Quanto Options

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1 Quanto Options

A quanto option can be any cash-settled option, whose payoff is converted into a third currency at maturity at a pre-specified rate, called the *quanto factor*. There can be quanto plain vanilla, quanto barriers, quanto forward starts, quanto corridors, etc. The Arbitrage pricing theory and the Fundamental theorem of asset pricing, also covered for example in [3] and [2], allow the computation of option values. Other references: Options: basic definitions, Option pricing: general principles, Foreign exchange market terminology.

1.1 FX Quanto Drift Adjustment

We take the example of a Gold contract with underlying XAU/USD in XAU-USD quotation that is quantoed into EUR. Since the payoff is in EUR, we let EUR be the numeraire or domestic or base currency and consider a Black-Scholes model

$$\begin{aligned}
\text{XAU-EUR: } dS_t^{(3)} &= (r_{EUR} - r_{XAU})S_t^{(3)}\,dt + \sigma_3 S_t^{(3)}\,dW_t^{(3)}, && (1)\\
\text{USD-EUR: } dS_t^{(2)} &= (r_{EUR} - r_{USD})S_t^{(2)}\,dt + \sigma_2 S_t^{(2)}\,dW_t^{(2)}, && (2)\\
dW_t^{(3)}dW_t^{(2)} &= -\rho_{23}\,dt, && (3)
\end{aligned}$$

where we use a minus sign in front of the correlation, because both $S^{(3)}$ and $S^{(2)}$ have the same base currency (DOM), which is EUR in this case. The scenario is displayed in Figure 1. The actual underlying is then

$$\text{XAU-USD: } S_t^{(1)} = \frac{S_t^{(3)}}{S_t^{(2)}}. \tag{4}$$

Using Itô's formula, we first obtain

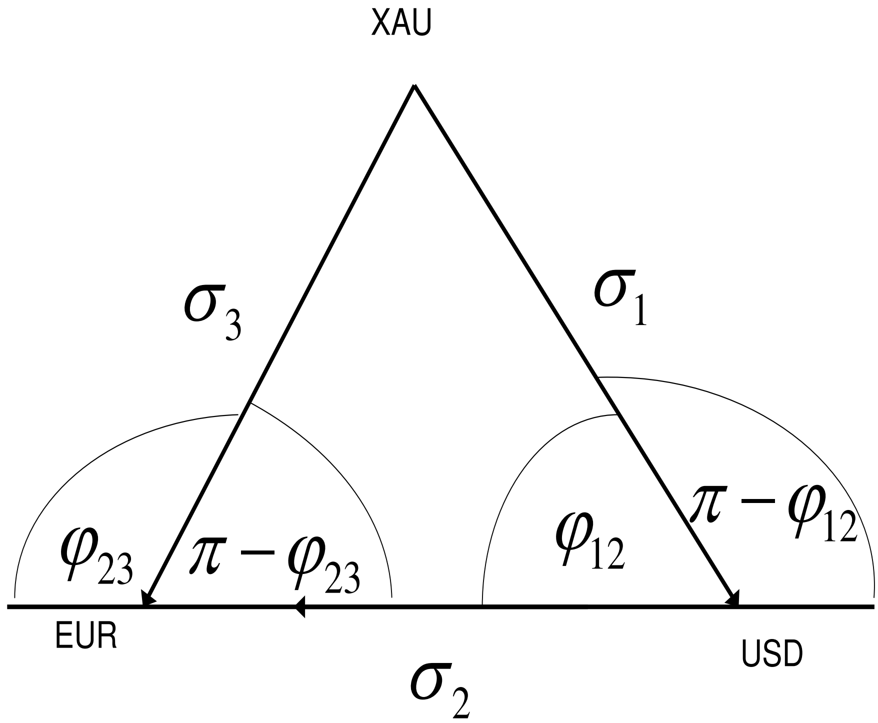

Figure 1: XAU-USD-EUR FX Quanto Triangle. The arrows point in the direction of the respective base currencies. The length of the edges represents the volatility. The cosine of the angles $\cos\phi_{ij} = \rho_{ij}$ represents the correlation of the currency pairs $S^{(i)}$ and $S^{(j)}$, *if* the base currency (DOM) of $S^{(i)}$ is the underlying currency (FOR) of $S^{(j)}$. If both $S^{(i)}$ and $S^{(j)}$ have the same base currency (DOM), then the correlation is denoted by $-\rho_{ij} = \cos(\pi - \phi_{ij})$.

