Corridor Variance Swap Spread

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Corridor Variance Swap Spread

Two assets:

$S_1$: typically SPX  
$S_2$: Asian or European index, NKY, HSI, SX5E

Payoff at maturity $T$:

$$\frac{1}{T} \int_0^T \left[ \sigma_2^2, t - \bar{\sigma}_2^2 \right] 1_{L \leq S_2, t \leq U} dt - \frac{1}{T} \int_0^T \left[ \sigma_1^2, t - \bar{\sigma}_1^2 \right] 1_{L \leq S_2, t \leq U} dt$$

for $0 \leq L < U \leq \infty$. The fixed volatilities $\bar{\sigma}_1$ and $\bar{\sigma}_2$ are typically chosen to make each leg of the swap spread have zero value at its inception.

The realized variance is accrued only when the level of the corridor index is within a predefined corridor. Vega exposures in specific range of market levels.

**Sell Side:** Natural hedge for the vega exposure of bank’s structured product business, especially products with barriers, such as autocallable notes

**Buy Side:** Construct relative value volatility trade in specific ranges of market levels
Autocallable Notes

- Popular structured product in Asian markets
- The investor receives a higher coupon
- Autocalled (the product terminates early) if the underlying hits a set target (typically 100% - 110% of the spot)
- Down-and-in ATM put option with barrier set at 60% - 70% of the spot
- The investor is effectively selling the embedded option to the issuer. The option premium provides the additional yield.
- The issuer has to short other options to hedge the long vega exposure, which puts downward pressure on implied volatilities in the listed options market.
- Digital risks of the barriers.
Autocallable Notes

Autocallable Note Payoff Structure

- Autocall level: 100, observed quarterly
- Coupon: 5% Quarterly, Coupon Barrier: 90
- Conversion Barrier: 75, observed daily
Variance Swap

Payoff at maturity $T$: \( \frac{1}{T} \int_{0}^{T} \sigma_t^2 dt - \bar{\sigma}^2 \)

The fixed volatility $\bar{\sigma}$ is chosen to make the swap have zero value at its inception.

In the absence of jumps, the payoff of a variance swap can be perfectly replicated by a static portfolio of vanilla options and a dynamic position in the underlying:

\[
\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t \quad \text{(diffusion process, no jumps)}
\]

\[
d \log S_t = \frac{1}{S_t} dS_t - \frac{1}{2} \sigma_t^2 dt
\]

\[
\int_{0}^{T} \sigma_t^2 dt = -2 \log \frac{S_T}{S_0} + \int_{0}^{T} \frac{2}{S_t} dS_t \quad \text{(replication formula)}
\]

\[
\mathbb{E}^Q \left[ \int_{0}^{T} \sigma_t^2 dt \right] = \mathbb{E}^Q \left[ -2 \log \frac{S_T}{S_0} \right] + 2 \int_{0}^{T} (r_t - q_t) dt = \mathbb{E}^Q \left[ -2 \log \frac{S_T}{F_{0,T}} \right]
\]

\[
= \int_{0}^{F_{0,T}} \frac{2}{K^2} \text{Put}(K, T) dK + \int_{F_{0,T}}^{\infty} \frac{2}{K^2} \text{Call}(K, T) dK = \int_{0}^{\infty} \text{OTMF}(K, T) dK
\]

where $\text{Call}(K, T) = \mathbb{E}^Q \left[ (S_T - K)^+ \right]$, $\text{Put}(K, T) = \mathbb{E}^Q \left[ (K - S_T)^+ \right]$ and $\text{OTMF}(K, T) = \text{Call}(K, T)$ if $K \geq F_{0,T}$ else $\text{Put}(K, T)$. 

Variance Swap Replication

\[ \tilde{\sigma}^2 = \frac{1}{T} \left[ \int_0^{F_0,T} \frac{2}{K^2} \text{Put}(K, T) dK + \int_{F_0,T}^{\infty} \frac{2}{K^2} \text{Call}(K, T) dK \right] \]

SPX Index, as of 03-20-2018, expiry 04-20-2018, \( \tilde{\sigma} = 18.9\% \)
Variance Swap and Smile Extrapolation

- Options are only traded in a finite range of strikes on the market
- Variance Swap replication is very sensitive to the prices/vols of deep out-of-the-money options, especially puts.
- Market quotes of variance swap rates can be used to select the extrapolation of implied volatilities outside of the range of traded strikes.

Example I - Mixed Lognormals

For a given maturity $T$ and $\mathbb{E}[X_T] = F$, the risk-neutral density function $\phi$ of $X_T$ is given by

$$
\phi(x) = \sum_{i=1}^{n} w_i \phi_i(x), \quad \text{with} \quad \phi_i = \text{pdf of Lognormal} \left( \log(\mu_i F) - \frac{1}{2} \sigma_i^2 T, \sigma_i^2 T \right)
$$

$$
\sum_{i=1}^{n} w_i = 1 \quad \text{and} \quad \sum_{i=1}^{n} w_i \mu_i = 1
$$

$$
\text{Call}(K) = \mathbb{E} \left[ (X_T - K)^+ \right] = \sum_{i=1}^{n} w_i \text{BlackCall}(K, T, \mu_i F, \sigma_i)
$$
Calibrate \((w_i, \mu_i, \sigma_i)_{1 \leq i \leq n}\) to vanilla option smiles.

