Fair Pricing in the presence of Local Stochastic Volatility

Francfort 16th-17th April

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The Fair price is the cost of Hedge of an instrument.

We motivate our study on the most actively traded instrument in the equity derivatives space i.e. Autocall Structure.

We present a classical model and highlight the right level of fine tuning parameter in order to price and hedge it in a consistent way.
Many thanks for the members of my team (Luc Mathieu, Camille Brossette, Marouen Messaoud, Sebastien Mollaret, Claude Muller, who supported this work).
The risk dynamic for the Autocall: vega spot ladder
Literature


[7] Lorenzo Bergomi : Stochastic Volatility, Wiley 2015,

And many more recent literature.
Litterature

• Most papers discuss the calibration of the local volatility with the presentation of several numerical techniques: Forward PDE, Fixed Point Algorithm, Particular Method

• In [7] the properties of the Skew Stickiness Ratio (SSR) are presented as a key property of the dynamic of the volatility

• In [6] a mix between implied market and statistics is done to establish a fair pricing approach
The objectives of this presentation are to:

1. Link a dynamic property of the market SSR with its equivalent mixing weight to pick the right model out of a group of Models
2. Derive a formula for the dynamic of the volatility in a LSV model
3. Propose a historical estimation of the parameter
4. Share Some other useful techniques in practice
Part I

Model Description
Model Exploration

We need to depart from the Black & Scholes model when there is a changing gamma sign. We have two alternatives:

1. Local volatility model
   a) Fits the volatility smile: vanilla options
   b) Permits to perform a vega transparisation on all strikes
   c) Takes into account the cost of static hedge with vanilla

2. Stochastic Volatility
   a) Fits a parametric smile
   b) Permits to perform a factor transparisation
   c) Takes into account the cost of dynamic vega Hedge
The general dynamic:

\[ \frac{dS_t}{S_t} = \sigma(t, S_t) dW_t \]

The pricing equation obtained through the PnL equation:

\[ \pi_{lv} \approx \pi_{bs} + \frac{1}{2} \mathbb{E}_Q \int_0^T S^2 \frac{\partial^2 \pi_{bs}}{\partial S^2} (\sigma^2(t, S_t) - \sigma_0^2) \, dt \]

The result from a static hedging argument:

\[ \pi_{lv} \approx \pi_{bs} + \int_0^T \int_0^\infty \frac{\partial \pi_{bs}}{\partial \Sigma(K, T)} (\Sigma(K, T) - \sigma_0) \, dK \, dT \]

The induced dynamic of the local volatility: **invariant** strike x spot

\[ \Sigma_{S_1}(K, T) \approx \Sigma_{S_0}\left(\frac{KS_1}{S_0}, T\right) \]
Market exploration

FX: low volatility, little skew, mostly curvature

Graph showing the comparison between Market, Local, and Model data.
Market exploration

Commodity: high volatility, positive skew (refuge), moderate curvature
Market exploration

Equity: moderate volatility, negative skew, moderate curvature
Stochastic Volatility: key concepts

The general dynamic:
\[ \frac{dS_t}{S_t} = \sigma_t dW_t, \quad \frac{d\sigma_t}{\sigma_t} = \alpha d\tilde{W}_t, \quad <dW_t, d\tilde{W}_t> = \rho dt \]

The pricing equation obtained through the PnL equation:
\[ \pi_{sv} = \pi_{bs} + \frac{1}{2} \mathbb{E}_Q \int_0^T \alpha^2 \sigma^2 \frac{\partial^2 \pi_{bs}}{\partial \sigma^2} dt + \mathbb{E}_Q \int_0^T \rho \alpha \sigma^2 S \frac{\partial^2 \pi_{bs}}{\partial \sigma \partial S} dt \]

The result from a static hedging argument:
\[ \pi_{sv} = \pi_{bs} + \frac{1}{6} T \alpha^2 \frac{\partial^2 \pi_{bs}}{\partial \ln \sigma^2} + \frac{1}{2} T \rho \alpha \sigma \frac{\partial^2 \pi_{bs}}{\partial \ln S \partial \ln \sigma} \]

The induced dynamic of the local volatility: **invariant** strike / spot
\[ \Sigma_{S_1} (K, T) = \Sigma_{S_0} \left( \frac{KS_0}{S_1}, T \right) \]
Stochastic Volatility : key concepts (II)

