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Unifying Exotic Option Closed Formulas

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Abstract

This paper aims to unify exotic option closed formulas by generalizing a large class of existing formulas and by setting a framework that allows for further generalizations. The formula presented covers options from the plain vanilla to most, if not all, mountain range exotic options and is developed in a multi-asset, multi-currency Black-Scholes model with time dependent parameters. The general formula not only covers existing cases but also enables the combination of diverse features from different types of exotic options. It also creates implicitly a language to describe payoffs that can be used in industrial applications to decouple the functions of payoff definition from

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pricing functions. Examples of several exotic options are presented, benchmarking the closed formulas' performance against Monte Carlo simulations. Results show a consistent over performance of the closed formula reducing calculation time by double digit factors.

Key words: exotic options, mountain range, discrete lookback, closed formula, payoff language, multi-asset multi-currency model

1 Introduction

The pricing of exotic options, defined in most references as every option type apart from the European and American vanilla options, is performed either by using a closed formula or by relying on a numerical method to evaluate the integral the pricing function involves. Whenever available, a closed formula is more precise and requires less computational effort. This is the reasoning behind our search for general closed formulas that unify exotic option pricing.

The closed formulas for pricing exotic options have mainly been developed to price options whose payoffs exhibit one, and only one, very specific feature, and they assume an elementary market setup. However, the industry requirements go well beyond these simplifications. Exotic options underlying assets spread across several currency zones, and exotic options payoff profiles include features from multiple exotic option types.

This need to account for multiple features in a computationally simple process calls for a unification of the existing closed exotic option pricing formulas. Thus, instead of proceeding to develop formulas for specific option types, we propose a general approach that is able to accommodate several of the features seen in most exotics. Hence, we produce a formula for a generic payoff, covering thus all exotic options whose features are included in it. The market setting underlying the formula is also able to accommodate very diverse market setups, covering as many currency zones as needed.

Finally, the general formula allows the development of payoff languages. Payoff languages are extremely useful in industrial pricing applications as they enable the decoupling the payoff definition process from the pricing routines. Thus, as long as the payoff only uses the features covered by this general formula, the development of a new payoff profile does not necessitate the development of a new pricing routine. This means that industry agents can freely combine the desired features, while using the same pricing routines.

This paper is divided into four sections. This first section covers the motivation for the paper and the literature review. The second section develops the model, the payoff of a generic claim and its pricing formula. Section three then discusses the applications, including performance matters, and provides examples and the final section concludes.

1.1 Literature Review

Literature on exotic options is vast and dates back to the late 1970s. It is not our intention to give a complete chronology of the works related to this field but just to refer some landmark contributions for each of the main threads of research. Compilations of exotic options descriptions and pricing formulas may be found in Nelken (1995), Zhang (1997), Haug (1998), and

Hakala and Wystup (2002).

According to our exotic option definition above, there are three threads of research in exotics, the first of which deals with options on multiple underlyings. The distinctive characteristic of these options is their high sensitivity to correlations. The landmark closed formulas were Margrabe (1978) - exchange options, Stulz (1982) maximum/minimum of two assets and Johnson (1987) for several assets. One other thread deals with path-dependent options, namely lookback and barrier, which this paper only includes in their discrete version. The main contributions on this thread are Rubinstein and Reiner (1991) for barrier options and Goldman et al.(1979) and Conze and Viswanathan (1991) for lookbacks. Further developments on barrier options were due to work by Heynen and Kat (1994), Carr (1995) and Wystup (2003). For a remarkable description of the barrier option problem see Björk (1998) whose general approach covers a wide class of payoffs. The last thread deals with Asian option and basket options. Their distinctive characteristic is the need to handle sums of geometric Brownian motions. Initial contributions for simpler geometric average problems are from Vorst (1992), and a major development for arithmetic average problems is due to Večeř (2001). The present paper extends previous work on this subject by Veiga (2004).

2 Formula Development

2.1 Model Description

The model on which we develop a closed formula can be classified as a multivariate Black–Scholes model. It is a multi-asset model in which all assets are tradable including for example stocks, currencies, precious metals and indexes composed by these.

We assume the existence of n assets A_i, and the respective bank accounts B_i where asset A_i may be deposited, with $i = 1, \dots, n$. Each of the accounts yields a return, in units of the same asset, at a continuosly compounded rate of r_i. Such a rate may be interpreted as an interest rate of a currency or as a repo rate[1] of a stock. Although it is also common also to use this rate to represent dividend payments for individual stocks, we advise against it since dividend payments are typically not payed continuously and are not proportional to the asset price, see [21] for details. Each bank account thus follows the dynamics

$$dB_i(t)/B_i(t) = r_i(t)\mathrm{d}t. \tag{1}$$

We furthermore assume the existence of one, and only one, price process for each asset A_i allowing its expression in units of another asset A_j. This structure is usually referred to as a tree structure. Though here the

[1] rate paid on a repurchase agreement or stock lending contract.

definition of the root (asset) of the tree is not critical, any asset can play that role, what is critical is to have one path, and only one path, to express the price of one asset in terms of any other. Such a structure excludes triangular relationships as for example EUR/USD, USD/JPY and EUR/JPY foreign exchange pairs. We exclude these relationships because they impose restrictions on the volatilities and correlations between the assets, see [9] for details.

Hence, we assume the existence of price processes S_{ij}, that is the price of one unit of A_i expressed in units of A_j, with the dynamics following the stochastic differential equation (SDE)

$$\mathrm{d}S_{ij}(t)/S_{ij}(t) = \mu_{ij}(t)\mathrm{d}t + \sigma_{ij}(t)\mathrm{d}W_i(t), \tag{2}$$

where $W_i(t)$ is a Brownian motion under the real world measure P, $\mu_{ij} \in \mathbb{R}$, $\sigma_{ij} \in \mathbb{R}_+$. Furthermore, $W_{ij}(t)$ is correlated with the Brownian motions that drive the other asset prices. Let $W_{kl}(t)$ be one such process, $\rho_{ij,kl}(t)$ its correlation with $W_{ij}(t)$, and $\varsigma_{ij,kl}(t) = \rho_{ij,kl}(t)\sigma_{ij}(t)\sigma_{kl}(t)$ the respective covariance.