SX5E Index, as of 03-23-2018, expiry 06-15-2018

Var swap rate from mixed lognormal fit \(20.75\%\)
Var swap rate mid market quote \(21.07\%\)
Variance Swap and Smile Extrapolation

Calibrate \((w_i, \mu_i, \sigma_i)_{1 \leq i \leq n}\) to both vanilla option smiles and var swap rate.

\[
VS^2 = \sum_{i=1}^{n} w_i \left( \sigma_i^2 - \frac{2}{T} \log \mu_i \right)
\]

SX5E Index, as of 03-23-2018, expiry 06-15-2018
Example II - Cubic Spline Regression

- The total implied variance function $x \mapsto \hat{\sigma}^2(x)T$ where $x = \log(K/F)$ is represented by a cubic spline, best fit to the vanilla smile, and linearly extrapolated for low and high strikes.
- Use var swap rate to find slopes for linear extrapolation at low and high strikes.

SX5E Index, as of 03-23-2018, expiry 06-15-2018

Var swap rate from cubic spline regression fit 20.85 %
Var swap rate mid market quote 21.73 %
Variance Spreads

Relative value volatility trades

- There exists a premium of implied volatility to realized volatility
- Buy variance on index with low premium (such as Asian or European indices) and sell variance on index with high premium (typically SPX)
- More stable P&L
Variance Spreads

Corridor Variance Swap Spread
Single-Asset Corridor Variance Swap

Payoff at maturity $T$:

$$\frac{1}{T} \int_0^T [\sigma_t^2 - \bar{\sigma}^2] 1_{L \leq S_t \leq U} dt, \quad 0 \leq L < U \leq \infty$$

The fixed volatility $\bar{\sigma}$ is chosen to make the swap have zero value at its inception.

- Range-limited vega exposure
- Excellent alternatives for investors who wish to long volatility and have a range-bound view on the index.

To replicate corridor variance swap, we define

$$g(S) = \begin{cases} 
\log(L) + \frac{1}{L}(S - L) & \text{if } S < L \\
\log(S) & \text{if } L \leq S \leq U \\
\log(U) + \frac{1}{U}(S - U) & \text{if } S > U 
\end{cases}$$

and

$$g'(S) = \begin{cases} 
\frac{1}{L} & \text{if } S < L \\
\frac{1}{S} & \text{if } L \leq S \leq U \\
\frac{1}{U} & \text{if } S > U 
\end{cases} \quad \text{and} \quad g''(S) = \begin{cases} 
0 & \text{if } S < L \\
-\frac{1}{S^2} & \text{if } L \leq S \leq U \\
0 & \text{if } S > U 
\end{cases}$$
The function \( g \) is equal to log function inside the corridor \([L, U]\) and linearly extrapolated beyond the corridor.

\[
dg(S_t) = g'(S_t) dS_t + \frac{1}{2} g''(S_t) S_t^2 \sigma_t^2 dt = g'(S_t) dS_t - \frac{1}{2} \sigma_t^2 1_{L \leq S_t \leq U} dt
\]
Single-Asset Corridor Variance Swap

\[ \int_0^T \sigma_t^2 1_{L \leq S_t \leq U} \, dt = \int_0^T 2g'(S_t) \, dS_t - 2 [g(S_T) - g(S_0)] \]

\[ E^Q \left[ \int_0^T 2g'(S_t) \, dS_t \right] = E^Q \left[ \int_0^T 2 \left( \frac{S_t}{L} 1_{S_t < L} + 1_{L \leq S_t \leq U} + \frac{S_t}{U} 1_{S_t > U} \right) (r_t - q_t) \, dt \right] \]

\[ = E^Q \left[ \int_0^T 2 \left( -\frac{(L - S_t)^+}{L} + 1 + \frac{(S_t - U)^+}{U} \right) (r_t - q_t) \, dt \right] \]

\[ = \int_0^T \left[ -\frac{2}{L} \text{Put}(L, t) + \frac{2}{U} \text{Call}(U, t) \right] (r_t - q_t) \, dt + 2 \int_0^T (r_t - q_t) \, dt \]

\[ E^Q \left[ \int_0^T \sigma_t^2 1_{L \leq S_t \leq U} \, dt \right] = \int_0^T \left[ -\frac{2}{L} \text{Put}(L, t) + \frac{2}{U} \text{Call}(U, t) \right] (r_t - q_t) \, dt \]
\[ + \int_L^U \frac{2}{K^2} \text{OTMF}(K, T) \, dK - 2 \left( g(F_0, T) - g(S_0) - \int_0^T (r_t - q_t) \, dt \right) \]

Note that the last term in the right-hand side of the equation vanishes if \( L \leq S_0 \leq U \).

