$

The diffusions with respect to which the expectations $\mathbb{E}_t^{Q}$ and $\mathbb{E}_t^{S^i}$ are taken, respectively equations (1) and (22), differ only in the drift term since $W_{i,t}$ and W_t are Brownian motions under their respective measures. Thus, to change the measure from S^i to Q, one can compensate the different drift by changing the starting value condition such that the solutions of the diffusions are equal.

Let, $S_{t,Q}$ and S_{t,S^i} be the initial conditions for diffusions (1) and (22) respectively. Then,

$$S_{t_n}=S_{t,Q}e^{\left(r-\frac{\sigma^2}{2}\right)(t_n-t)+\sigma W_{t_n}}\qquad\text{under } Q\text{, and} \tag{27}$$

$$S_{t_n}=S_{t,S^i}e^{\left(r+i\sigma^2-\frac{\sigma^2}{2}\right)(t_n-t)+\sigma W_{i,t_n}}\qquad\text{under } S^i. \tag{28}$$

If we now make $S_{t,S^i}=S_{t,Q}e^{-i\sigma^2(t_n-t)}$ then, under S^i,

$$S_{t_n}=S_{t,Q}e^{\left(r-\frac{\sigma^2}{2}\right)(t_n-t)+\sigma W_{i,t_n}}. \tag{29}$$

This solution is now equivalent, in a weak sense, to (27), since the distribution of S_{t_n} is, in both cases, a log-normal distribution with parameters $\mu_{LN}=\left(r-\frac{\sigma^2}{2}\right)(t_n-t)$ and $\sigma_{LN}=\sigma\sqrt{t_n-t}$.

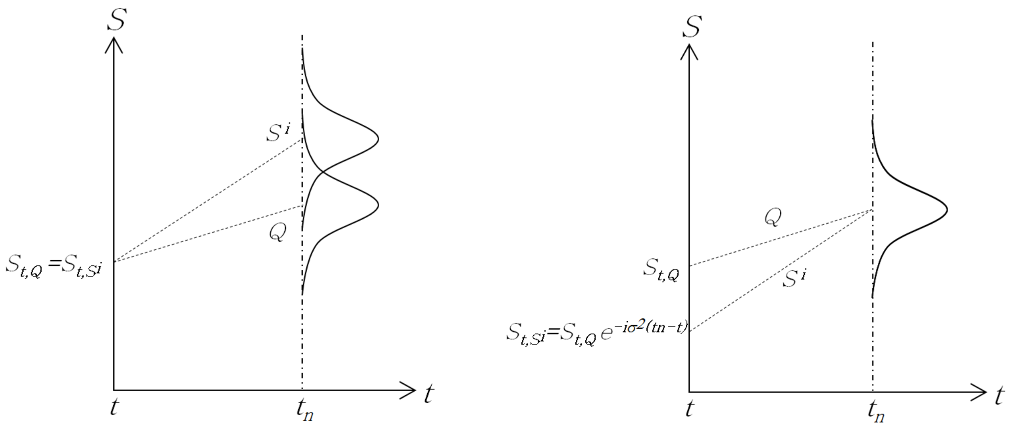

Figure 1: Illustration of the weak convergence argument that relates two expectations of the same random variable under different measures.

The equality of distributions is enough to state the following equality of expectations,

$$\mathbb{E}_t^Q \left[X(S_{t_n})\right] = \mathbb{E}_t^{S^i} \left[X(S_{t_n})\right]\Big|_{S_{t,S^i} = S_{t,Q} e^{-i\sigma^2(t_n - t)}} . \tag{30}$$

Figure 1 illustrates the relationship between two expectation of the same random variables under the measures Q and S^i.

Recalling equation (26), we have

$$\begin{aligned}
\mathbb{E}_t^Q &\left[C^i\left(S_{t_n}, t_n\right)\right] = e^{\left(r+\frac{1}{2}i\sigma^2\right)(i-1)(T-t_n)} \mathbb{E}_t^Q \left[\mathbb{E}_{t_n}^{S^i}\left[f^{(i)}\left(S_T\right)\right]\right] \\
&= e^{\left(r+\frac{1}{2}i\sigma^2\right)(i-1)(T-t_n)}\ \mathbb{E}_t^{S^i} \left[\mathbb{E}_{t_n}^{S^i}\left[f^{(i)}\left(S_T\right)\right]\right]\Big|_{S_{t,S^i} = S_{t,Q} e^{-i\sigma^2(t_n - t)}} \\
&= e^{\left(r+\frac{1}{2}i\sigma^2\right)(i-1)(T-t_n)}\ \mathbb{E}_t^{S^i} \left[f^{(i)}\left(S_T\right)\right]\Big|_{S_{t,S^i} = S_{t,Q} e^{-i\sigma^2(t_n - t)}} ,
\end{aligned}$$

where we again used the tower property to prove the proposition. □

Combining equation (21) and proposition 2.1 we obtain

$$\mathbb{E}_t^Q \left[C^i\left(S_{t_n}, t_n\right)\right] = e^{-\left(r+\frac{1}{2}i\sigma^2\right)(i-1)(t_n-t)} C^i \left(e^{-i\sigma^2(t_n-t)} S_t, t\right) . \tag{31}$$

We can now proceed to state the formula.

Closed Formula

Despite the series' convergence within the referred radius, Estrella [7] mentions also significant instability if one takes a low order approximation. We will be returning to this issue when we consider a specific example.