Although other setups are also plausible, we choose this one for three reasons: it is general enough to accommodate most exotic options we have encountered, the formulas generated are still manageable, and the volatilities and correlations can be freely specified. Figure 1 illustrates a model setup that would underlie the valuation of a typical structured product that depends on several equity indexes spread across the world.

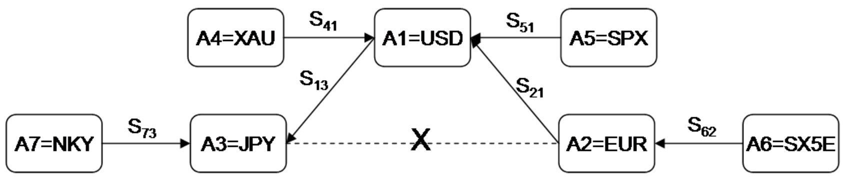

Figure 1: Example of market setup. The abbreviations refer to the following: USD to United States dollars, EUR to the euro currency, JPY to the Japanese yen, XAU to the gold ounce, SPX to the S&P500 index, SX5E to the DJ Eurostoxx 50 index, and NKY to the Nikkei index.

It shows a market with seven assets and six prices. It includes the currencies of the three main monetary zones and the most popular indexes of each. The currency pairs S_{21} and S_{13} are the most liquid and are defined according to market standards, EUR/USD and USD/JPY respectively. The prices of the baskets of stocks that compose each of the equity indexes A_5, A_6 and A_7 are naturally expressed in terms of their respective currencies.

It is well known that a market with the same number of random sources driven by (correlated) Brownian motions $W_{ij}(t)$ as of tradable assets A_i is complete and arbitrage free. See Björk [1] or Shreve [17] for details. Therefore, there exists a unique martingale measure Q_k, equivalent to measure P. In such a measure Q_k, all portfolios expressed in terms of units of the *numéraire* portfolio $B_k(t)$ are martingales. In this model, the transformation from measure P to Q_k is found by solving a simple system of equations in which the matrix is triangular. This system yields a transformation of the type

$$\mathrm{d}W_{ij}(t) = \mathrm{d}W_{ij,k}(t) - \frac{1}{\sigma_{ij}(t)}\left(r_i(t) + \mu_{ij}(t) - r_j(t) + \sum_{h=1}^{n} \lambda_{j_h} \varsigma_{ij,i_h}(t)\right)\mathrm{d}t. \tag{3}$$

where n is the number of price conversions needed to express the asset in which the price of A_i is expressed, i.e. A_j, to A_k; λ_{j_h} accounts for the direction of each of the prices, which may be natural ($\lambda_{j_h} = 1$) or inverse ($\lambda_{j_h} = -1$). A price expressed in the natural direction, with respect to the price path from A_j to A_k, is one that multiplies the previous quantity to yield the next. Conversely, a price expressed in the inverse direction is one that divides. Finally, i_h are the indexes of the assets that stand between assets A_j and A_k. The indexes are unique as the tree structure implies that there is one, and only one, shortest conversion path connecting the assets.

Applying the transformation to $S_{ij}(t)$ in equation (2) we get the dynamics of $S_{ij}(t)$ under the measure Q_k as

$$\mathrm{d}S_{ij}(t)/S_{ij}(t) = \left(r_j(t) - r_i(t) - \sum_{h=1}^{n} \lambda_{j_h} \varsigma_{ij,i_h}(t)\right)\mathrm{d}t + \sigma_{ij}(t)\mathrm{d}W_{ij,k}(t). \tag{4}$$

In the example above, if the EUR bank account is chosen as *numéraire*, the dynamics of the index NKY are

$$\begin{aligned}\mathrm{d}S_{73}(t)/S_{73}(t) = (r_3(t) - r_7(t) - (-\varsigma_{73,13}(t) - \varsigma_{73,21}(t)))\,\mathrm{d}t \\ + \sigma_{73}(t)\mathrm{d}W_{73,2}(t).\end{aligned} \tag{5}$$

Now that we have all dynamics of all prices S under one arbitrary martingale measure Q_k, the relevant information concerning the location of A_i in the tree structure is condensed in the summation $\sum_{h=1}^{n} \lambda_{j_h} \varsigma_{ij,i_h}(t)$. Therefore, we can suppress the letter in the subscript of S, σ, ρ and ς that tracks the asset in which the price is expressed. Thus, ij will be only i from now on. Furthermore, we will also assign the symbol $d_i(t)$ to the drift term function, yielding (4) in a more economic form as

$$\mathrm{d}S_i(t)/S_i(t) = d_i(t)\mathrm{d}t + \sigma_i(t)\mathrm{d}W_{ik}(t), \tag{6}$$

where we also removed the comma on the diffusion term because, from now on, we shall only need one symbol to refer to an asset.

2.2 Abstract Assets

Apart from the physical assets A_i, we assume the existence of a new set of abstract assets $\mathbb{A}_i$. The conditions these assets need to fulfill are that (i) their price is a function of the prices S_i, and (ii) their price follows, under any given measure Q_k, a dynamic of the type

$$\mathrm{d}\mathbb{S}_i(t)/\mathbb{S}_i(t) = d_{\mathbb{S}_i}(t)\mathrm{d}t + \sigma_{\mathbb{S}_i}(t)\mathrm{d}W_{\mathbb{S}_i k}(t). \tag{7}$$

Thus, we define $\mathbb{S}_i$ as the most general case we can conceive

$$\mathbb{S}_i(t) = \prod_{j=1}^{m} \left(S_{i_j,t_j}(t)\right)^{\alpha_j}, \tag{8}$$

where $S_{i_j,t_j}(t)$ is the process $S_{i_j}(t)$ frozen at time t_j, i.e., $S_{i_j}(t \wedge t_j)$ and i_j is an index of an asset. Hence, the process $S_{i_j,t_j}(t)$ has zero drift and diffusion after time t_j, and we write its dynamics as

$$\mathrm{d}S_{i_j,t_j}(t)/S_{i_j,t_j}(t) = \delta_{i_j}(t)\mathrm{d}t + \theta_{i_j}(t)\mathrm{d}W_{ik}(t), \tag{9}$$

with $\delta_{i_j}(t) = d_{i_j}(t)$ and $\theta_{i_j}(t) = \sigma_{i_j}(t)$ for $t < t_j$ and both equal to zero otherwise. The covariance is also redefined as $\zeta_{i_j i_h}(t) = \theta_{i_j}(t)\theta_{i_h}(t)\rho_{i_j,i_h}(t)$. Without loss of generality we assume $t_1 < t_2 < \ldots < t_n$.