Equivalent Local Volatility :

\[ \mathbb{E}_Q(\sigma_t^2 | S_t = S) = \frac{\mathbb{E}_Q\left(\frac{K^2}{\sqrt{\int_0^T \sigma_s^2 ds}} e^{\frac{2(1-\rho^2) \int_0^T \sigma_s^2 ds}{\sqrt{\int_0^T \sigma_s^2 ds}}} \right)}{\mathbb{E}_Q\left(\frac{e^{\frac{2(1-\rho^2) \int_0^T \sigma_s^2 ds}{\sqrt{\int_0^T \sigma_s^2 ds}}} \right)} \]

With \( K = \ln \left( \frac{S}{S_0} \right) + \frac{1}{2} \int_0^T \sigma_s^2 ds - \rho \int_0^T \sigma_s d\bar{W}_s ds \)

Short Term Asymptotics \( T \to 0 \):

\[ \alpha^2 \ln^2 \left( \frac{S_t}{S_0} \right) + 2\rho \sigma_0 \alpha \ln \left( \frac{S_t}{S_0} \right) + \sigma_0^2 \]
Hot Start Stochastic Volatility: key concepts

The general dynamic:

\[
\frac{dS_t}{S_t} = \sigma_t dW_t, \quad \frac{d\sigma_t}{\sigma_t} = \alpha d\tilde{W}_t, \quad <dW_t, d\tilde{W}_t> = \rho dt,
\]

\[
\sigma_0^+ = \sigma_0^- e^{\beta Z - \frac{1}{2} \beta^2}, \quad Z = N(0,1)
\]

This can be seen as the limit of a fast mean reverting model (proof in [5])

A recent paper in Risk magazine applies it for Heston Model
Market exploration: implied Volatility

![Implied Volatility chart]

FOR INFORMATION ONLY

16-17 April, 2018
Market exploration: local Volatility
Hot Start Stochastic Volatility: key concepts

The general dynamic: it is a special 2 Factor Bergomi Model with:

1\textsuperscript{st} mean reversion to infinity $\Rightarrow$ super fast short term

2\textsuperscript{nd} mean reversion to zero $\Rightarrow$ super slow term

**Strategy to fit the parameters from Bergomi approach!**

**Fits the market quite nicely with parsimony.**

**This model is simpler and numerics are more robust.**

**It is easily extendible with a local volatility component.**
Neither the local volatility nor the Stochastic volatility have the right volatility dynamic (we shall quantify this with the SSR).

We shall identify in a group of LSV model the one that is the closest to the observed market dynamic.

The Choice of the pure stochastic volatility model needs to be close enough to the market volatility smile in order to make the adjustment due to local volatility as small as possible.
Part II

Market Description
Market Exploration : key concept

In the beginning we had two models that match the market. However, they differ on the smile dynamic. Stochastic Volatility keeps the moneyness constant whereas the Local Volatility keeps the product strike by maturity constant. Where does reality stand?

We measure reality thanks to the SSR Skew Stickiness Ratio defined as follows:

\[
R_T = \frac{\mathbb{E}(d\Sigma_t(F,T)d\ln S_t)}{\frac{d\Sigma_t(K,T)}{d\ln K} \bigg|_{K=F} \mathbb{E}((d\ln S_t)^2)}
\]

Where \( \Sigma_t(K,T) \) is the implied volatility at time \( t \) for residual maturity \( T \) and strike \( K \)
Market Exploration : key concept

This ratio quantifies the variation of the ATMF volatility when the spot moves, in units of ATMF skew.

The SSR allows classifying the stochastic volatility models:

\[ R_T = 0 \] corresponds to the sticky-delta regime,
\[ R_T = 1 \] corresponds to the sticky-strike regime,
\[ R_T = 2 \] corresponds to the sticky-Dupire regime.
SSR Model Results

Stochastic Volatility: \( R_T = 0 \)
Local Volatility: \( R_T = 2 \)
Rough Volatility: \( R_T = \frac{3}{2} + H \)
Shifted Log Normal: \( R_T = 0 \)
Fast Stochastic Volatility: \( R_T = 0 \)

How does it compare with reality?
Market Exploration : SSR estimation

<table>
<thead>
<tr>
<th>Underlying</th>
<th>SSR</th>
</tr>
</thead>
<tbody>
<tr>
<td>SX5E</td>
<td>130,38%</td>
</tr>
<tr>
<td>KOSPI</td>
<td>107,52%</td>
</tr>
<tr>
<td>SPX</td>
<td>87,50%</td>
</tr>
<tr>
<td>EURUSD</td>
<td>115,50%</td>
</tr>
<tr>
<td>GOLD</td>
<td>72,00%</td>
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</tbody>
</table>
Do we really need to use complex models?