For a call option with only one dividend payment during the time until maturity, we refer to the case of equation (12) that we recall here,

$$C_n(S_t, t) = C\left(S_t, t\right) + e^{-r(t_n - t)} \sum_{i=1}^{\eta_n} \frac{(-D_n)^i}{i!} \mathbb{E}_t^Q \left[C^i\left(S_{t_n^-}, t_n\right)\right] .$$

We now use equation (31) and cancel the discount factor $e^{-r(t_n-t)}$ to get

$$\begin{aligned}C_n(S_t,t) =& C\left(S_t,t\right)+ \\ &\sum_{i=1}^{\eta_n}\frac{(-D_n)^i}{i!}e^{-\left(r+\frac{1}{2}(i-1)\sigma^2\right)i(t_n-t)}C^i\left(e^{-i\sigma^2(t_n-t)}S_t,t\right),\end{aligned} \tag{32}$$

with C^i as in (20), and C as in (18).

The derivative order η_n is the necessary order to obtain convergence with respect to each specific option valuation.

For options with two dividend payments before maturity, we recall equation (16), which translates to

$$\begin{aligned}C_{n-1}(S_t,t) =& C_n(S_t,t)+ \\ &\sum_{i=1}^{\eta_{n-1}}\frac{(-D_{n-1})^i}{i!}e^{-\left(r+\frac{1}{2}(i-1)\sigma^2\right)i(t_{n-1}-t)}C^i\left(e^{-i\sigma^2(t_{n-1}-t)}S_t,t\right)+ \\ &\sum_{j=1}^{\eta_{n-1}}\frac{(-D_{n-1})^j}{j!}\sum_{i=1}^{\eta_n}\frac{(-D_n)^i}{i!}d\cdot C^{i+j}\left(e^{-i\sigma^2(t_n-t_{n-1})-(i+j)\sigma^2(t_{n-1}-t)}S_t,t\right),\end{aligned} \tag{33}$$

in the Black-Scholes model, with

$$\begin{aligned}d = \exp\Bigg\{ &-\left(r+\frac{1}{2}(i-1)\sigma^2\right)i(t_n-t_{n-1}) \\ &-\left(r+\frac{1}{2}(i+j-1)\sigma^2\right)(i+j)(t_{n-1}-t) \\ &-i\sigma^2 j(t_n-t_{n-1}) \Bigg\}.\end{aligned}$$

Again here, the derivatives orders η_n, η_{n-1} are the necessary orders to obtain convergence with respect to each specific option valuation. A step-by-step derivation of formula (33) can be found in appendix B.

2.3 The Greeks

A closed formula for the derivative [2] of the option price of arbitrary order is a straightforward application of the chain rule. Thus, for the g^{th} derivative of the call price with one discrete dividend payment we have

$$\begin{aligned}C_n^g(S_t,t) =& C^g\left(S_t,t\right)+ \\ &\sum_{i=1}^{\eta_n}\frac{(-D_n)^i}{i!}e^{-\left(r+\left(\frac{1}{2}(i-1)+g\right)\sigma^2\right)i(t_n-t)}C^{i+g}\left(e^{-i\sigma^2(t_n-t)}S_t,t\right).\end{aligned} \tag{34}$$

[2]The derivatives of the option price are usually called *Greeks* because Greek alphabet letters are commonly used to denote them.

Similarly, for the call with two dividend payments we have

$$
\begin{aligned}
&C^g_{n-1}(S_t,t) = C^g_n(S_t,t) + \\
&\sum_{i=1}^{\eta_{n-1}} \frac{(-D_{n-1})^i}{i!} e^{-\left(r+\left(\frac{1}{2}(i-1)+g\right)\sigma^2\right)i(t_{n-1}-t)} C^{i+g}\left(e^{-i\sigma^2(t_{n-1}-t)}S_t,t\right) + \\
&\sum_{j=1}^{\eta_{n-1}} \frac{(-D_{n-1})^j}{j!} \sum_{i=1}^{\eta_n} \frac{(-D_n)^i}{i!} d_g \cdot C^{i+j+g}\left(e^{-i\sigma^2(t_n-t_{n-1})-(i+j)\sigma^2(t_{n-1}-t)}S_t,t\right),
\end{aligned}
\tag{35}
$$

with

$$
\begin{aligned}
d_g = \exp\Bigg\{ \quad & -\left(r+\left(\frac{1}{2}(i-1)+g\right)\sigma^2\right) i(t_n-t_{n-1}) \\
& -\left(r+\left(\frac{1}{2}(i+j-1)+g\right)\sigma^2\right)(i+j)(t_{n-1}-t) \\
& - i\sigma^2 j(t_n-t_{n-1}) \quad \Bigg\}.
\end{aligned}
\tag{36}
$$

The derivatives of the call price with respect to other variables than the spatial variable S can be obtained as a function of the latter. In particular

$$
\begin{aligned}
\frac{\partial C}{\partial \sigma} &= \sigma S_t^2 C^2(S_t,t)(T-t), \\
\frac{\partial C}{\partial r} &= \left(S_t C^1(S_t,t) - C(S_t,t)\right)(T-t), \\
\frac{\partial C}{\partial t} &= rC(S_t,t) - rS_t C^1(S_t,t) - \frac{1}{2}S_t^2 C^2(S_t,t).
\end{aligned}
$$

Further details can be found in Carr [6] and in Reiß and Wystup [14].