This form exploits the geometric nature of the asset prices S_i, and that linear combinations of normally distributed random variables are still normal.

To characterize the asset $\mathbb{A}_i$ and its price $\mathbb{S}_i$ we need to determine the following quantities: the correlation with any other asset $\zeta_{\mathbb{S}_i k}(t)$, the drift term $d_{\mathbb{S}_i}(t)$, the volatility term $\sigma_{\mathbb{S}_i}(t)$, and the rate of return of deposits of $\mathbb{A}_i$, i.e., $r_{\mathbb{A}_i(t)}$.

Using the Itô formula we obtain the dynamic of $\mathbb{S}_i$ as

$$\begin{aligned}\mathrm{d}\mathbb{S}_i(t)/\mathbb{S}_i(t) =& \sum_{j=1}^{m} \alpha_j \left(\delta_{i_j}(t) - \frac{\theta^2_{i_j}(t)}{2} + \sum_{h=1}^{m} \frac{\alpha_h \zeta_{i_j i_h}(t)}{2} \right) \mathrm{d}t \\ &+ \sum_{j=1}^{m} \alpha_j \theta_{i_j}(t) \mathrm{d}W_{i_j k}(t).\end{aligned} \tag{10}$$

We thus have $d_{\mathbb{S}_i}$ and $\sigma^2_{\mathbb{S}_i}$ as

$$\sigma^2_{\mathbb{S}_i}(t) = \sum_{j=1}^{m}\sum_{h=1}^{m} \alpha_j \alpha_h \zeta_{i_j i_h}(t) = \sum_{j=1}^{m}\sum_{h=1}^{m} \alpha_j \alpha_h \varsigma_{i_j,i_h}(t) \mathbb{I}_{\{t<t_j, t<t_h\}}, \tag{11}$$

$$d_{\mathbb{S}_i}(t) = \sum_{j=1}^{m} \alpha_j \left(\delta_{i_j}(t) - \frac{\theta^2_{i_j}(t)}{2} \right) + \frac{\sigma^2_{\mathbb{S}_i}(t)}{2}, \tag{12}$$

with $\sigma^2_{\mathbb{S}_i}(t)$ the variance of a sum of correlated normals.

As to the covariance of $\mathbb{S}_i$ with any other price $\mathbb{S}_j$, we make use of the relationship $\zeta_{\mathbb{S}_i\mathbb{S}_j}(t)\mathrm{d}t = \sigma_{\mathbb{S}_i}(t)\sigma_{\mathbb{S}_j}(t)\mathrm{d}W_{\mathbb{S}_ik}(t)\mathrm{d}W_{\mathbb{S}_jk}(t)$ and, using (10), conclude that

$$\zeta_{\mathbb{S}_i\mathbb{S}_j}(t) = \sum_{a=1}^{m_i}\sum_{b=1}^{m_j}\alpha_a\alpha_b\zeta_{i_ai_b}(t) = \sum_{a=1}^{m_i}\sum_{b=1}^{m_j}\alpha_a\alpha_b\varsigma_{a,b}(t)\mathbb{I}_{\{t<t_a,t<t_b\}}, \tag{13}$$

and, in the special case where one of the abstract assets is equal to an asset price S_x, we also have

$$\zeta_{\mathbb{S}_ix}(t) = \sum_{j=1}^{m}\alpha_j\zeta_{i_jx}(t) = \sum_{j=1}^{m}\alpha_j\varsigma_{j,x}(t)\mathbb{I}_{\{t<t_j\}}. \tag{14}$$

The return rate $r_{\mathbb{S}_i}$ of the bank account $\mathbb{B}_i$ associated with asset $\mathbb{A}_i$ can be easily calculated using the fact that the dynamics of $\mathbb{B}_i$ expressed in terms of B_k, under the measure Q_k, is a martingale. A simple application of the Itô formula yields

$$\mathrm{d}\left(\frac{\mathbb{B}_i\mathbb{S}_i}{B_k}\right)(t) = \left(\frac{\mathbb{B}_i\mathbb{S}_i}{B_k}\right)(t)\left(\left(r_{\mathbb{S}_i}(t) + d_{\mathbb{S}_i}(t) - r_k(t)\right)\mathrm{d}t + \sigma_{\mathbb{S}_i}(t)\mathrm{d}W_{\mathbb{S}_ik}(t)\right). \tag{15}$$

Consequently,

$$r_{\mathbb{S}_i}(t) = r_k(t) - d_{\mathbb{S}_i}(t). \tag{16}$$

We can now even write the SDE of the price process S_l under the martingale measure associated with $\mathbb{S}_i$, $Q_{\mathbb{S}_i}$, just adding one extra price conversion to the path from A_j to A_k, described in section 2.1, yielding

$$\mathrm{d}S_l(t)/S_l(t) = \left(d_l(t) + \zeta_{\mathbb{S}_il}(t)\right)\mathrm{d}t + \sigma_l(t)\mathrm{d}W_{l\mathbb{S}_i}(t), \tag{17}$$

and consequently

$$\mathrm{d}\mathbb{S}_l(t)/\mathbb{S}_l(t) = \left(d_{\mathbb{S}_l}(t) + \zeta_{\mathbb{S}_i\mathbb{S}_l}(t)\right)\mathrm{d}t + \sigma_{\mathbb{S}_l}(t)\mathrm{d}W_{l\mathbb{S}_i}(t). \tag{18}$$