The payoff of a corridor variance swap can be replicated by vanilla options with strikes within the corridor at the final maturity, a continuous stream of vanilla options of intermediate maturities, and a dynamic portfolio in the underlying \( S \).
Two-Asset Corridor Variance Swap

Two legs of a corridor variance swap spread:

Leg 1: \[ \frac{1}{T} \int_0^T \left[ \sigma_{2,t}^2 - \bar{\sigma}_2^2 \right] 1_{L \leq S_{2,t} \leq U} dt \] Single-Asset Corridor Var Swap

Leg 2: \[ \frac{1}{T} \int_0^T \left[ \sigma_{1,t}^2 - \bar{\sigma}_1^2 \right] 1_{L \leq S_{2,t} \leq U} dt \] Two-Asset Corridor Var Swap

The challenge of valuation/risk management mainly comes from the two-asset leg.

Notations

\[ VS_i^2(T) = \mathbb{E} \left[ \frac{1}{T} \int_0^T \sigma_{i,t}^2 dt \right] \] Vanilla var swap on \( i \)-th asset

\[ VS_{i,j}^2(L, U, T) = \mathbb{E} \left[ \frac{1}{T} \int_0^T \sigma_{i,t}^2 1_{L \leq S_{j,t} \leq U} dt \right] \] Two-asset corridor var swap

\[ VS_i^2(L, U, T) = VS_{i,i}^2(L, U, T) \] Single-asset corridor var swap

Consistency Conditions

1. As \( S_2 \to S_1 \), \( VS_{1,2}(L, U, T) \to VS_1(L, U, T) \)
2. As \( L \downarrow 0 \) and \( U \uparrow \infty \), \( VS_{1,2}(L, U, T) \uparrow VS_1(T) \)
3. As \( U \downarrow L \), \( VS_{1,2}(L, U, T) \downarrow 0 \)
Two-Asset Corridor Variance Swap - Cross Local Variance

**Question:** What is the risk exposure of a two-asset corridor var swap?

\[
\mathbb{E} \left[ \int_0^T \sigma^2_{1,t} 1_{L \leq S_{2,t} \leq U} dt \right] = \int_0^T \mathbb{E} \left[ \mathbb{E} \left[ \sigma^2_{1,t} | S_{2,t} \right] 1_{L \leq S_{2,t} \leq U} \right] dt
\]

The key is to evaluate the *cross local variance*:

\[
\mathbb{E} \left[ \sigma^2_{1,t} | S_{2,t} \right]
\]

Two simple cases:

1. **\( S_1 \) and \( S_2 \) are identical:**

\[
\mathbb{E} \left[ \sigma^2_{1,t} | S_{2,t} \right] = \sigma^2_{1,\text{loc}} (t, S_{1,t})
\]

2. **\( S_1 \) and \( S_2 \) are independent:**

\[
\mathbb{E} \left[ \sigma^2_{1,t} | S_{2,t} \right] = \mathbb{E} \left[ \sigma^2_{1,t} \right] = \mathbb{E} \left[ \sigma^2_{1,\text{loc}} (t, S_{1,t}) \right] \quad \text{(const)}
\]
Two-Asset Corridor Variance Swap - Cross Local Variance

As difference of two down-variance payoffs (lower bound=0):

\[ \mathbb{E} \left[ \int_0^T \sigma_{1,t}^2 1_{S_2,t \leq U} \, dt \right] = \mathbb{E} \left[ \int_0^T \sigma_{1,t}^2 1_{S_2,t \leq U} \, dt \right] - \mathbb{E} \left[ \int_0^T \sigma_{1,t}^2 1_{S_2,t \leq L} \, dt \right] \]

\[ =: V_{1,2} (U,T) - V_{1,2} (L,T) \]

where \( V_{1,2} (K,T) \) is the undiscounted value of the down-variance payoff:

\[ V_{1,2} (K,T) = \int_0^T \mathbb{E} \left[ \mathbb{E} [\sigma_{1,t}^2 | S_2,t] 1_{S_2,t \leq K} \right] \, dt \]

\[ \frac{\partial V_{1,2}}{\partial K} = \int_0^T \mathbb{E} \left[ \mathbb{E} [\sigma_{1,t}^2 | S_2,t] \delta (S_2,t - K) \right] \, dt = \int_0^T \mathbb{E} [\sigma_{1,t}^2 | S_2,t = K] \frac{\partial^2 C_2}{\partial K^2} (K,t) \, dt \]

\[ \mathbb{E} \left[ \sigma_{1,t}^2 | S_2,T = K \right] = \frac{\partial^2 V_{1,2}}{\partial T \partial K} \frac{\partial T \partial K}{\partial^2 C_2} \]  

where \( C_2 (K,T) = \mathbb{E} \left[ (S_{2,T} - K)^+ \right] \).

The cross local variance \( \mathbb{E} \left[ \sigma_{1,t}^2 | S_2,t = K \right] \) can be locked in by trading two-asset corridor variance swaps.