3 Results

For ease of reference we reproduce the results stated in Vellekoop and Nieuwenhuis [19] for European call options with seven discrete dividend payments. The model parameters are set at $S_0 = 100$, $\sigma = 25\%$ and $r = 6\%$. Furthermore the stock will pay one dividend per year, with each dividend one year after the previous, of amount 6, 6.5, 7, 7.5, 8, 8 and 8 for the first seven years respectively. We consider three different scenarios of dividend stream payments referenced by the payment date of the first dividend t_1, set at 0.1, 0.5 and 0.9. With respect to the call option specifications, we consider three different options, all with seven years maturity, with strikes of 70, 100 and 130. The calculations reported in Table 1 were performed taking a second order approximation for each of the dividend payments, i.e., $\eta_1, ..., \eta_7 = 2$. This approximation order proved to be very effective in this case, producing errors of 0.01 in the worst cases when compared to the results reported in Vellekoop and Nieuwenhuis [19]. For reference we also report the values that the two most frequently used methods in practice produce under *Modified stock price* and *Modified strike price*. It should be noted that the values these methods produce differ rather strongly from the closed formula approximation.

To illustrate the importance of several terms present in the formula, we aggregate them by the number of dividends involved in each term. We thus have direct terms, combined effect terms of two dividends, and combined effect terms of a number of dividends up to n (in this case seven).

The derivatives of the call price are also displayed starting with the first two spatial derivatives, delta and gamma, and followed by the derivatives with respect to σ, t and r, vega, theta and rho. All calculations were performed in *Mathematica*© using the routines stated in appendix D.

Table 1: European call, $\sigma = 25\%$, $r = 6\%$, $S_0 = 100$, $T = 7$

$t_1 = 0.1$			$t_1 = 0.5$			$t_1 = 0.9$		
Strikes								
70	100	130	70	100	130	70	100	130
Call with no dividends								
56.5642	42.5839	31.9696	56.5642	42.5839	31.9696	56.5642	42.5839	31.9696
Terms with dividends – direct effect, combined effect of 2 divs,..., combined effect of n divs.								
-36.6298	-30.6458	-24.7611	-35.4202	-29.4067	-23.5880	-34.2355	-28.2015	-22.4556
4.0040	5.0787	5.1579	4.1083	5.0173	4.9599	4.1882	4.9289	4.7451
1.0166	0.5521	0.1572	0.9272	0.4264	0.0547	0.8234	0.3053	-0.0336
-0.0335	-0.1165	-0.1114	-0.0752	-0.1257	-0.1024	-0.1078	-0.1258	-0.0889
-0.0358	-0.0148	-0.0021	-0.0311	-0.0080	0.0021	-0.0236	-0.0018	0.0048
0.0002	0.0017	0.0013	0.0018	0.0018	0.0010	0.0027	0.0015	0.0006
0.0003	0.0001	0.0000	0.0002	0.0000	-0.0001	0.0001	0.0000	-0.0001
Total – option price								
24.8862	17.4394	12.4114	26.0752	18.4890	13.2968	27.2117	19.4905	14.1419
Option price reported in Vellekoop and Nieuwenhuis [19]								
24.90	17.43	12.40	26.08	18.48	13.29	27.21	19.48	14.13
Modified stock price method								
20.1576	12.3709	7.7556	20.1576	12.3709	7.7556	20.1576	12.3709	7.7556
Modified strike price method								
30.7358	23.1768	17.5976	30.7358	23.1768	17.5976	30.7358	23.1768	17.5976
Spatial derivatives – delta (per cent), gamma (per ten thousand).								
70.6821	56.0090	43.8271	71.1645	56.9270	44.9643	71.6629	57.8120	46.0412
60.6487	73.2547	75.1486	57.4643	70.0730	72.6798	54.6009	67.2527	70.5040
Other derivatives – vega, theta and rho.								
35.5295	42.9144	44.0239	33.6640	41.0505	42.5776	31.9866	39.3989	41.3030
-1.5950	-1.6643	-1.5629	-1.5071	-1.5999	-1.5234	-1.4263	-1.5411	-1.4874
112.0643	104.7648	90.6232	105.6951	101.1355	89.0311	99.7581	97.7093	87.4769

In any scenario of the case being analyzed, the number of evaluations of the Black-Scholes pricing formula or any of its derivatives amounts to 2187. To understand how the different terms contribute to the total number of evaluations, Table 2 breaks it down by terms aggregated by the number of dividend payments they have.

In the simplest case of only one dividend payment until maturity, Table 2 would be much different. The call with no dividends plus two derivatives would yield a total of only three terms.

Naturally, one should expect that different cases may require different orders of approximation. Problems with larger individual dividends, with dividends very close to maturity or in the presence of very low volatilities should require a higher approximation order. The issue at stake is how smooth the function being approximated is - the smoother the function the lower the required approximation order.

4 Conclusion

Departing from the well known behavior of the option price at the dividend payment date, we approximate it by Taylor series expansion and successfully manipulate it to arrive at a closed form pricing formula.