Before we conclude this subsection, we would like to make a remark on how these abstract assets fit the arbitrage theory framework. As in Björk [1], Chapter 24, arbitrage theory requires that the *numéraire* of a given model definition must be a *traded* asset. Clearly these abstract assets are not traded *per se*, and they cannot be replicated by any self-financing portfolio. However, we were able to find the $Q_{\mathbb{S}_i}$ measure in which all the portfolios ϑ of the above assets and abstract assets, expressed in units of $\mathbb{B}_k$, are martingales when translated to units of $\mathbb{B}_i$, by $\frac{\vartheta(t)B_k(t)}{\mathbb{B}_i(t)\mathbb{S}_i(t)}$,

2.3 Generic Contract

Now we need an abstract definition of a contract, or a claim, that should include as many features and existing contracts as possible. Hence, we propose the following payoff definition expressed in terms of asset A_k

$$\Phi_k = \sum_{i=1}^{n} c_i \mathbb{S}_{I_i,t_i}(T_i)\mathbb{I}_{C_i}, \tag{19}$$

with $c_i \in \mathbb{R}$, $\mathbb{S}_{I_i,t_i}(T_i)$ as the price of an abstract asset $\mathbb{A}_{I_i}$ expressed in terms of A_k, observed at time t_i, to be settled at time T_i, and $\mathbb{I}_{C_i}$ the indicator function of the set C_i that will be defined in section 2.4. For the payoff to be adapted, we need $t_i \leq T_i$. We note that $\mathbb{S}_{I_i,t_i}(t)$ is actually a stopped process as above and, likewise, $d_{\mathbb{S}_{I_i,t_i}}(t)$, $\sigma_{\mathbb{S}_{I_i,t_i}}(t)$ and $\zeta_{\mathbb{S}_{I_i,t_i}x}(t)$ are zero for $t > t_i$. Likewise, the return rate $r_{\mathbb{S}_{I_i,t_i}}(t) = r_k(t)$ for $t > t_i$.

To be able to price this claim, we assume the existence of a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ and that all prices S_i are adapted. The arbitrage free price of such a contract is, as usual, the discounted expected payoff under the unique equivalent martingale measure Q_k, thus

$$V(t_0) = \sum_{i=1}^{n} c_i B_k(t_0) E_{t_0}^{Q_k}\left[\frac{\mathbb{S}_{I_i,t_i}(T_i)\mathbb{I}_{C_i}}{B_k(T_i)}\right], \tag{20}$$

where $E_{t_0}^{Q_k}$ is the conditional expectation, under the measure Q_k, conditioned on the σ-algebra $\mathcal{F}_{t_0}$. We also use the fact that the conditional expectation is a linear operator to interchange it with the summation.

For each term of the summation we may write, with $V(t_0) = \sum_{i=1}^{n} v_i(t_0)$,

$$\frac{v_i(t_0)}{B_k(t_0)} = E_{t_0}^{Q_k}\left[c_i \frac{\mathbb{S}_{I_i,t_i}(T_i)\mathbb{I}_{C_i}}{B_k(T_i)}\right], \tag{21}$$

which is a martingale by definition of Q_k.

We now translate the price and the payoff expressed in units of B_k in units of $\mathbb{B}_{I_i,t_i}(T_i)$. These new quantities are martingales under the measure $Q_{\mathbb{S}_{I_i,t_i}}$, and therefore

$$\frac{v_i(t_0)}{B_k(t_0)} \frac{B_k(t_0)}{\mathbb{S}_{I_i,t_i}(t_0)\mathbb{B}_{I_i,t_i}(t_0)} = E_{t_0}^{Q_{\mathbb{S}_{I_i,t_i}}}\left[c_i \frac{\mathbb{S}_{I_i,t_i}(T_i)\mathbb{I}_{C_i}}{B_k(T_i)} \frac{B_k(T_i)}{\mathbb{S}_{I_i,t_i}(T_i)\mathbb{B}_{I_i,t_i}(T_i)}\right]. \tag{22}$$

This step can be view as a change of *numéraire* from B_k to $\mathbb{B}_{I_i,t_i}$ as in Geman *et al.* [6].

Canceling terms and rearranging we get

$$V(t_0) = \sum_{i=1}^{n} c_i \mathbb{S}_{I_i,t_i}(t_0)\mathbb{B}_{I_i,t_i}(t_0) E_{t_0}^{Q_{\mathbb{S}_{I_i,t_i}}}\left[\frac{\mathbb{I}_{C_i}}{\mathbb{B}_{I_i,t_i}(T_i)}\right]. \tag{23}$$

Additionally, we know that $\mathbb{B}_i$ is a deterministic processes under the measure $Q_{\mathbb{S}_{I_i,t_i}}$ and can thus be taken out of the expectation, yielding

$$\begin{aligned} V(t_0) &= \sum_{i=1}^{n} c_i \mathbb{S}_{I_i,t_i}(t_0) \frac{\mathbb{B}_{I_i,t_i}(t_0)}{\mathbb{B}_{I_i,t_i}(T_i)} P_{t_0}^{Q_{\mathbb{S}_{I_i,t_i}}}(C_i) \\ &= \sum_{i=1}^{n} c_i \mathbb{S}_{I_i,t_i}(t_0) \exp\left\{-\int_{t_0}^{T_i} r_{\mathbb{S}_{I_i,t_i}}(u)\mathrm{d}u\right\} P_{t_0}^{Q_{\mathbb{S}_{I_i,t_i}}}(C_i), \end{aligned} \tag{24}$$

where $P_{t_0}^{Q_{\mathbb{S}_{I_i,t_i}}}(C_i)$ is the probability of the set C_i, under the risk neutral measure $Q_{\mathbb{S}_{I_i,t_i}}$, and considering the prices at time t_0.

However, in general the expression on the right hand side of (24) does not lead to a closed formula and may require numerical integration. Hence, we need to impose some restrictions on the shape of the set C_i to make sure the probability terms $P_{t_0}^{Q_{\mathbb{S}_{I_i,t_i}}}(C_i)$ can be evaluated using a closed form expression. More specifically, we will constrain the set C_i in a way that guarantees that $P_{t_0}^{Q_{\mathbb{S}_{I_i,t_i}}}(C_i)$ can be evaluated by a sum of multivariate normal cumulative distribution functions.