The similar analysis can be applied to single-asset corridor variance swaps where we can lock in the local variance \( \sigma^2_{\text{loc}} (t,S_t) = \mathbb{E} [\sigma_t^2 | S_t] \).
Two-Asset Corridor Variance Swap  LV + LV

Basket Local Volatility Model (LV+LV)

\[
\begin{align*}
\frac{dS_{1,t}}{S_{1,t}} &= (r_{1,t} - q_{1,t}) \, dt + \sigma_{1,\text{loc}}(t, S_{1,t}) \, dW_{1,t}^S \\
\frac{dS_{2,t}}{S_{2,t}} &= (r_{2,t} - q_{2,t}) \, dt + \sigma_{2,\text{loc}}(t, S_{2,t}) \, dW_{2,t}^S \\
dW_{1,t}^S dW_{2,t}^S &= \rho_{S_1 S_2} \, dt
\end{align*}
\]

\[
VS_{1,2}^2 (L, U; LV+LV) = \frac{1}{T} \int_0^T \mathbb{E} \left[ \mathbb{E} \left[ \sigma_{1,\text{loc}}^2(t, S_{1,t}) \middle| S_{2,t} \right] 1_{L \leq S_{2,t} \leq U} \right] \, dt
\]

Pros

- Simple multi-asset model, consistent with vanilla smile of each asset
- The correlation $\rho_t$ can be calibrated to the market prices of basket options or other products with exposure to $S_1$ v.s. $S_2$ correlation.

Cons: What about exotic risks such as volatility-of-volatility risk, (cross) spot/volatility correlation risk?
Two-Asset Corridor Variance Swap  SLV + LV

**One-Factor SLV Model + Local Vol Model (SLV+LV)**

\[
\frac{dS_{1,t}}{S_{1,t}} = (r_{1,t} - q_{1,t}) dt + \sigma_{1,loc}(t, S_{1,t}) \frac{\sqrt{V_{1,t}}}{\sqrt{\mathbb{E}[V_{1,t} | S_{1,t}]}} dW_{1,t}^{S} \\

\frac{dV_{1,t}}{V_{1,t}} = a_{1}(t, V_{1,t}) dt + b_{1}(t, V_{t}) dW_{1,t}^{V} \\

\frac{dS_{2,t}}{S_{2,t}} = (r_{2,t} - q_{2,t}) dt + \sigma_{2,loc}(t, S_{2,t}) dW_{2,t}^{S}
\]

where

\[dW_{1,t}^{S} dW_{2,t}^{S} = \rho_{t}^{S_{1}S_{2}} dt, \quad dW_{1,t}^{S} dW_{1,t}^{V} = \rho_{t}^{S_{1}V_{1}} dt, \quad dW_{2,t}^{S} dW_{1,t}^{V} = \rho_{t}^{S_{2}V_{1}} dt.\]

Three correlations need to be specified and they must satisfy the positive-definiteness constraint:

\[1 - \left( \rho_{t}^{S_{1}S_{2}} \right)^2 - \left( \rho_{t}^{S_{1}V_{1}} \right)^2 - \left( \rho_{t}^{S_{2}V_{1}} \right)^2 + 2\rho_{t}^{S_{1}V_{1}} \rho_{t}^{S_{1}S_{2}} \rho_{t}^{S_{2}V_{1}} > 0 \quad (D)\]

The SDE for \(S_{1,t}\) is a McKean SDE and can be efficiently simulated by particle method [GL].
We assume that the two models (SLV+LV and LV+LV) have the same local vol functions $\sigma_{1,\text{loc}}$ and $\sigma_{2,\text{loc}}$, and the same spot/spot correlation $\rho_{S_1 S_2}^t$.

We shall compare the following three functions:

1. $\mathbb{E}\left[\sigma^2_{1,\text{loc}}(t, S_1,t) \frac{V_{1,t}}{\mathbb{E}[V_{1,t}|S_1,t]} | S_2,t\right]$ in SLV + LV

2. $\mathbb{E}\left[\sigma^2_{1,\text{loc}}(t, S_1,t) | S_2,t\right]$ in SLV + LV

3. $\mathbb{E}\left[\sigma^2_{1,\text{loc}}(t, S_1,t) | S_2,t\right]$ in LV + LV

Note that (2) and (3) are not the same in general even if $S_1,t$ and $S_2,t$ have the same marginals and the same correlation $\rho_{S_1 S_2}^t$ drives $W^S_{1,t}$ and $W^S_{2,t}$, the joint distribution of $S_1,t$ and $S_2,t$ are not the same in SLV+LV and LV+LV due to stochastic volatility $V_{1,t}$ and cross spot-vol correlation $\rho^{S_2 V_1}$ in SLV+LV.
Two-Asset Corridor Variance Swap  SLV+LV vs LV+LV

Example 1.