Table 2: Number of Calls to the Black–Scholes pricing formula or its derivative.

n	Number of terms with n dividends (1) $\binom{7}{n}$	Number of parcels in each term (2) $\eta_1 \cdot \ldots \cdot \eta_n$	Number of evaluations of BS or its derivative (1)x(2)
0	1	1	1
1	7	2	14
2	21	4	84
3	35	8	280
4	35	16	560
5	21	32	672
6	7	64	448
7	1	128	128
		Total:	2187

We do so by making use of the results from Carr [6] and by an appropriate reparametrization of the Black–Scholes stochastic differential equation. The procedure is described for options on stocks with up to two dividend payments and is applied to an example with seven dividend payments. Formulas for more dividend payments may be developed using the same procedure. By the same means we also derive the derivatives, of any order, of the option price with respect to the underlying asset; and from those the remaining derivatives with respect to other parameters in the model.

The results show that a low approximation order is enough to attain reasonable results in most problems observed in practice.

The extension of this results to other models than the Black–Scholes model is left for future research. That is the case also for the extension of this approach to multi-asset options.

Appendices

A Condition's Verification

Conditions' Verification

Condition 2.1 is easily satisfied but it hints at the limitations that this method will have. It reads: *Let* $C\left(S_{t_n^-}, t_n\right)$ *be a function infinitely differentiable on* $S_{t_n^-}$*, and let the corresponding infinite Taylor series expansion be convergent.*

Proof. From equation (18) one can see that the call price $C\left(S_{t_n^-}, t_n\right)$ is the difference of products of infinitely differentiable functions if $t < T$ and thus satisfies the condition. If $t = T$, then $N(x)$ turns into a step function and thus is no longer differentiable. Thus, the first condition determines that $S_{t_n^-} \in \mathbb{R}_+$. The convergence of $C\left(S_{t_n^-}, t_n\right)$ was proved by Estrella [7] for shifts $-D_n$ such that $-S_{t_n^-} < D_n < S_{t_n^-}$. Thus, the interval on which the function is differentiable and whose series is convergent is

$$-S_{t_n^-} < D_n < S_{t_n^-}, \tag{37}$$

which includes the relevant interval for dividend shifts $[0, S_{t_n^-})$. □

Condition 2.2 reads:
let $\sum_{i=1}^{\infty} \frac{(-D_n)^i}{i!} C^i\left(S_{t_n^-}, t_n\right) \cdot \mathbb{I}_{A_n} \cdot \phi\left(S_{t_n^-}\right)$ *converge uniformly on any interval* $(a,b) \in [S_{t_n^-}, S_{t_n^-} - D_n]$*, and may* $\mathbb{E}^Q_{t_{n-1}}\left[\left|\sum_{i=1}^{\infty} \frac{(-D_n)^i}{i!} C^i\left(S_{t_n^-}, t_n\right) \cdot \mathbb{I}_{A_n}\right|\right]$ *or* $\sum_{i=1}^{\infty} \mathbb{E}^Q_{t_{n-1}}\left[\left|\frac{(-D_n)^i}{i!} C^i\left(S_{t_n^-}, t_n\right) \cdot \mathbb{I}_{A_n}\right|\right]$ *be*

finite. With $\phi\left(S_{t_n^-}\right)$ *the distribution function of* $S_{t_n^-}$ *given* $\mathcal{F}_{t_{n-1}}$.

Proof. The convergence interval of the product of the functions is the intersection of both intervals. The convergence of $C\left(S_{t_n^-}, t_n\right)$ has already been established in (37) and so we only need to verify the convergence properties of $\mathbb{I}_{A_n}$ and $\phi\left(S_{t_n^-}\right)$. Function $\mathbb{I}_{A_n}$ is trivially convergent either on $D_n < S_{t_n^-}$ or on $D_n > S_{t_n^-}$. With respect to $\phi\left(S_{t_n^-}\right)$, it is well known that in the Black Scholes model, the distribution of $S_{t_n^-}$ is lognormal and thus $\phi\left(S_{t_n^-}\right)$ is the composition of $\log(S_{t_n^-})$ and the normal probability density function. Apart from the constants, the normal density $\exp\left\{(x-\mu)^2/(2\sigma^2)\right\}/\sqrt{2\pi\sigma}$ can by described as the successive composition of e^x and x^2. Both functions are convergent everywhere, so is the normal density. The composition with $\log(S_{t_n^-})$ restricts the convergence radius to that of the log function, i.e., $(0, 2S_{t_n^-})$. In terms of shifts D_n with respect to $S_{t_n^-}$ we have $-S_{t_n^-} < D_n < S_{t_n^-}$. Thus no restrictions must be imposed on the domain of convergence found in (37).

For the final statement of the condition, it suffices to show that $\mathbb{E}^Q_{t_{n-1}}\left[\left|\sum_{i=1}^{\infty}\frac{(-D_n)^i}{i!}C^i\left(S_{t_n^-}, t_n\right)\cdot\mathbb{I}_{A_n}\right|\right]$ is finite.

Since the series $\sum_{i=1}^{\infty}\frac{(-D_n)^i}{i!}C^i\left(S_{t_n^-}, t_n\right)$ converges to $C\left(S_{t_n^-} - D_n, t_n\right) - C\left(S_{t_n^-}, t_n\right)$ and $C\left(S_{t_n^-}, t_n\right)$ is finite for every $S_{t_n^-}$ as long as $t_n < T$, the condition is verified. □

Finally, condition 2.3 states: *let* $C_n\left(S_{t_{n-1}^-}, t_{n-1}\right)$ *be a function infinitely differentiable on* $S_{t_{n-1}^-}$ *and let the corresponding infinite Taylor series expansion be convergent.*

Proof. The differentiability of C_n is determined by that of C, proved above with respect to condition 2.1, and $\frac{(-D_n)^i}{i!}\mathbb{E}^Q_{t_{n-1}}\left[C^i\left(S_{t_n^-}, t_n\right)\cdot\mathbb{I}_{A_n}\right]$. For the differentiability of the last, we note, by (31), that it is determined by that of $C^i_{\mathbb{I}_{A_n}}\left(S_{t_n^-}, t_n\right)$, which is satisfied by way of (57) and (58).