2.4 Set Definition and its Probability

Before we state the main result of this section we need the following:

Proposition 2.1. *The processes* $\log(S_{l,s}(t))$*, with* l *iterating over all asset prices in the model, and* $s \geq 0$*, are jointly normally distributed for any time* $t > 0$.

Proof. Standard stochastic calculus, applied to equation (17), yields its solution as

$$\begin{aligned} S_{l,s}(t) = S_l(t_0 \wedge s) \exp\Bigg\{ \int_{t_0\wedge s}^{t\wedge s} \left(d_l(u) + \zeta_{\mathbb{S}_i l}(u) - \frac{\sigma_l^2(u)}{2} \right) \mathrm{d}u \\ + \int_{t_0\wedge s}^{t\wedge s} \sigma_l(u) \mathrm{d}W_{l\mathbb{S}_i}(u) \Bigg\}. \end{aligned} \tag{25}$$

From this solution, it follows immediately that $\log(S_{l,s}(t))$ is normally distributed with mean μ and variance ψ as

$$\begin{aligned} \mu &= \log(S_l(t_0 \wedge s)) + \int_{t_0\wedge s}^{t\wedge s} \left(d_l(u) + \zeta_{\mathbb{S}_i l}(u) - \frac{\sigma_l^2(u)}{2} \right) \mathrm{d}u, \\ \psi &= \int_{t_0\wedge s}^{t\wedge s} \sigma_l^2(t) \mathrm{d}u. \end{aligned}$$

Furthermore, any linear combination of the logs of frozen processes is still normally distributed. Without loss of generality, let $0 = s_0 \leq s_1 \leq \ldots \leq s_m$. Then

$$\sum_{g=1}^{m} \alpha_g \log(S_{l_g,s_g}(t)) = \sum_{g=1}^{m} \alpha_g \mu_g + \sum_{g=1}^{m} \int_{t_0 \wedge s_g}^{t \wedge s_g} \alpha_g \sigma_{l_g}(u) \mathrm{d}W_{l_g \mathbb{S}_i}(u) \tag{26}$$

is normally distributed, with mean $\sum_{g=1}^{m} \alpha_g \mu_g$ and variance

$$\sum_{g=1}^{m} \int_{t \wedge s_{g-1}}^{t \wedge s_g} \sum_{a=g}^{m} \sum_{b=g}^{m} \alpha_a \alpha_b \varsigma_{l_a,l_b}(u) \, \mathrm{d}u. \tag{27}$$

By theorem 9.5.13 of [5], this is enough to prove that any set of random variables $\log(S_{l,s}(t))$ is jointly normally distributed. □

Proposition 2.2. *Let the set C_i be of the form*

$$\bigcap_{l=1}^{m_i} \left\{ \frac{\mathbb{S}_{I_{l_u},t_{l_u}}(T_i)}{\mathbb{S}_{I_{l_d},t_{l_d}}(T_i)} < h_l \right\}, \tag{28}$$

with I_{l_u}, I_{l_d} denoting the indexes of abstract assets, $h_l \geq 0$ and t_{l_u}, $t_{l_d} \leq T_i$.

Then $P_t^{Q_{\mathbb{S}_{I_i,t_i}}}(C_i)$ is of the form

$$\mathcal{N}_{m_i}^{\mathbb{S}_{I_i,t_i}}(v; \phi, \Sigma), \tag{29}$$

with $\mathcal{N}_{m_i}$ denoting the m_i-dimensional multivariate normal cumulative distribution function, with covariance matrix Σ and mean vector ϕ, evaluated at vector v.

Proof. By (8) and recalling that all $\mathbb{S}_i(t)$ are positive by definition, we have

$$\bigcap_{l=1}^{m_i} \left\{ \frac{\mathbb{S}_{I_{l_u},t_{l_u}}(T_i)}{\mathbb{S}_{I_{l_d},t_{l_d}}(T_i)} < h_l \right\} = \tag{30}$$

$$\bigcap_{l=1}^{m} \left\{ \sum_{a=1}^{m_{l_u}} \alpha_a \log(S_{i_a,t_a}(t_{l_u})) - \sum_{b=1}^{m_{l_d}} \alpha_b \log(S_{i_b,t_b}(t_{l_d})) < \log(h_l) \right\}. \tag{31}$$

Proposition 2.1 tells us that all $\log(S_{l,s}(t))$ are jointly normally distributed. Therefore, the random vector X with m_i elements

$$X_l = \sum_{a=1}^{m_{l_u}} \alpha_a \log(S_{i_a,t_a}(t_{l_u})) - \sum_{b=1}^{m_{l_d}} \alpha_b \log(S_{i_b,t_b}(t_{l_d})), \tag{32}$$

with $l = 1, \ldots, m_i$, is just a linear transformation of the vector of variables of the type of $\log(S_{l,s}(t))$ and, therefore, is normally distributed (or its elements are jointly normally distributed).

From proposition 2.1 we can derive their mean. Thus, $\phi = [\phi_1, \ldots, \phi_{m_i}]^T$ with

$$\phi_l = \sum_{a=1}^{m_{l_u}} \alpha_a \left(\log(S_{i_a}(t_0 \wedge t_a)) + \int_{t_0 \wedge t_a}^{t_{l_u} \wedge t_a} d_{i_a}(s) + \zeta_{\mathbb{S}_{I_i,t_i} i_a}(s) - \frac{\sigma_{i_a}^2(s)}{2} \,\mathrm{d}s \right) -$$
$$- \left(\sum_{b=1}^{m_{l_d}} \alpha_b \left(\log(S_{i_b}(t_0 \wedge t_b)) + \int_{t_0 \wedge t_b}^{t_{l_d} \wedge t_b} d_{i_b}(s) + \zeta_{\mathbb{S}_{I_i,t_i} i_b}(s) - \frac{\sigma_{i_b}^2(s)}{2} \,\mathrm{d}s \right) \right) \tag{33}$$