$$
\begin{aligned}
d\frac{1}{S_t^{(2)}} &= -\frac{1}{(S_t^{(2)})^2}\, dS_t^{(2)} + \frac{1}{2}\cdot 2\cdot \frac{1}{(S_t^{(2)})^3}\,(dS_t^{(2)})^2 \\
&= (r_{USD} - r_{EUR} + \sigma_2^2)\frac{1}{S_t^{(2)}}\, dt - \sigma_2 \frac{1}{S_t^{(2)}}\, dW_t^{(2)}, \qquad (5)
\end{aligned}
$$

and hence

$$
\begin{aligned}
dS_t^{(1)} &= \frac{1}{S_t^{(2)}}\,dS_t^{(3)} + S_t^{(3)}\,d\frac{1}{S_t^{(2)}} + dS_t^{(3)}\,d\frac{1}{S_t^{(2)}} \\
&= \frac{S_t^{(3)}}{S_t^{(2)}}(r_{EUR} - r_{XAU})\,dt + \frac{S_t^{(3)}}{S_t^{(2)}}\sigma_3\,dW_t^{(3)} \\
&\quad + \frac{S_t^{(3)}}{S_t^{(2)}}(r_{USD} - r_{EUR} + \sigma_2^2)\,dt - \frac{S_t^{(3)}}{S_t^{(2)}}\sigma_2\,dW_t^{(2)} + \frac{S_t^{(3)}}{S_t^{(2)}}\rho_{23}\sigma_2\sigma_3\,dt \\
&= (r_{USD} - r_{XAU} + \sigma_2^2 + \rho_{23}\sigma_2\sigma_3)S_t^{(1)}\,dt + S_t^{(1)}(\sigma_3\,dW_t^{(3)} - \sigma_2\,dW_t^{(2)}).
\end{aligned}
$$

Since $S_t^{(1)}$ is a geometric Brownian motion with volatility σ_1, we introduce a new Brownian motion $W_t^{(1)}$ and find

$$
dS_t^{(1)} = (r_{USD} - r_{XAU} + \sigma_2^2 + \rho_{23}\sigma_2\sigma_3)S_t^{(1)}\,dt + \sigma_1 S_t^{(1)}\,dW_t^{(1)}. \tag{6}
$$

Now Figure 1 and the *law of cosine* imply

$$
\sigma_3^2 = \sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2, \tag{7}
$$
$$
\sigma_1^2 = \sigma_2^2 + \sigma_3^2 + 2\rho_{23}\sigma_2\sigma_3, \tag{8}
$$

which yields

$$
\sigma_2^2 + \rho_{23}\sigma_2\sigma_3 = \rho_{12}\sigma_1\sigma_2. \tag{9}
$$

As explained in the *currency triangle* in Figure 1, ρ_{12} is the correlation between XAU-USD and USD-EUR, whence $\rho \stackrel{\Delta}{=} -\rho_{12}$ is the correlation between XAU-USD and EUR-USD. Inserting this into Equation (6), we obtain the usual formula for the drift adjustment

$$
dS_t^{(1)} = (r_{USD} - r_{XAU} - \rho\sigma_1\sigma_2)S_t^{(1)}\,dt + \sigma_1 S_t^{(1)}\,dW_t^{(1)}. \tag{10}
$$

This is the Risk Neutral Pricing process that can be used for the valuation of any derivative depending on $S_t^{(1)}$ which is quantoed into EUR.

1.1.1 Extensions to other Models

The previous derivation can be extended to the case of term-structure of volatility and correlation. However, introduction of volatility smile would distort the relationships. Nevertheless, accounting for smile effects is important in real market scenarios. See Foreign exchange smile: conventions and empirical facts and Foreign exchange smile modeling for details. To do this, one could, for example, capture the smile for a multi-currency model with a *weighted Monte Carlo technique* as described by Avellaneda et al. in [1]. This would still allow to use the previous result.