The convergence of $C_n\left(S_{t_{n-1}^-}, t_{n-1}\right)$ is determined by that of C, determined in the proof of 2.1, and by that of the series

$$\sum_{j=1}^{\infty}\frac{(-D_{n-1})^j}{j!}\frac{\partial^j\left(\sum_{i=1}^{\eta}\frac{(-D_n)^i}{i!}\mathbb{E}^Q_{t_{n-1}}\left[C^i\left(S_{t_n^-}, t_n\right)\cdot\mathbb{I}_{A_n}\right]\right)}{\partial S^j_{t_{n-1}}}. \tag{38}$$

The corresponding convergence interval of C with respect to shifts D_{n-1} is the following

$$-S_{t_{n-1}^-} < D_{n-1} < S_{t_{n-1}^-}. \tag{39}$$

The remaining term (38) can be restated making use of (31)

$$\sum_{j=1}^{\infty}\frac{(-D_{n-1})^j}{j!}\sum_{i=1}^{\eta}\frac{(-D_n)^i}{i!}\cdot \frac{\partial^j\left(e^{-\left(r+\frac{1}{2}i\sigma^2\right)(i-1)(t_n-t_{n-1})}C^i_{\mathbb{I}_{A_n}}\left(e^{-i\sigma^2(t_n-t_{n-1})}S_{t_{n-1}^-}, t_{n-1}\right)\right)}{\partial S^j_{t_{n-1}}},$$

which then yields,

$$e^{r(t_n-t_{n-1})}\sum_{i=1}^{\eta}\frac{\left(-D_n e^{-\left(r+\frac{1}{2}(i-1)\sigma^2\right)(t_n-t_{n-1})}\right)^i}{i!}\sum_{j=1}^{\infty}\frac{\left(-D_{n-1}e^{-i\sigma^2(t_n-t_{n-1})}\right)^j}{j!}\cdot$$
$$C^{i+j}_{\mathbb{I}_{A_n}}\left(e^{-i\sigma^2(t_n-t_{n-1})}S_{t^-_{n-1}},t_{n-1}\right). \quad (40)$$

Now, for each i, the series above is the Taylor series expansion of the i^{th} derivative of $C_{\mathbb{I}_{A_n}}$, or $C^i_{\mathbb{I}_{A_n}}$, multiplied by the constant

$$\frac{\left(-D_n e^{-\left(r+\frac{1}{2}(i-1)\sigma^2\right)(t_n-t_{n-1})}\right)^i}{i!}.$$

The constant does not affect the convergence of $C^i_{\mathbb{I}_{A_n}}$.

The convergence of $C^i_{\mathbb{I}_{A_n}}$ is assured by the same arguments that yield the convergence of the call price function found in Estrella [7]. Following the same reasoning, we take one specific derivative to perform the analysis, $i=3$, since it carries the same convergence property of any other derivative. We note then that $C^3_{\mathbb{I}_{A_n}}$ is a product of constants, ψ_1, ψ_2 and of division by S^3. Since the division and the constants do not alter the convergence, we need to check the functions ψ_1 and ψ_2. Again, we have multiplicative constants and the following factors that depend on S: $N'(d_-)$ and $H_h(d_-)$. Both of the factors can be obtained by successive composition of some or all of the functions e^x, x^2 and $log(x)$ and thus the convergence radius shall be the intersection of the convergence radius of all of these functions. Since the exponential and the square functions are convergent everywhere, the radius of convergence of $C^i_{\mathbb{I}_{A_n}}$ is determined by the convergence of $log(x)$ and that is x or, in our case, $S_{t^-_{n-1}}$. The composition of $C^i_{\mathbb{I}_{A_n}}$ with $f\left(S_{t^-_{n-1}}\right)=e^{-i\sigma^2(t_n-t_{n-1})}S_{t^-_{n-1}}$ does not change the convergence radius since $\log\left(f(S_{t^-_{n-1}})\right)=\log\left(S_{t^-_{n-1}}\right)-i\sigma^2(t_n-t_{n-1})$ and thus $\log\left(S_{t^-_{n-1}}\right)$ is still the function that determines the convergence interval. Furthermore, the shifts $-D_{n-1}e^{-i\sigma^2(t_n-t_{n-1})}$ are smaller that $-D_{n-1}$ for any i. Thus, we can take the following convergence interval for the Taylor series expansion of $C^i_{\mathbb{I}_{A_n}}$ for all i,

$$-S_{t^-_{n-1}}<D_{n-1}<S_{t^-_{n-1}}. \quad (41)$$

□

Thus, the fulfillment of conditions necessary for the development of a closed formula have rendered the following restrictions on the size of dividends:

$$-S_{t^-_n}<D_n<S_{t^-_n} \text{ and } -S_{t^-_{n-1}}<D_{n-1}<S_{t^-_{n-1}}.$$

These restrictions overlap with the dividend payment policy that determines dividend payments if $0\le D_n<S_{t^-_n}$ or $0\le D_{n-1}<S_{t^-_{n-1}}$ and thus are fulfilled by construction.