We could stop here!!

Why are we going to use another level of complexity?

We take the risk of slower implementation, unstable greeks?

We take the risk to increase our regulatory capital!

→ We propose a test to decide on increasing complexity or not?
We introduce a measure that gives us how sensitive is a Product to the difference between two models Local Volatility and Stochastic Volatility which price the same market vanilla but differ in the smile dynamic:

$$\phi = \frac{|\pi_{sv} - \pi_{lv}|}{\pi_{sv} + \pi_{lv}}$$

1. When this index is zero, it tells us that the payoff is simply replicable by vanilla European options. In other words, this product depends only on marginal distributions.

2. When the index approaches one, we have a very toxic product. Indeed, not only does it tell us that the cheapest model sees no value in the product, but also that the other models (the most expensive model) sees a lot of value. Typically, the cheapest model is the local volatility model and the most expensive the stochastic volatility.
Part III

Model Extension
In the case of non zero toxicity index, we need to build a model that matches vanilla options and the dynamic of the volatility surface?

We introduce a group of Local Stochastic Volatility Models with a $\epsilon$ mixing weight
Stochastic Local Volatility: key concepts

The general dynamic LSV (P1):

\[
\frac{dS_t}{S_t} = \sigma_t f_\epsilon(t, S_t) dW_t
\]

\[
\frac{d\sigma_t}{\sigma_t} = \cdots dt + \epsilon \alpha e^{-\lambda(T-t)} d\overline{W}_t, \quad <dW_t, d\overline{W}_t> = \rho dt
\]

The pricing PDE equation obtained through the PnL equation:

\[
\pi_{lsv} = \pi_{lv} + \epsilon (\pi_{sv} - \pi_{lv})
\]

The induced dynamic of the local volatility: geometric average

\[
\Sigma_{S_1}(LSV)(K, T) = \Sigma_{S_0} \left( \frac{KS_0}{S_1}, T \right) \epsilon \Sigma_{S_0} \left( \frac{KS_1}{S_0}, T \right)^{1-\epsilon}
\]
Stochastic Local Volatility: key concepts

The general dynamic LSV (P2):
\[
\frac{dS_t}{S_t} = \sigma_t f_\epsilon(t, S_t) dW_t \\
\frac{d\sigma_t}{\sigma_t} = \cdots dt + \epsilon \alpha e^{-\lambda(T-t)} d\bar{W}_t, <dW_t, d\bar{W}_t> = \epsilon \rho dt
\]

The pricing equation obtained through the PnL equation:
\[
\pi_{LSV} = \pi_{lv} + \epsilon^2 (\pi_{sv} - \pi_{lv})
\]

The induced dynamic of the local volatility: geometric average
\[
\Sigma_{S_1} (LSV)(K,T) = \Sigma_{S_0} \left( \frac{KS_0}{S_1}, T \right) ^{\epsilon^2} \Sigma_{S_0} \left( \frac{KS_1}{S_0}, T \right) ^{1-\epsilon^2}
\]
Sketch of a Proof cf[5]

Start with a PDE description

\[
\partial_t u + \frac{1}{2} \sigma^2 f_{\varepsilon} S^2 \partial_{SS} u + \frac{1}{2} \varepsilon^2 \alpha^2 \partial_{\sigma\sigma} u + \varepsilon^2 \rho \alpha \sigma^2 f_{\varepsilon} \partial_{\sigma} u = 0
\]

P2 assumption makes a balance between volga and vanna

Perturbation theory: we search \( u = u_0 + \varepsilon^2 u_1 + \cdots \) and \( f_{\varepsilon} = f_0 + \varepsilon^2 f_1 + \cdots \)

\[
\partial_t u_0 + \frac{1}{2} \sigma_0^2 f_0^2 S^2 \partial_{SS} u_0 = 0 \quad \text{\( \Rightarrow \)} \quad \text{classical local volatility model}
\]

Feynmann-Kacc equation

\[
u_1 = \mathbb{E} \int_0^T \frac{1}{2} \sigma^2 f_1^2 S^2 \partial_{SS} u_0 + \frac{1}{2} \alpha^2 \partial_{\sigma\sigma} u_0 + \rho \alpha \sigma^2 f_0 \partial_{\sigma} u_0
\]