1.2 Quanto Vanilla

Common among Foreign exchange options is a quanto plain vanilla paying

$$Q[\phi(S_T - K)]^+, \tag{11}$$

where K denotes the strike, T the expiration time, ϕ the usual put-call indicator taking the value $+1$ for a call and -1 for a put, S the underlying in FOR-DOM quotation and Q the quanto factor from the domestic currency into the quanto currency. We let

$$\tilde{\mu} \stackrel{\Delta}{=} r_d - r_f - \rho\sigma\tilde{\sigma}, \tag{12}$$

be the *adjusted drift*, where r_d and r_f denote the risk free rates of the domestic and foreign underlying currency pair respectively, $\sigma = \sigma_1$ the volatility of this currency pair, $\tilde{\sigma} = \sigma_2$ the volatility of the currency pair DOM-QUANTO and

$$\rho = \frac{\sigma_3^2 - \sigma^2 - \tilde{\sigma}^2}{2\sigma\tilde{\sigma}} \tag{13}$$

the correlation between the currency pairs FOR-DOM and DOM-QUANTO in this quotation. Furthermore we let r_Q be the risk free rate of the quanto currency. With the same principles as in Pricing formulae for foreign exchange options we can derive the formula for the value as

$$v = Qe^{-r_Q T}\phi[S_0 e^{\tilde{\mu}T}\mathcal{N}(\phi d_+) - K\mathcal{N}(\phi d_-)], \tag{14}$$

$$d_\pm = \frac{\ln\frac{S_0}{K} + \left(\tilde{\mu} \pm \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}, \tag{15}$$

where $\mathcal{N}$ denotes the cumulative standard normal distribution function and n its density.

1.3 Quanto Forward

Similarly, we can easily determine the value of a quanto forward paying

$$Q[\phi(S_T - K)], \tag{16}$$

where K denotes the strike, T the expiration time, ϕ the usual long-short indicator, S the underlying in FOR-DOM quotation and Q the quanto factor from the domestic currency into the quanto currency. Then the formula for the value can be written as

$$v = Qe^{-r_Q T}\phi[S_0 e^{\tilde{\mu}T} - K]. \tag{17}$$

This follows from the vanilla quanto value formula by taking both the normal probabilities to be one. These normal probabilities are exercise probabilities under some measure. Since a forward contract is always exercised, both these probabilities must be equal to one.

1.4 Quanto Digital

A European style quanto digital pays

$$Q I\!I_{\{\phi S_T \geq \phi K\}}, \tag{18}$$

where K denotes the strike, S_T the spot of the currency pair FOR-DOM at maturity T, ϕ takes the values $+1$ for a digital call and -1 for a digital put, and Q is the pre-specified conversion rate from the domestic to the quanto currency. The valuation of European style quanto digitals follows the same principle as in the quanto vanilla option case. The value is

$$v = Qe^{-r_Q T}\mathcal{N}(\phi d_-). \tag{19}$$

We provide an example of European style digital put in USD/JPY quanto into EUR in Table 1.

Notional	100,000 EUR
Maturity	3 months (92days)
European style Barrier	108.65 USD-JPY
Theoretical value	71,555 EUR
Fixing source	ECB

Table 1: Example of a quanto digital put. The buyer receives 100,000 EUR if at maturity, the ECB fixing for USD-JPY (computed via EUR-JPY and EUR-USD) is below 108.65. Terms were created on Jan 12 2004 with the following market data: USD-JPY spot ref 106.60, USD-JPY ATM vol 8.55%, EUR-JPY ATM vol 6.69%, EUR-USD ATM vol 10.99% (corresponding to a correlation of -27.89% for USD-JPY against JPY-EUR), USD rate 2.5%, JPY rate 0.1%, EUR rate 4%.

1.5 Hedging of Quanto Options

Hedging of quanto options can be done by running a multi-currency options book. All the usual Greeks can be hedged. Delta hedging is done by trading in the underlying spot market. An exception is the correlation risk, which can only be hedged with other derivatives depending on the same correlation. This is normally not possible. In FX the correlation risk can be translated into a vega position as shown in [4] or in the section on Foreign exchange basket options. We illustrate this approach for quanto plain vanilla options now.