B Derivation of the Two Dividends Formula

For two options with two dividends to maturity, we fall back on the case of equation (16) that we recall

$$C_{n-1}(S_{t_{n-2}}, t_{n-2}) = C\left(S_{t_{n-2}}, t_{n-2}\right) + \tag{42a}$$

$$e^{-r(t_n - t_{n-2})} \sum_{i=1}^{\infty} \frac{(-D_n)^i}{i!} \mathbb{E}^Q_{t_{n-2}} \left[C^i\left(S_{t_n}, t_n\right)\right] + \tag{42b}$$

$$e^{-r(t_{n-1} - t_{n-2})} \sum_{j=1}^{\infty} \frac{(-D_{n-1})^j}{j!} \mathbb{E}^Q_{t_{n-2}} \left[C^j\left(S_{t_{n-1}}, t_{n-1}\right)\right] + \tag{42c}$$

$$e^{-r(t_n - t_{n-2})} \sum_{j=1}^{\infty} \frac{(-D_{n-1})^j}{j!} \sum_{i=1}^{\infty} \frac{(-D_n)^i}{i!} \mathbb{E}^Q_{t_{n-2}} \left[\frac{\partial^j\left(\mathbb{E}^Q_{t_{n-1}}\left[C^i\left(S_{t_n}, t_n\right)\right]\right)}{\partial S^j_{t_{n-1}}}\right]. \tag{42d}$$

The evaluation of the terms (42a), (42b) and (42c) are done as in the one dividend case (32). The solution for the term (42d) is obtain by the following procedure:

- by equation (31)

$$\mathbb{E}^Q_{t_{n-2}} \left[\frac{\partial^j\left(\mathbb{E}^Q_{t_{n-1}}\left[C^i\left(S_{t_n}, t_n\right)\right]\right)}{\partial S^j_{t_{n-1}}}\right] = \tag{43}$$

$$\mathbb{E}^Q_{t_{n-2}} \left[\frac{\partial^j\left(e^{-\left(r+\frac{1}{2}i\sigma^2\right)(i-1)(t_n - t_{n-1})} C^i\left(e^{-i\sigma^2(t_n - t_{n-1})} S_{t_{n-1}}, t_{n-1}\right)\right)}{\partial S^j_{t_{n-1}}}\right], \tag{44}$$

- by taking the constant $c_1 = e^{-\left(r+\frac{1}{2}i\sigma^2\right)(i-1)(t_n - t_{n-1})}$ out of the expectation and by resolving the derivative,

$$c_1 \mathbb{E}^Q_{t_{n-2}} \left[C^{i+j}\left(e^{-i\sigma^2(t_n - t_{n-1})} S_{t_{n-1}}, t_{n-1}\right) e^{-i\sigma^2 j(t_n - t_{n-1})}\right], \tag{45}$$

- by taking the constant $c_2 = e^{-i\sigma^2 j(t_n - t_{n-1})}$ out of the expectation and equation (30)

$$c_1 c_2\, \mathbb{E}^{S^{i+j}}_{t_{n-2}} \left[C^{i+j}\left(e^{-i\sigma^2(t_n - t_{n-1})} S_{t_{n-1}}, t_{n-1}\right)\right]\Big|_{S_{t_{n-2}, S^{i+j}}}, \tag{46}$$

$$S_{t_{n-2}, S^{i+j}} = S_{t_{n-2}, Q} e^{-(i+j)\sigma^2(t_{n-1} - t_{n-2})}, \tag{47}$$

- by equation (21) and by taking out of the expectation the constant c_3, with $c_3 = e^{\left(r+\frac{1}{2}(i+j)\sigma^2\right)(i+j-1)(T - t_{n-1})}$

$$c_1 c_2 c_3\, \mathbb{E}^{S^{i+j}}_{t_{n-2}} \left[\mathbb{E}^{S^{i+j}}_{t_{n-1}}\left[f^{(i+j)}\left(S_T\right)\right]\Big|_{S_{t_{n-1}, S^{i+j}}}\right]\Bigg|_{S_{t_{n-2}, S^{i+j}}}, \tag{48}$$

$$S_{t_{n-1}, S^{i+j}} = S_{t_{n-1}, Q} e^{-i\sigma^2(t_n - t_{n-1})}, \tag{49}$$

$$S_{t_{n-2}, S^{i+j}} = S_{t_{n-2}, Q} e^{-(i+j)\sigma^2(t_{n-1} - t_{n-2})}, \tag{50}$$

- by aggregating both corrections of the starting value $S_{t_{n-1},S^{i+j}}$ and $S_{t_{n-2},S^{i+j}}$ in the starting value $S_{t_{n-2},S^{i+j}}$

$$c_1c_2c_3\,\mathbb{E}^{S^{i+j}}_{t_{n-2}}\left[\mathbb{E}^{S^{i+j}}_{t_{n-1}}\left[f^{(i+j)}\left(S_T\right)\right]\right]\Big|_{S_{t_{n-2},S^{i+j}}}, \tag{51}$$

$$S_{t_{n-2},S^{i+j}} = S_{t_{n-2},Q}e^{-i\sigma^2(t_n-t_{n-1})-(i+j)\sigma^2(t_{n-1}-t_{n-2})}, \tag{52}$$