(SLV+LV) \[
\begin{align*}
\frac{dS_{1,t}}{S_{1,t}} &= \sqrt{V_{1,t}} dW_{1,t}^S \\
\frac{dV_{1,t}}{V_{1,t}} &= \kappa_1 (\theta_1 - V_{1,t}) dt + \omega_1 \sqrt{V_{1,t}} dW_{1,t}^V \\
\frac{dS_{2,t}}{S_{2,t}} &= \sigma_{2,\text{loc}} (t, S_{2,t}) dW_{2,t}^S
\end{align*}
\]

with \( dW_{1,t}^S dW_{1,t}^V = \rho_{S1V1} dt \)

LV+LV \[
\begin{align*}
\frac{dS_{1,t}}{S_{1,t}} &= \sigma_{1,\text{loc}} (t, S_{1,t}) dW_{1,t}^S \\
\frac{dS_{2,t}}{S_{2,t}} &= \sigma_{2,\text{loc}} (t, S_{2,t}) dW_{2,t}^S
\end{align*}
\]

with \( dW_{1,t}^S dW_{2,t}^S = \rho_{S1S2} dt \)

Numerical values of model parameters:

\[
\begin{align*}
S_{1,0} &= 100, \ S_{2,0} = 100, \ \rho_{S1S2} = 0.3 \\
V_{1,0} &= 0.02, \ \kappa_1 = 3, \ \theta_1 = 0.05, \ \omega_1 = 0.5, \ \rho_{S1V1} = -0.75 \\
\sigma_{1,\text{loc}} \text{ is the local vol surface in Heston model (} V_0 = V_{1,0}, \ \kappa = \kappa_1, \ \theta = \theta_1, \ \omega = \omega_1, \ \rho = \rho_{S1V1} \text{)} \\
\sigma_{2,\text{loc}} &\equiv 0.2 \\
The \text{ positive definiteness constraints (D) gives } -0.856 \leq \rho_{S2V1} \leq 0.406 \\
\rho_*^2 V_1 &= \rho_{S1S2} \rho_{S1V1} = -0.225.
\end{align*}
\]
Two-Asset Corridor Variance Swap  \( SLV+LV \) vs \( LV+LV \)

\[ t = 0.5 \]

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Two-Asset Corridor Variance Swap \( SLV+LV \) vs \( LV+LV \)

\[
\sqrt{\frac{1}{T} \int_0^T \sigma_1^2(t) 1_{L(t) \leq S_2(t) \leq U(t)} dt} \text{ in SLV+LV}
\]

\[
\sqrt{\frac{1}{T} \int_0^T \sigma_{1, \text{loc}}^2(t, S_1(t)) 1_{L(t) \leq S_2(t) \leq U(t)} dt} \text{ in SLV+LV}
\]

\[
\sqrt{\frac{1}{T} \int_0^T \sigma^2(t, S_1(t)) 1_{L(t) \leq S_2(t) \leq U(t)} dt} \text{ in LV+LV}
\]
Question: In SLV + LV, when do we have
\[ \mathbb{E} \left[ \sigma^2_{1,t} \mid S_{2,t} \right] = \mathbb{E} \left[ \sigma^2_{1,\text{loc}}(t, S_{1,t}) \mid S_{2,t} \right] ? \]

One sufficient condition that makes the above equality hold is
\[ \mathbb{E} \left[ \sigma^2_{1,t} \mid S_{1,t}, S_{2,t} \right] = \mathbb{E} \left[ \sigma^2_{1,t} \mid S_{1,t} \right] \tag{A} \]

Assume (A) holds. Then
\[
\mathbb{E} \left[ \sigma^2_{1,t} \mid S_{2,t} \right] = \mathbb{E} \left[ \mathbb{E} \left[ \sigma^2_{1,t} \mid S_{1,t}, S_{2,t} \right] \mid S_{2,t} \right] = \mathbb{E} \left[ \mathbb{E} \left[ \sigma^2_{1,t} \mid S_{1,t} \right] \mid S_{2,t} \right] \\
= \mathbb{E} \left[ \sigma^2_{1,\text{loc}}(t, S_{1,t}) \mid S_{2,t} \right]
\]

The assumption (A) says that given \( S_{1,t} \), the other variable \( S_{2,t} \) provides no additional information in estimating \( \sigma^2_{1,t} \).

In other words, the dependence of \( \sigma^2_{1,t} \) on \( S_{2,t} \) is only through the dependence of \( \sigma^2_{1,t} \) on \( S_{1,t} \) and then the dependence of \( S_{1,t} \) on \( S_{2,t} \).
Let \((W^1_t, W^2_t, W^3_t)\) be the 3-dimensional uncorrelated standard Brownian motion. In the SLV+LV model, if we let

\[
\begin{align*}
    dW^S_{1,t} &= dW^1_t \\
    dW^S_{2,t} &= \rho_t^{S_1S_2} dW^1_t + \sqrt{1 - (\rho_t^{S_1S_2})^2} dW^2_t \\
    dW^V_{1,t} &= \rho_t^{S_1V_1} dW^1_t + \sqrt{1 - (\rho_t^{S_1V_1})^2} dW^3_t
\end{align*}
\]

\[
\implies \quad \begin{align*}
    dW^S_{2,t} dW^V_{1,t} &= \rho_t^{S_1S_2} \rho_t^{S_1V_1} dt.
\end{align*}
\]

If we choose

\[
\rho_t^{S_2V_1} = \rho_t^{S_1S_2} \rho_t^{S_1V_1}
\]

(C)

Numerical results show that

\[
E \left[ \sigma^2_{1,loc}(t, S_1, t) \bigg| S_2, t \right] \text{ in SLV + LV} \approx E \left[ \sigma^2_{1,loc}(t, S_1, t) \bigg| S_2, t \right] \text{ in LV + LV}
\]

However, \(E \left[ \sigma^2_{1,t} \bigg| S_2, t \right] \) in SLV+LV may differ even if (C) holds.
Example 1a.