1.5.1 Vega Positions of Quanto Plain Vanilla Options

Starting from Equation (14), we obtain the sensitivities

$$
\begin{aligned}
\frac{\partial v}{\partial \sigma} &= QS_0 e^{(\tilde{\mu}-r_Q)T}\left[n(d_+)\sqrt{T} - \phi\mathcal{N}(\phi d_+)\rho\tilde{\sigma}T\right], \\
\frac{\partial v}{\partial \tilde{\sigma}} &= -QS_0 e^{(\tilde{\mu}-r_Q)T}\phi\mathcal{N}(\phi d_+)\rho\sigma T, \\
\frac{\partial v}{\partial \rho} &= -QS_0 e^{(\tilde{\mu}-r_Q)T}\phi\mathcal{N}(\phi d_+)\sigma\tilde{\sigma}T, \\
\frac{\partial v}{\partial \sigma_3} &= \frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial \sigma_3} \\
&= \frac{\partial v}{\partial \rho}\frac{\sigma_3}{\sigma\tilde{\sigma}} \\
&= -QS_0 e^{(\tilde{\mu}-r_Q)T}\phi\mathcal{N}(\phi d_+)\sigma\tilde{\sigma}T\frac{\sigma_3}{\sigma\tilde{\sigma}} \\
&= -QS_0 e^{(\tilde{\mu}-r_Q)T}\phi\mathcal{N}(\phi d_+)\sigma_3 T \\
&= -QS_0 e^{(\tilde{\mu}-r_Q)T}\phi\mathcal{N}(\phi d_+)\sqrt{\sigma^2+\tilde{\sigma}^2+2\rho\sigma\tilde{\sigma}}T.
\end{aligned}
$$

Note that the computation is standard calculus and repeatedly using the identity

$$S_0 e^{\tilde{\mu}T} n(\phi d_+) = K n(\phi d_-). \tag{20}$$

The understanding of these Greeks is that σ and $\tilde{\sigma}$ are both risky parameters, independent of each other. The third independent risk is either σ_3 or ρ, depending on what is more likely to be known.

This shows exactly how the three vega positions can be hedged with plain vanilla options in all three legs, provided there is a liquid vanilla options market in all three legs. In the example with XAU-USD-EUR the currency pairs XAU-USD and EUR-USD are traded, however, there is no liquid vanilla market in XAU-EUR. Therefore, the correlation risk remains unhedgeable. Similar statements would apply for quantoed stocks or stock indices. However, in FX, there are situations with all legs being hedgeable, for instance EUR-USD-JPY.

The signs of the vega positions are not uniquely determined in all legs. The FOR-DOM vega is smaller than the corresponding vanilla vega in case of a call and positive correlation or put and negative correlation, larger in case of a put and positive correlation or call and negative correlation. The DOM-Q vega takes the sign of the correlation in case of a call and its opposite sign in case of a put. The FOR-Q vega takes the opposite sign of the put-call indicator ϕ.

We provide an example of pricing and vega hedging scenario in Table 2, where we notice, that dominating vega risk comes from the FOR-DOM pair, whence most of the risk can be hedged.

		data set 1	data set 2	data set 3
FX pair	FOR-DOM	XAU-USD	XAU-USD	XAU-USD
spot	FOR-DOM	800.00	800.00	800.00
strike	FOR-DOM	810.00	810.00	810.00
quanto	DOM-Q	1.0000	1.0000	1.0000
volatility	FOR-DOM	10.00%	10.00%	10.00%
quanto volatility	DOM-Q	12.00%	12.00%	12.00%
correlation	FOR-DOM – DOM-Q	25.00%	25.00%	-75.00%
domestic interest rate	DOM	2.0000%	2.0000%	2.0000%
foreign interest rate	FOR	0.5000%	0.5000%	0.5000%
quanto currency rate	Q	4.0000%	4.0000%	4.0000%
time in years	T	1	1	1
1=call -1=put	FOR	1	-1	1
quanto vanilla option	value	30.81329	31.28625	35.90062
quanto vanilla option	vega FOR-DOM	298.14188	321.49308	350.14600
quanto vanilla option	vega DOM-Q	-10.07056	9.38877	33.38797
quanto vanilla option	vega FOR-Q	-70.23447	65.47953	-35.61383
quanto vanilla option	correlation risk	-4.83387	4.50661	-5.34207
quanto vanilla option	vol FOR-Q	17.4356%	17.4356%	8.0000%
vanilla option	value	32.6657	30.7635	32.6657
vanilla option	vega	316.6994	316.6994	316.6994

Table 2: Example of a quanto plain vanilla.

1.6 Applications

The standard application are performance linked deposit or performance notes as in [5]. Any time the performance of an underlying asset needs to be converted into the notional currency invested, and the exchange rate risk is with the seller, we need a quanto product. Naturally, an underlying like gold, which is quoted in USD, would be a default candidate for a quanto product, when the investment is in a currency other than USD.