- by the tower law

$$c_1c_2c_3\,\mathbb{E}^{S^{i+j}}_{t_{n-2}}\left[f^{(i+j)}\left(S_T\right)\right]\Big|_{S_{t_{n-2},S^{i+j}}}, \tag{53}$$

$$S_{t_{n-2},S^{i+j}} = S_{t_{n-2},Q}e^{-i\sigma^2(t_n-t_{n-1})-(i+j)\sigma^2(t_{n-1}-t_{n-2})}, \tag{54}$$

- again here, by equation (21) and by taking out the constant c_4, with $c_4 = e^{-\left(r+\frac{1}{2}(i+j)\sigma^2\right)(i+j-1)(T-t_{n-2})}$

$$c_1c_2c_3c_4C^{i+j}\left(S_{t_{n-2}}e^{-i\sigma^2(t_n-t_{n-1})-(i+j)\sigma^2(t_{n-1}-t_{n-2})}, t_{n-2}\right), \tag{55}$$

with

$$\begin{aligned} c_1c_2c_3c_4 = d = \exp\{ &-\left(r+\frac{1}{2}i\sigma^2\right)(i-1)(t_n-t_{n-1}) \\ &-\left(r+\frac{1}{2}(i+j)\sigma^2\right)(i+j-1)(t_{n-1}-t_{n-2}) \\ &- i\sigma^2 j(t_n-t_{n-1})\,\}\,. \end{aligned}$$

C Alternative to the Black–Scholes Formula

Following the same reasoning as in Carr [6], we were able to derive the following closed formulas for the price and a generic derivative of a claim with payoff $\max\left(S_T-K,0\right)\cdot\mathbb{I}_{\{S_{t_n}>D_n\}}$.

The pricing formula is

$$\begin{aligned} C_{\mathbb{I}_{A_n}}\left(s,t\right) =& s\cdot N_2\left(d_+, \frac{\log\frac{s}{D_n}+(r+\frac{\sigma^2}{2})(t_n-t)}{\sigma\sqrt{t_n-t}}\right) - \\ & Ke^{-r(T-t)}\cdot N_2\left(d_-, \frac{\log\frac{s}{D_n}+(r-\frac{\sigma^2}{2})(t_n-t)}{\sigma\sqrt{t_n-t}}\right). \end{aligned} \tag{56}$$

For the first derivative we have

$$C^1_{\mathbb{I}_{A_n}}\left(s,t\right) = N_2\left(d_+, \frac{\log\frac{s}{D_n}+(r+\frac{\sigma^2}{2})(t_n-t)}{\sigma\sqrt{t_n-t}}\right), \tag{57}$$

with N_2 the bivariate standard normal distribution with correlation parameter $\rho = \sqrt{\frac{t_n-t}{T-t}}$.

For derivatives of order greater than one,

$$\begin{aligned} C^i_{\mathbb{I}_{A_n}}\left(S_t,t\right) =& e^{\left(r+\frac{1}{2}i\sigma^2\right)(i-1)(T-t)}(-1)^{i-2}\cdot \\ & \frac{K}{s^i}\sum_{j=0}^{i-2}\binom{i-2}{j}\psi_1(s,t,j)\psi_2(s,t,i-2-j). \end{aligned} \tag{58}$$

with

$$\psi_1(s,t,j) = (-1)^j e^{-\left(r+\frac{1}{2}(j-1)\sigma^2\right)j(T-t)} \frac{N'\left(d_-\right)}{\sigma\sqrt{T-t}} \cdot \sum_{h=0}^{j} \frac{H_h\left(d_-\right) U_i(j+1,h+1)}{\left(-\sigma\sqrt{T-t}\right)^h},$$

and

$$\psi_2(s,t,j) = \begin{cases} N(z) & \text{if } j=0 \\ N'(-z) \sum_{h=0}^{j-1} \mathcal{S}_1\left(j,h+1\right) H_h\left(-z\right) \left(\frac{\rho}{\sigma\sqrt{T-t_n}}\right)^{h+1} & \text{if } j \geq 0 \end{cases}.$$

Every symbol with the same meaning as above and additionally

$$z = \frac{d_{D,i} - \rho \cdot d_{K,i}}{\sqrt{\frac{T-t_n}{T-t}}},$$

$$d_{D,i} = \frac{\log \frac{s}{D_n} + (r + i\sigma^2 - \frac{\sigma^2}{2})(t_n - t)}{\sigma\sqrt{t_n - t}},$$

$$d_{K,i} = \frac{\log \frac{s}{K} + (r + i\sigma^2 - \frac{\sigma^2}{2})(T - t)}{\sigma\sqrt{T - t}}.$$