$$= \log\left(\mathbb{S}_{I_{l_u},t_{l_u}}(t_0)\right) + \int_{t_0}^{T_i} d_{\mathbb{S}_{I_{l_u},t_{l_u}}}(s) - \frac{\sigma^2_{\mathbb{S}_{I_{l_u},t_{l_u}}}(s)}{2} + \zeta_{\mathbb{S}_{I_i,t_i}\mathbb{S}_{I_{l_u},t_{l_u}}}(s)\mathrm{d}s$$
$$- \left(\log\left(\mathbb{S}_{I_{l_d},t_{l_d}}(t_0)\right) + \int_{t_0}^{T_i} d_{\mathbb{S}_{I_{l_d},t_{l_d}}}(s) - \frac{\sigma^2_{\mathbb{S}_{I_{l_d},t_{l_d}}}(s)}{2} + \zeta_{\mathbb{S}_{I_i,t_i}\mathbb{S}_{I_{l_d},t_{l_d}}}(s)\mathrm{d}s \right). \tag{34}$$

Let us define Σ_{ef}, the elements of Σ, with $e, f = 1, \ldots, m_i$. The covariance between two of the random variables X_e, X_f is, by definition,

$$\Sigma_{ef} = E_{t_0}^{Q_{\mathbb{S}_{I_i,t_i}}} \left[(X_e - \phi_e)(X_f - \phi_f) \right], \tag{35}$$

which yields, after some simple algebra,

$$\int_{t_0}^{T_i} \zeta_{\mathbb{S}^e_{I_{l_u},t_{l_u}} \mathbb{S}^f_{I_{l_u},t_{l_u}}}(s) - \zeta_{\mathbb{S}^e_{I_{l_u},t_{l_u}} \mathbb{S}^f_{I_{l_d},t_{l_d}}}(s)$$
$$- \zeta_{\mathbb{S}^e_{I_{l_d},t_{l_d}} \mathbb{S}^f_{I_{l_u},t_{l_u}}}(s) + \zeta_{\mathbb{S}^e_{I_{l_d},t_{l_d}} \mathbb{S}^f_{I_{l_d},t_{l_d}}}(s)\mathrm{d}s, \tag{36}$$

where $\mathbb{S}^e_{\cdot}$ and $\mathbb{S}^f_{\cdot}$ are the assets in conditions $l = e$ and $l = f$ respectively. It is worth noting that the covariance is the same whatever the measure under which the expectation is taken. This follows the well known fact that changes of martingale measures only modify the location of the distribution and not its shape.

The variance of X_e is obtained with $f = e$. The elements of the vector v are $v_l = \log(h_l)$, with $l = 1, \ldots, m_i$. □

2.5 Pricing Formula

Proposition 2.2 together with Equation (24) allow us to write the following:

Theorem 2.1. *The arbitrage free price of the claim with payoff Φ_k as in* (19) *and the sets C_i as in* (28) *can be calculated using the formula*

$$V(t_0) = \sum_{i=1}^{n} c_i \mathbb{S}_{I_i,t_i}(t_0) \exp\left\{ - \int_{t_0}^{T_i} r_{\mathbb{S}_{I_i,t_i}}(u)\mathrm{d}u \right\} \mathcal{N}_{m_i}^{\mathbb{S}_{I_i,t_i}}(v; \phi, \Sigma). \tag{37}$$

Finally, we may also consider the complement of sets C_i in proposition 2.2, as we have $P_t^{Q_{\mathbb{S}_{I_i,t_i}}}\left(\overline{C_i}\right) = 1 - \mathcal{N}_{m_i}^{\mathbb{S}_{I_i,t_i}}(v;\phi,\Sigma)$, by the properties of cumulative distribution functions.

2.6 Derivatives

Developing a closed pricing formula has immediate benefits when it comes to pricing the claims and also opens the possibility of allowing the calculation of the quantities relevant for hedging strategies and risk management, i.e., partial derivatives, also by closed formulas. This assumes special importance as the numeric methods to calculate the price typically show significant degradation when used to evaluate partial derivatives.

The approach we take to calculate the relevant partial derivatives relies on the works of Carr [3] and of Reiß and Wystup [16]. The first paper shows how to calculate spatial derivatives, i.e., derivatives with respect to the asset prices, by deriving the payoff function instead of the pricing formula. The second enables us to write the derivatives with respect to the other parameters in the model as functions of the spatial derivatives, in particular with respect to correlation parameters.

We start by writing the partial differential equation (PDE) implicit in the pricing formula (20) by using the Feynman-Kac theorem

$$V_t + \frac{1}{2}\sum_{i,j=1}^{n} \zeta_{\mathbb{S}_{I_i}\mathbb{S}_{I_j}} \mathbb{S}_{I_i}\mathbb{S}_{I_j} V_{\mathbb{S}_{I_i}\mathbb{S}_{I_j}} + \sum_{i=1}^{n} d_{\mathbb{S}_{I_i}} \mathbb{S}_{I_i} V_{\mathbb{S}_{I_i}} = r_k V, \tag{38}$$

where we removed the parameters of all the functions and processes to promote clarity and also the freeze time subscript making $\mathbb{S}_{I_i,t_i}(t) = \mathbb{S}_{I_i}$. n is the number of abstract assets in the model and the subscripts of V denote partial derivatives. See Björk [1] for details.

If we derive PDE (38) with respect to any $\mathbb{S}_{I_i}$ we get a PDE for the derivative function, the quantity needed for delta-hedging the claim. Consecutive derivations yield PDEs for all higher order spatial derivatives.