1.6.1 Performance Linked Deposits

A performance linked deposit is a deposit with a participation in an underlying market. The standard is that a GBP investor waives her coupon that the money market would pay and instead buys a EUR-GBP call with the same maturity date as the coupon, strike K and notional N in EUR. These parameters have to be chosen in such a way that the offer price of the EUR call equals the money market interest rate plus the sales margin. The strike is often chosen to be the current spot. The notional is often a percentage p of the deposit amount A, such as 50% or 25%. The annual coupon paid to the investor is then a pre-defined minimum coupon plus the participation

$$p \cdot \frac{\max[S_T - S_0, 0]}{S_0}, \tag{21}$$

which is the return of the exchange rate viewed as an asset, where the investor is protected against negative returns. So, obviously, the investor buys a EUR call GBP put with strike $K = S_0$ and notional $N = pA$ GBP or $N = pA/S_0$ EUR. Thus, if the EUR goes up by 10% against the GBP, the investor gets a coupon of $p \cdot 10\%$ p.a. in addition to the minimum coupon.

Example. We consider the example shown in Table 3. In this case, if the EUR-GBP spot fixing is 0.7200, the additional coupon would be 0.8571% p.a. The break-even point is at 0.7467, so this product is advisable for a very strong EUR bullish view. For a weakly bullish view an alternative would be to buy an up-and-out call with barrier at 0.7400 and 75% participation, where we would find the best case to be 0.7399 with an additional coupon of 4.275% p.a., which would lead to a total coupon of 6.275% p.a.

Composition

- From the money market we get 49,863.01 GBP at the maturity date.
- The investor buys a EUR call GBP put with strike 0.7000 and with notional 1.5 Million GBP.
- The offer price of the call is 26,220.73 GBP, assuming a volatility of 8.0% and a EUR rate of 2.50%.
- The deferred premium is 24,677.11 GBP.
- The investor receives a minimum payment of 24,931.51 GBP.

Notional	5,000,000 GBP
Start date	3 June 2005
Maturity	2 September 2005 (91 days)
Number of days	91
Money market reference rate	4.00% act/365
EUR-GBP spot reference	0.7000
Minimum rate	2.00% act/365
Additional coupon	$30\% \cdot \frac{100 \max[S_T - 0.7000, 0]}{0.7000}$ act/365
S_T	EUR-GBP fixing on 31 August 2005 (88 days)
Fixing source	ECB

Table 3: Example of a performance linked deposit, where the investor is paid 30% of the EUR-GBP return. Note that in GBP the daycount convention in the money market is act/365 rather than act/360.

- Subtracting the deferred premium and the minimum payment from the money market leaves a sales margin of 254.40 GBP (awfully poor I admit).

- Note that the option the investor is buying must be cash-settled.

Variations. There are many variations of the performance linked notes. Of course, one can think of European style knock-out calls or window-barrier calls. For a participation in a downward trend, the investor can buy puts. One of the frequent issues in Foreign Exchange, however, is the deposit currency being different from the domestic currency of the exchange rate, which is quoted in FOR-DOM (foreign-domestic), meaning how many units of domestic currency are required to buy one unit of foreign currency. So if we have a EUR investor who wishes to participate in a EUR-USD movement, we have a problem, the usual *quanto confusion* that can drive anybody up the wall in FX at various occasions. What is the problem? The payoff of the EUR call USD put

$$[S_T - K]^+ \tag{22}$$

is in domestic currency (USD). Of course, this payoff can be converted into the foreign currency (EUR) at maturity, but at what rate? If we convert at rate S_T, which is what we could do in the spot market at no cost, then the investor buys a vanilla EUR call. But here, the investor receives a coupon given by

$$p \cdot \frac{\max[S_T - S_0, 0]}{S_T}. \tag{23}$$

If the investor wishes to have performance of Equation (21) rather than Equation (23), then the payoff at maturity is converted at a rate of 1.0000 into EUR, and this rate is set at the beginning of the trade. This is the *quanto factor*, and the vanilla is actually a *self-quanto* vanilla, i.e., a EUR call USD put, cash-settled in EUR, where the payoff in USD is converted into EUR at a rate of 1.0000. This self quanto vanilla can be valued by inverting the exchange rate, i.e., looking at USD-EUR. This way the valuation can incorporate the smile of EUR-USD. Similar considerations need to be taken into account if the currency pair to participate in does not contain the deposit currency at all. A typical situation is a EUR investor, who wishes to participate in the gold price, which is measured in USD, so the investor needs to buy a XAU call USD put quantoed into EUR. So the investor is promised a coupon as in Equation (21) for a XAU-USD underlying, where the coupon is paid in EUR, this implicitly means that we must use a quanto plain vanilla with a quanto factor of 1.0000.

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