The numbers $U_i(j+1,h+1)$ follow the recursion

$$U_i(m,n) = U_i(m-1,n-1) - (i-(m-1))U_i(m-1,n),$$

with starting values

$$U_i(0,0) = 1, \quad U_i(\cdot,0) = 0, \quad U_i(0,\cdot) = 0.$$

D *Mathematica*© Code

```
In[1]:= << Statistics`NormalDistribution`

In[2]:= ND = NormalDistribution[0, 1];

In[3]:= CallDivs[t_, S_, σ_, r_, k_, T_, Divs_, ηMax_, precision_, grkOrder_] := (
         result = Table[0, {grkOrder + 1}];
         n = Length[Divs];
         η = Table[ηMax, {i, n}];

         (* Terms with i dividends *)
         For[i = 0, i ≤ n, i++,
          m = Table[n + 1 - z, {z, 1, i}];

          (* Number of Terms with i dividends *)
          For[j = 1, j ≤ Binomial[n, i], j++,
           sdiv = Table[Divs⟦m⟦z⟧⟧⟦2⟧, {z, 1, i}];
           sη = Table[η⟦m⟦z⟧⟧, {z, 1, i}];
           sηN = Fold[Times, 1, sη];
           sη = Table[1, {z, 1, i}];

           (* All terms of the summation *)
           For[h = 1, h ≤ sηN, h++,

            (* Discount Term and Shift Composition *)
            sηi = 0;
            discExp = 0;
            shiftExp = 0;
            Δti = 0;
            For[a = 1, a ≤ i, a++,
             Δti = If[a < i, Divs⟦m⟦a⟧⟧⟦1⟧ - Divs⟦m⟦a + 1⟧⟧⟦1⟧, Divs⟦m⟦a⟧⟧⟦1⟧ - t];
             sηi += sη⟦a⟧;
             (*chain rule terms*)
             discExp += shiftExp * sη⟦a⟧ ;
             (*change of measure terms*)
             discExp += -(r + 0.5 (sηi - 1) σ^2) sηi Δti ;
             (*shift*)
             shiftExp += -sηi σ^2 Δti;
            ];
            discount = Exp[discExp];
            shift = Exp[shiftExp];
            dFactor = Fold[Times, 1, (-sdiv)^sη / sη!];
            derivNum = Fold[Plus, 0, sη];

            (* Call the Black Scholes Formula or Derivative *)
            If[i == 1,
             contrib = dFactor discount BSDer[1, t, shift S, T, k, r, σ, derivNum];
             If[Abs[contrib] < precision, η⟦m⟦i⟧⟧ = h - 1; Break[];, result⟦1⟧ += contrib],
             result⟦1⟧ += dFactor discount BSDer[1, t, shift S, T, k, r, σ, derivNum]];

            (* Greeks *)
            For[gg = 1, gg ≤ grkOrder, gg++,
             discExp += shiftExp * gg;
             discount = Exp[discExp];
             result⟦gg + 1⟧ += dFactor discount BSDer[1, t, shift S, T, k, r, σ, derivNum + gg];
            ];

            (*Increase sη to cover all exponents of the divs*)
            For[g = i, g ≥ 1, g--,
             If[sη⟦g⟧ + 1 ≤ η⟦m⟦g⟧⟧,
              sη⟦g⟧++; Break[];,
              sη⟦g⟧ = 1]
            ];
           ];

           (*Decrease m to run all combinations*)
           For[f = i, f ≥ 1, f--,
```

```
      If[m[[f]] > i - f + 1,
        m[[f]] --;
        For[ff = f + 1, ff ≤ i, ff ++,
          m[[ff]] = m[[ff - 1]] - 1
        ]; Break[];
       ];
     ];
    ];
   ];
   result)
```

In[4]:=
```
BSDer[cp_, t_, S_, T_, k_, r_, σ_, n_] :=
  If[n == 0, cp S CDF[ND, cp d1[t, S, T, k, r, σ]] - k ⅇ^(-r (T-t)) CDF[ND, cp d2[t, S, T, k, r, σ]],
```
$$\frac{1}{S^n}\left(\sum_{i=1}^{n}\texttt{StirlingS1[n, i] BSDerLog[cp, t, S , T, k, r, σ, i]}\right)]$$

In[5]:=
```
BSDerLog[cp_, t_, S_, T_, k_, r_, σ_, n_] :=
```
$$\texttt{cp S CDF[ND, cp d1[t, S, T, k, r, σ]]} + k\, e^{-r\,(T-t)}\,\frac{\texttt{PDF[ND, d2[t, S, T, k, r, σ]]}}{\sigma\sqrt{T-t}}\sum_{i=0}^{n-2}\frac{\sum_{g=0}^{i/2}\frac{\texttt{d2[t,S,T,k,r,σ]}^{i-2g}}{(i-2g)!}\,\frac{i!}{g!\,(-2)^g}}{\left(-\sigma\sqrt{T-t}\right)^i}$$

In[6]:=
$$\texttt{d1[t\_, S\_, T\_, k\_, r\_, σ\_] :=}\ \frac{\texttt{Log}[\frac{S}{k}] + \left(r + \frac{\sigma^2}{2}\right)\ (T-t)}{\sigma\sqrt{T-t}}$$

In[7]:=
$$\texttt{d2[t\_, S\_, T\_, k\_, r\_, σ\_] :=}\ \frac{\texttt{Log}[\frac{S}{k}] + \left(r - \frac{\sigma^2}{2}\right)\ (T-t)}{\sigma\sqrt{T-t}}$$

In[8]:=
```
t1 = 0.1;
```

In[9]:=
```
Divs := {{t1, 6}, {1 + t1, 6.5}, {2 + t1, 7}, {3 + t1, 7.5}, {4 + t1, 8}, {5 + t1, 8}, {6 + t1, 8}};
```

In[10]:=
```
CallDivs[0., 100, 0.25, 0.06, 70, 7, Divs, 5, 0.01, 1]
```

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