Same model parameters as in Example 1, except that the correlations now are time-varying:

\[ \rho_{SV_1}^S = -0.75, \quad \rho_{S_1S_2}^S = 0.3 \text{ if } t \leq 0.25 \text{ else } 0, \quad \rho_{SV_2}^S = \rho_{S_1S_2}^S \rho_{SV_1}^S \]

\( \mathbb{E}[V_{1,t} | S_{2,t}] \) in SLV+LV differs markedly from \( \mathbb{E} \left[ \sigma_{1,loc}^2(t, S_{1,t}) | S_{2,t} \right] \) in SLV+LV and LV+LV.
Two-Asset Corridor Variance Swap  SLV+LV vs LV+LV

Example 2.

\[
\begin{align*}
\text{SLV + LV} & \quad \left\{ \begin{array}{l}
\frac{dS_{1,t}}{S_{1,t}} = \bar{\sigma}_1 \frac{\sqrt{V_{1,t}}}{\sqrt{\mathbb{E}[V_{1,t} | S_{1,t}]}} \, dW_{1,t}^S \\
\quad dV_{1,t} = \kappa (\theta - V_{1,t}) \, dt + \omega \sqrt{V_{1,t}} \, dW_{1,t}^V \\
\frac{dS_{2,t}}{S_{2,t}} = \bar{\sigma}_2 \, dW_{2,t}^S
\end{array} \right. \\
& \text{with} \quad dW_{1,t}^S \, dW_{2,t}^S = \rho_{S_1 S_2}^t \, dt
\end{align*}
\]

Here \( \bar{\sigma}_1 \) and \( \bar{\sigma}_2 \) are constants. In particular, the corresponding local vol functions are constants: \( \sigma_{1,loc}(t, S_{1,t}) = \bar{\sigma}_1 \) and \( \sigma_{2,loc}(t, S_{2,t}) = \bar{\sigma}_2 \).

\[
\text{LV + LV} \quad \left\{ \begin{array}{l}
\frac{dS_{1,t}}{S_{1,t}} = \bar{\sigma}_1 \, dW_{1,t}^S \\
\frac{dS_{2,t}}{S_{2,t}} = \bar{\sigma}_2 \, dW_{2,t}^S \\
\end{array} \right. \\
\text{with} \quad dW_{1,t}^S \, dW_{2,t}^S = \tilde{\rho}_{S_1 S_2}^t \, dt.
\]

Since \( \bar{\sigma}_1 \) is constant, the correlation \( \tilde{\rho}_{S_1 S_2}^t \) has no impact on \( \text{VS}_{1,2}(L, U, T) \) in LV+LV.
Two-Asset Corridor Variance Swap  SLV+LV vs LV+LV

Numerical Values of model parameters:
\[ \bar{\sigma}_1 = 0.15, \bar{\sigma}_2 = 0.2, S_{1,0} = 100, S_{2,0} = 100, L = 70, U = 110, T = 1 \]
\[ V_0 = 0.0165, \kappa = 3, \theta = 0.05, \omega = 0.5 \text{ and } 0.1, \rho^{S_1V_1} = -0.75, \rho^{S_1S_2} = 0.3 \]

LV+LV is unable to capture the vol-of-vol risk, cross spot/vol correlation risk.

\[ \rho^{S_1V_1} = -0.75, \rho^{S_1S_2} = 0.3 \]
Two-Asset Corridor Variance Swap - Approximation I

Postulate:
\[
\mathbb{E} \left[ \sigma^2_{1,t} \mid S_{2,t} \right] \approx \alpha_t + \beta_t \sigma^2_{2,\text{loc}}(t, S_{2,t}) \tag{PI}
\]

Motivation: if \( \sigma^2_{1,t} = \alpha_t + \beta_t \sigma^2_{2,t} + \epsilon_t \) and \( \mathbb{E}[\epsilon_t | S_{2,t}] = 0 \), then the postulate holds.

\[
VS^2_{1,2}(L, U, T) = \mathbb{E} \left[ \frac{1}{T} \int_0^T \sigma^2_{1,t} 1_{L \leq S_{2,t} \leq U} dt \right] = \frac{1}{T} \int_0^T \mathbb{E} \left[ \mathbb{E} \left[ \sigma^2_{1,t} \mid S_{2,t} \right] 1_{L \leq S_{2,t} \leq U} \right] dt
\]

\[
\approx \frac{1}{T} \int_0^T \alpha_t \mathbb{E} \left[ 1_{L \leq S_{2,t} \leq U} \right] dt + \frac{1}{T} \int_0^T \beta_t \mathbb{E} \left[ \sigma^2_{2,\text{loc}}(t, S_{2,t}) 1_{L \leq S_{2,t} \leq U} \right] dt
\]

If \( \alpha_t \equiv \alpha \) and \( \beta_t \equiv \beta \) are both constants, then the formula reduces to

\[
VS^2_{1,2}(L, U, T) \approx \alpha \text{RA}_2(L, U, T) + \beta \text{VS}^2_2(L, U, T)
\]

The parameters \( \alpha \) and \( \beta \) can be estimated from historical data (regressing \( \sigma^2_{1,t} \) on \( \sigma^2_{2,t} \)) or calibrated to the current market prices of relevant instruments.
Two-Asset Corridor Variance Swap - Approximation I

For a given set of maturities $T_1 < \cdots < T_n$, we assume $\alpha_t$ and $\beta_t$ take constant values between each $T_i$ and $T_{i+1}$.