Matching the extreme point with a mixing weight of 100%

Deduce that for any option: \( u = u_{LV} + \varepsilon^2 (u_{SV} - u_{LV}) \)

Apply previous to Vanilla

\[
\Delta_{LSV} = \Delta_{LV} + \varepsilon^2 (\Delta_{SV} - \Delta_{LV})
\]

Use general formula for delta of Vanilla \( \Delta(K,T) = \Delta_{BS}(K,T) + Vega(K,T) \partial_S \Sigma_{S_0}(S_0, K,T) \)

Deduce \( \partial_S \Sigma_{LSV,S_0}(K,T) = \partial_S \Sigma_{LV,S_0}(K,T) + \varepsilon^2 \left( \partial_S \Sigma_{SV,S_0}(K,T) - \partial_S \Sigma_{LV,S_0}(K,T) \right) \)

Divide by common term \( \Sigma_{Market}(S_0, K,T) \)

Integrate and obtain the dynamic formula

\[
\Sigma_{S_1}(LSV)(K,T) = \Sigma_{S_0} \left( \frac{KS_0}{S_1}, T \right) \varepsilon^2 \Sigma_{S_0} \left( \frac{KS_1}{S_0}, T \right)^{1-\varepsilon^2}
\]
Stochastic Local Volatility: Numerical Illustration

We compare the ATM volatility evolution for the 3M tenor based on the following assumptions:

1. Exact recomputation under P2 of pricing and then implying the volatility,
2. Proxy under assumption P2,
3. Verification of Proxy under assumption P1,

![Graph showing volatility comparison](image)
Stochastic Local Volatility: Numerical Illustration

We compare the ATM volatility evolution for the 2Y tenor based on the following assumptions:

1. Exact recomputation under P2 of pricing and then implying the volatility,
2. Proxy under assumption P2,
3. Verification of Proxy under assumption P1,
Under (P2) : Vanna and Volga are well balanced:

The link between the smile dynamic expressed as the $R_T$ and a particular LSV model with a parameter $\epsilon$ is given by the following formula:

$$\epsilon^2 = 1 - \frac{1}{2} R_T$$
Mixing Weight

\( \epsilon = 1 \) corresponds to the **full stochastic volatility**, 
\( \epsilon = \frac{1}{\sqrt{2}} \) corresponds to the in between model, 
\( \epsilon = 0 \) corresponds to the **full local volatility model**.
Mixing Weight time series

SX5E MW

SPX MW
## Market Exploration : Mixing Weight estimation

<table>
<thead>
<tr>
<th>Underlying</th>
<th>MW</th>
</tr>
</thead>
<tbody>
<tr>
<td>SX5E</td>
<td>59.00%</td>
</tr>
<tr>
<td>KOSPI</td>
<td>68.00%</td>
</tr>
<tr>
<td>SPX</td>
<td>75.00%</td>
</tr>
<tr>
<td>EURUSD</td>
<td>65.00%</td>
</tr>
<tr>
<td>GOLD</td>
<td>80.00%</td>
</tr>
</tbody>
</table>
1. Rather than being a universal constant the mixing weight is something like a useful statistical invariant that describes the right amount of smile dynamic with the volatility surface.

2. It is an essential element of pricing and hedging.

3. The graph represents the evolution of the term structure of the mixing weight over time.
Part IV

Multi Asset SV
Requirements

Building a multi-dimensional process from the particular mono building blocks needs to be performed with care.

Must at least satisfy the following conditions:

**Condition 1**: be consistent with the mono dimensional world

**Condition 2**: be consistent with market prices of ATM basket vanillas

**Condition 3**: be consistent with mathematical constraints concerning semi definite positiveness of the global correlation matrix
In our setting, we consider m stock prices with their respective volatility.

\[
\frac{dS_i}{S_i} = \ldots + \sigma_i \, dW_i^S, \quad i = 1 \ldots m
\]

For each stock price, volatility is assumed to follow a stochastic process as defined in the following:

\[
\frac{d\sigma_i}{\sigma_i} = \ldots + \alpha_i \, dW_i^{*\sigma}, \quad i = 1 \ldots m
\]