We now need to write the PDE (38) derivative with respect to an arbitrary sequence of variables. Hence, we write it as

$$V_{tD_p} + \frac{1}{2}\sum_{i,j=1}^{n} \zeta_{\mathbb{S}_{I_i}\mathbb{S}_{I_j}} \mathbb{S}_{I_i}\mathbb{S}_{I_j} V_{\mathbb{S}_{I_i}\mathbb{S}_{I_j}D_p} + \sum_{i=1}^{n} a_i(D_p)\mathbb{S}_{I_i} V_{\mathbb{S}_{I_i}D_p} = b(D_p)\, V_{D_p}, \tag{39}$$

with D_p denoting the sequence of derivations, formally $D_p = \prod_{h=1}^{p} \mathbb{S}_{I_h}$, and I_h an index of an abstract asset in the model. Additionally, a_i and b are functions of time t defined by

$$a_i(D_p) = d_{\mathbb{S}_{I_i}} + \sum_{h=1}^{p} \zeta_{\mathbb{S}_{I_i}\mathbb{S}_{I_h}}, \tag{40}$$

$$b(D_p) = r_k - \sum_{h=1}^{p} d_{\mathbb{S}_{I_h}} - \sum_{f=1}^{p} \sum_{g=f+1}^{p} \zeta_{\mathbb{S}_{I_f}\mathbb{S}_{I_g}}. \tag{41}$$

We list a_i and b in Table 1 for the first and second order derivatives.

Table 1: a_i and b for the first and second order derivatives.

	$p = 0$	$p = 1$	$p = 2$
D_p	1	$\mathbb{S}_{I_1}$	$\mathbb{S}_{I_1}\mathbb{S}_{I_2}$
$a_i(D_p)$	$d_{\mathbb{S}_{I_i}}$	$d_{\mathbb{S}_{I_i}} + \zeta_{\mathbb{S}_{I_i}\mathbb{S}_{I_1}}$	$d_{\mathbb{S}_{I_i}} + \zeta_{\mathbb{S}_{I_i}\mathbb{S}_{I_1}} + \zeta_{\mathbb{S}_{I_i}\mathbb{S}_{I_2}}$
$b(D_p)$	r_k	$r_k - d_{\mathbb{S}_{I_1}}$	$r_k - d_{\mathbb{S}_{I_1}} - d_{\mathbb{S}_{I_2}} - \zeta_{\mathbb{S}_{I_1}\mathbb{S}_{I_2}}$

It is worth noting that a_i and b of the first order derivatives recover the trend of all $\mathbb{S}_{I_i}$s under the measure $Q_{\mathbb{S}_{I_1}}$ in equation (18) and the deposit rate $r_{\mathbb{S}_{I_1}}$ in equation (16), respectively. The volatilities and covariances are trivially recovered since they do not change. Hence, the measure under which we should take the expectation of the first derivative of the payoff, with respect to $\mathbb{S}_{I_1}$, is the measure where $\mathbb{S}_{I_1}$ itself is the *numéraire*, i.e., $Q_{\mathbb{S}_{I_1}}$. Therefore, we denote the measure produced by the p-th order derivative as Q_{D_p} with the respective *numéraire* $D_p = \prod_{h=1}^{p} \mathbb{S}_{I_h}$ and deposit return rate $r_{D_p} = b(D_p)$.

To apply the Feynman-Kac theorem to the PDE (39), all we need is to calculate the respective boundary condition. We do so on a term by term basis of contract function (19)

$$\frac{\partial^p \Phi_k}{\partial D_p} = \sum_{i=1}^{n} c_i \frac{\partial^p \left(\mathbb{S}_{I_i,t_i}(T_i)\mathbb{I}_{C_i}\right)}{\partial D_p}. \tag{42}$$

We can now proceed to write an expression for the spatial derivatives.

Theorem 2.2. *Spatial derivatives of the pricing function of the $V(t(0))$ are given by the expression*

$$\frac{\partial^p V(t_0)}{\partial D_p(t_0)} = \sum_{i=1}^{n} c_i \exp\left\{-\int_{t_0}^{T_i} r_{D_p}(u)\mathrm{d}u\right\} E_{t_0}^{Q_{D_p}}\left[\frac{\partial^p \mathbb{S}_{I_i,t_i}(T_i)\mathbb{I}_{C_i}}{\partial D_p}\right]. \tag{43}$$

Despite its condensed look, this formula harbors some amount of complexity. To clarify, and for completeness, we write the evaluation formulas for the first order derivatives. We start by writing explicitly the first derivative of the contract function. To consider only sets C_i that yield closed form solutions, we use the definition in Proposition 2.2 as a product of indicator functions. Thus,

$$\mathbb{I}_{C_i} = \prod_{l=1}^{m_i} \mathbb{I}_{\left\{ \frac{\mathbb{S}_{I_{l_u},t_{l_u}}(T_i)}{\mathbb{S}_{I_{l_d},t_{l_d}}(T_i)} < h_l \right\}}. \tag{44}$$

To derive the contract function, all we need is to apply the product rule and to recall that

$$\frac{\partial \mathbb{I}_{\left\{ \frac{a}{b} < c \right\}}}{\partial b} = \delta(a - bc), \quad \text{and} \quad \frac{\partial \mathbb{I}_{\left\{ \frac{a}{b} < c \right\}}}{\partial a} = -\delta(a - bc), \tag{45}$$

with $\delta(x)$ the Dirac delta function.[2]

We find

$$\frac{\partial \Phi_k}{\partial \mathbb{S}_{I_x,t_x}(T_i)} = \sum_{i=1}^{n} c_i \Bigg(\mathbb{I}_{\{(I_x,t_x)=(I_i,t_i)\}} \mathbb{I}_{C_i}$$

$$+ \mathbb{S}_{I_i,t_i}(T_i) \Bigg(\sum_{j=1}^{m_i} \mathbb{I}_{\left\{(I_x,t_x)=(I_{j_d},t_{j_d})\right\}} \delta\left(\mathbb{S}_{I_{j_u},t_{j_u}}(T_i) - h_j \mathbb{S}_{I_{j_d},t_{j_d}}(T_i) \right)$$

$$\prod_{l \neq j}^{m_i} \mathbb{I}_{\left\{ \frac{\mathbb{S}_{I_{l_u},t_{l_u}}(T_i)}{\mathbb{S}_{I_{l_d},t_{l_d}}(T_i)} < h_l \right\}}$$

$$- \sum_{j=1}^{m_i} \mathbb{I}_{\{(I_x,t_x)=(I_{j_u},t_{j_u})\}} \delta\left(\mathbb{S}_{I_{j_u},t_{j_u}}(T_i) - h_j \mathbb{S}_{I_{j_d},t_{j_d}}(T_i) \right)$$

$$\prod_{l \neq j}^{m_i} \mathbb{I}_{\left\{ \frac{\mathbb{S}_{I_{l_u},t_{l_u}}(T_i)}{\mathbb{S}_{I_{l_d},t_{l_d}}(T_i)} < h_l \right\}} \Bigg) \Bigg).$$