Then we can calibrate $\alpha_t$ and $\beta_t$, if for each $T_i$, there exists a set of lower and upper bounds $(L_i, U_i)$ such that the prices of the following instruments are available:

1. Vanilla var swaps, $\text{VS}_1(T_i)$ and $\text{VS}_2(T_i)$
2. Range accrual and single-asset corridor var swap on $S_2$, $\text{RA}_2(L_i, U_i, T_i)$ and $\text{VS}_2(L_i, U_i, T)$
3. Two-asset corridor var swap $\text{VS}_{1,2}(L_i, U_i, T)$

Note that $\text{VS}_1(T_i)$, $\text{VS}_2(T_i)$, $\text{RA}_2(L_i, U_i, T_i)$ and $\text{VS}_2(L_i, U_i, T)$ can be easily computed (say, by replication) from an implied vol surface or local volatility model calibrated to the market prices of vanilla call and put options.

Now let $T = T_1$ and $\alpha_t \equiv \alpha$ and $\beta_t \equiv \beta$ for $0 \leq t \leq T$, then we have

$$\text{VS}_{1,2}^2(L, U, T) = \alpha \text{RA}_2(L, U, T) + \beta \text{VS}_2^2(L, U, T) \quad (A1)$$

Letting $L = 0$ and $U = \infty$ gives another constraint

$$\text{VS}_1^2(T) = \alpha + \beta \text{VS}_2^2(T) \quad (A2)$$

We can solve equations (A1) and (A2) to obtain $\alpha$ and $\beta$.

The values of $\alpha$ and $\beta$ at further maturities can be obtained by bootstrap.
Example 3. How good is the approximation?

\[
\frac{dS_1}{S_1} = \sqrt{V_1} dW_{1_s} \\
dV_1 = \kappa_1 (\theta_1 - V_1) dt + \omega_1 \sqrt{V_1} dW_{1_v} \\
\frac{dS_2}{S_2} = \sigma_{2,loc}(t, S_2) dW_{2_s}
\]

with

\[
dW_{1_s} dW_{1_v} = \rho_{S_1 S_1} dt \\
dW_{1_s} dW_{2_s} = \rho_{S_1 S_2} dt \\
dW_{2_s} dW_{1_v} = \rho_{S_2 S_1} dt
\]

Numerical values of model parameters:

\[
S_{1,0} = 100, \ S_{2,0} = 100, \ \rho_{S_1 S_2} = 0.3, \ \rho_{S_2 V_1} = -0.5 \\
V_{1,0} = 0.02, \ \kappa_1 = 3, \ \theta_1 = 0.05, \ \omega_1 = 0.5, \ \rho_{S_1 V_1} = -0.75 \\
\sigma_{2,loc} \text{ is the local vol surface in another Heston model with parameters } V_0 = 0.04, \\
\kappa = 1, \ \theta = 0.06, \ \omega = 0.25 \text{ and } \rho = -0.85
\]

For a list of maturities \(T_i = \frac{i}{12}, \ i = 1, \cdots, 12\), and corridor \(L = 70\) and \(U = 110\), the above model is used to compute the values of \(VS_1(T_i), \ VS_2(T_i), \ RA_2(L, U, T_i), \ VS_2(L, U, T_i)\) and \(VS_{1,2}(L, U, T_i)\). Then we calibrate piece-wise constants \(\alpha_t\) and \(\beta_t\) to these values by bootstrapping.
We apply the approximation formula with calibrated $\alpha_t$ and $\beta_t$ to other corridors.

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<th>L/U</th>
<th>56/88</th>
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<th>70/110</th>
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<td>-0.30</td>
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Two-Asset Corridor Variance Swap - Approximation I

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Corridor Variance Swap Spread
Two-Asset Corridor Variance Swap - Approximation I

The accuracy can be improved with more market quotes. For example, if there are market quotes for two sets of corridors \((L, U)\) and \((\tilde{L}, \tilde{U})\) (typically \((70\%, 110\%)\) and \((70\%, 130\%)\) of the spot), we can modify the postulate (PI) as follows:

\[
E\left[\sigma_{1,t}^2 \mid S_{2,t}\right] \approx \alpha_t + \left(\beta_t + \gamma_t \frac{S_{2,t}}{S_{2,0}}\right) \sigma_{2,loc}^2(t, S_{2,t})
\]  

\((\text{PIa})\)