Driving Brownian Motions of both processes are assumed to be correlated, which allows us to write:

\[
dW_i^{*\sigma} = \rho_i^{\sigma S} \, dW_i^S + \sqrt{1 - (\rho_i^{\sigma S})^2} \, dW_i^\sigma
\]

Where \(W_i^\sigma\) and \(W_i^S\) are independent random variables. The parameter \(\rho_i^{\sigma S}\) is essentially driving the mono dimensional LSV calibration.
Fitting conditions 1 and 3

To fit the Condition 1 (be consistent with the mono dimensional case) we shall build a correlation matrix Spot – Vol \times Spot – Vol we shall build a global correlation matrix which does not move the parameter \( \rho_i^{\sigma S} \). Also to fit the condition 3, we propose a parametric correlation matrix that is the result of process.

Our idea to fit both conditions 1 and 3 is to decompose all factors \( W_i^{\sigma} \), for \( i = 1 \ldots m \) according to:

\[
dW_i^{\sigma} = \sqrt{\rho^{\sigma}} dZ + \sqrt{1 - \rho^{\sigma}} dB_i^{\sigma}, \quad i = 1 \ldots m
\]

We have introduced only one new parameter \( \rho^{\sigma} \), with \( B_i^{\sigma} \) and \( Z \) independent.

\[
S \begin{pmatrix}
\rho_{ji}^{SS} \\
\rho_{ij}^{SS} \\
\rho_{ij}^{\sigma S} \\
\rho_{ij}^{\sigma S} + \sqrt{1 - (\rho_i^{\sigma S})^2} \sqrt{1 - (\rho_j^{\sigma S})^2} (\rho^{\sigma} + (1 - \rho^{\sigma})1\{i = j\})
\end{pmatrix}
\]
How are we doing with Condition 2

Multi SV decorrelates and makes the basket options cheaper than their corresponding LV price.

Comparison of basket prices for a model satisfying just conditions 1 and 3.
Fitting Condition 2 by recorrelating the spots

We use the lambda mechanism $\rho^{S,S}_{i,j} \rightarrow (1 - \epsilon^S) \rho^{S,S}_{i,j} + \epsilon^S$ to recorrelate the underlyings and here are the results:

Comparison of basket prices for a model satisfying conditions 1, 2 and 3.

what is remarkable is that not only do we match the ATM basket prices but also the other strikes 90% and 110%.

This result is very interesting as it say that matching the ATM basket prices with this particular matrix of correlation ends up by matching the basket distribution;

In some cases, this could be applied to worst of depending on the hedging instruments.
Part V

Conclusion
1. In case of non zero toxicity index:
\[
\phi = \frac{\mid \pi_{sv} - \pi_{lv} \mid}{\pi_{sv} + \pi_{lv}}
\]

2. Estimate Skew Stickiness Ratio:
\[
R_T = \frac{\mathbb{E}(d\Sigma_t(F,T)d\ln S_t)}{\int \frac{d\Sigma_t(K,T)}{d\ln K}_{K=F} \mathbb{E}((d\ln S_t)^2)}
\]

3. Estimate the mixing weight:
\[
\epsilon^2 = 1 - \frac{1}{2} R_T
\]

4. Pricing Proxy Formula:
\[
\pi_{lsv} = \pi_{lv} + \epsilon^2 (\pi_{sv} - \pi_{lv})
\]
Conclusion Multi

Build a multi SV model which is

1. consistent with Mono LSV

2. Fitting basket options or any other hedging instruments

3. Mathematically based on a semi definite postive matrix

1. Introduce idiosyncratic volatility factor $\rho^\sigma$

2. Build a global correlation Matrix

$$
S \left( \begin{array}{ccc}
\rho_{ji}^{SS} & \rho_j^{\sigma S} & \rho_j^{\sigma S} \\
\rho_j^{\sigma S} & \rho_i^{\sigma S} & \rho_i^{\sigma S} \\
\rho_j^{\sigma S} & \rho_i^{\sigma S} & \rho_i^{\sigma S} + \sqrt{1 - (\rho_i^{\sigma S})^2} \left( 1 - (\rho_j^{\sigma S})^2 (\rho^\sigma + (1 - \rho^\sigma)1_{i = j}) \right)
\end{array} \right)
$$

3. Recorrelate to match ATM and remarkably basket skew

$$
\rho_{i,j}^{S,S} \rightarrow (1 - \epsilon^S) \rho_{i,j}^{S,S} + \epsilon^S
$$
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