If \(\alpha_t \equiv \alpha\), \(\beta_t \equiv \beta\) and \(\gamma_t \equiv \gamma\), for \(0 \leq t \leq T\), then

\[
VS_{1,2}^2(L, U, T) = \alpha RA_2(L, U, T) + \beta VS_{2}^2(L, U, T) + \gamma GS_{2}^2(L, U, T)
\]

\[
VS_{1,2}^2(\tilde{L}, \tilde{U}, T) = \alpha RA_2(\tilde{L}, \tilde{U}, T) + \beta VS_{2}^2(\tilde{L}, \tilde{U}, T) + \gamma GS_{2}^2(L, U, T)
\]

\[
VS_1^2(T) = \alpha + \beta VS_2^2(T) + \gamma GS_2^2(T)
\]

where

\[
GS_i^2(L, U, T) = E\left[\frac{1}{T} \int_0^T \frac{S_{i,t}}{S_{i,0}} \sigma_{i,t}^2 1_{L \leq S_{i,t} \leq U} dt\right]
\]

is the corridor Gamma swap on \(S_i\) with corridor \((L, U)\) and \(GS_i(T) = GS_i(0, \infty, T)\) is the vanilla Gamma swap.

The single-asset corridor Gamma swap can be perfectly replicated by static positions of vanilla options (replicating truncated and linearly extrapolated payoff \(S_T \log S_T\)) and dynamically rebalanced delta-hedging portfolio. In particular, if \(r_t \equiv q_t\), then

\[
GS_i^2(L, U, T) = \frac{1}{T} \int_L^U \frac{2}{K} \text{OTM}(K, T) dK.
\]

For a list of maturities \(\{T_i\}\), we shall assume \(\alpha_t, \beta_t\) and \(\gamma_t\) take on constant values between \(T_i\) and \(T_{i+1}\) so that their values can be calibrated by bootstrapping.
Simple and fast calibration and pricing, no Monte Carlo simulation needed

Establishes explicit dependence on volatilities and correlations, well-suited for risk management

Once the values of $\alpha_t$, $\beta_t$ (and $\gamma_t$) are calibrated, the approximation formula (PI) (or (PIa)) for the cross local variance can be used to price two-asset corridor var swaps with other corridors (for deals entered earlier when the corridors were set relative to the spot at the time of the trade), or even other general cross weighted variance payoffs, i.e.,

$$
\mathbb{E} \left[ \frac{1}{T} \int_0^T \sigma_{1,t}^2 w(S_{2,t}) \, dt \right] = \frac{1}{T} \int_0^T \mathbb{E} \left[ \mathbb{E} \left[ \sigma_{1,t}^2 \mid S_{2,t} \right] w(S_{2,t}) \right] \, dt
$$

$$
= \mathbb{E} \left[ \frac{1}{T} \int_0^T \left( \alpha_t + \left( \beta_t + \gamma_t \frac{S_{2,t}}{S_{2,0}} \right) \sigma_{2,loc}^2 (t, S_{2,t}) \right) w(S_{2,t}) \, dt \right]
$$

The last expectation can be easily evaluated by replication, PDE or Monte Carlo simulation in the local vol model.
**Postulate:** The variance index $S_{1,t}$ follows a local volatility dynamics.

\[
\frac{dS_{1,t}}{S_{1,t}} = \sigma_{1,\text{loc}}(t, S_{1,t}) \, dW_{1,t}
\]

Then

\[
\text{VS}_{1,2}^{2}(L, U) = \mathbb{E} \left[ \frac{1}{T} \int_{0}^{T} \sigma_{1,\text{loc}}^{2}(t, S_{1,t}) \, 1_{L \leq S_{2,t} \leq U} \, dt \right]
\]

\[
= \frac{1}{T} \int_{0}^{T} \mathbb{E} \left[ \sigma_{1,\text{loc}}^{2}(t, S_{1,t}) \right] \mathbb{E} \left[ 1_{L \leq S_{2,t} \leq U} \, | \, S_{1,t} \right] \, dt
\]

\[
= \mathbb{E} \left[ \frac{1}{T} \int_{0}^{T} \sigma_{1,\text{loc}}^{2}(t, S_{1,t}) \left( \Phi_{t}(U \, | \, S_{1,t}) - \Phi_{t}(L \, | \, S_{1,t}) \right) \, dt \right]
\]

where

\[
\Phi_{t}(K \, | \, S_{1,t}) = \mathbb{E} \left[ 1_{S_{2,t} \leq K} \, | \, S_{1,t} \right]
\]

is the conditional CDF of $S_{2,t}$ given $S_{1,t}$.

In this case we price the two-asset corridor variance swap as a single-asset weighted variance swap on $S_{1}$ with weight $\Phi_{t}(U \, | \, S_{1,t}) - \Phi_{t}(L \, | \, S_{1,t})$.

The conditional CDF $\Phi_{t}$ may be derived from the copula of $S_{1,t}$ and $S_{2,t}$, which in turn can be calibrated to the market prices of basket payoffs $S_{1,t}$ and $S_{2,t}$ if available.
References