Taking advantage of the fact that $Q_{D_p} = Q_{\mathbb{S}_{I_x,t_x}}$, the first order derivative formula turns out to be

[2] The Dirac delta function is characterized by the two properties

$$\delta(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{+\infty} \delta(x) \mathrm{d}x = 1.$$

$$\frac{\partial V(t_0)}{\partial \mathbb{S}_{I_x,t_x}(t_0)} = \exp\left\{-\int_{t_0}^{T_i} r_{\mathbb{S}_{I_x,t_x}}(u)\mathrm{d}u\right\}\cdot \tag{46a}$$

$$\left(\sum_{i=1}^{n} \mathbb{I}_{\{(I_x,t_x)=(I_i,t_i)\}} c_i \mathcal{N}_{m_i}^{\mathbb{S}_{I_x,t_x}}(v;\phi,\Sigma)\right. \tag{46b}$$

$$+\sum_{i=1}^{n} c_i \mathbb{S}_{I_i,t_i}(t_0) \sum_{j=1}^{m_i} \mathbb{I}_{\{(I_x,t_x)=(I_{j_d},t_{j_d})\}} \mathcal{N}_{m_i}^{\mathbb{S}_{I_x,t_x}}\left(v;\phi,\Sigma \mid \frac{\mathbb{S}_{I_{j_u},t_{j_u}}(T_i)}{\mathbb{S}_{I_{j_d},t_{j_d}}(T_i)} = h_j\right) \tag{46c}$$

$$\left.-\sum_{i=1}^{n} c_i \mathbb{S}_{I_i,t_i}(t_0) \sum_{j=1}^{m_i} \mathbb{I}_{\{(I_x,t_x)=(I_{j_u},t_{j_u})\}} \mathcal{N}_{m_i}^{\mathbb{S}_{I_x,t_x}}\left(v;\phi,\Sigma \mid \frac{\mathbb{S}_{I_{j_u},t_{j_u}}(T_i)}{\mathbb{S}_{I_{j_d},t_{j_d}}(T_i)} = h_j\right)\right). \tag{46d}$$

For performance reasons, it is important to observe that the probabilities $\mathcal{N}_{m_i}^{\mathbb{S}_{I_x,t_x}}(v;\phi,\Sigma)$ in (46b) are also calculated in the context of the pricing function.

In order to recover the derivatives with respect to real asset prices S_l, all we need is to apply the chain rule. Thus,

$$\frac{\partial V(t_0)}{\partial S_l(t_0)} = \sum_{x=1}^{n} \frac{\partial V(t_0)}{\partial \mathbb{S}_{I_x,t_x}(t_0)} \frac{\partial \mathbb{S}_{I_x,t_x}(t_0)}{\partial S_l(t_0)}. \tag{47}$$

The first factor in the summation is the one we derived above; the second factor is a simple derivative that either yields zero, if $S_l(t_0)$ no longer affects $\mathbb{S}_{I_x,t_x}(t_0)$, or yields $\mathbb{S}_{I_x,t_x}(t_0)\frac{\alpha_l}{S_l(t_0)}$ otherwise, with α_l as in the definition (8).

Finally, we can use the result from Reiß and Wystup [16] to calculate the derivatives with respect to the other model parameters. As an example, in a model with constant volatilities and correlations, a derivative with respect to the correlation between two asset prices is given by

$$\frac{\partial V(t_0)}{\partial \rho_{jk}} = S_j(t_0)S_k(t_0)\sigma_j\sigma_k \frac{\partial^2 V(t_0)}{\partial S_j(t_0)\partial S_k(t_0)}(t^* - t_0), \tag{48}$$

With t^* the maximum t, with $t_0 \leq t \leq T$, such that both $S_j(t)$ and $S_k(t)$ still influence the payoff function Φ_k.

3 Applications

We believe that this paper provides relevant contributions to several practical problems. First of all, it offers a multi-currency, multi-asset model description fit for implementation. The model itself is of the Black–Scholes type with time dependent parameters. The general description of the contract payoff allows for implementations where each instrument is defined through a payoff language. Such a payoff language enables addition of new instruments without additional development of the application. The payoff profiles that are covered by the general form of the contract, in (19) and (28), are the following: European style vanilla options, exchange options, digital options, forward start and cliquet options, options on the n^{th}-best/worst, options on the discretely observed maximum/minimum, most types of mountain range options, discrete barriers and lookbacks, power options and combinations of these. It allows the use of the following prices as underlying assets: stocks prices denominated on domestic currency, foreign currency (quanto), and foreign currency translated to domestic, as well as geometric averages of stocks prices to produce geometric Asian options or geometric basket options. These last two types are not as common in the industry as their arithmetic counterparts, but their prices are still very useful as control variates, which are very effective in reducing the variance of Monte Carlo simulations of the arithmetic versions. To illustrate the breadth of instruments covered by the contract definition above, we provide below a series of examples.

3.1 Performance

As the pricing formula for the contract requires several evaluations of multivariate normal probabilities, it is crucial to weight its computational cost against that of the alternative methods. To calculate the multivariate normal, we used the method developed by Genz [7]. The alternative, as far as we know, is only a Monte Carlo simulation that may, or may not, include variance reduction techniques. However, due to the fact that the convergence of Monte Carlo simulations depends strongly on the payoff profile of the contract, it is impossible to run a performance comparison valid for the contract's general form (28). Therefore, we shall provide only case based performance analyses in each example of section 3.2. For a performance comparison focused only on the calculation of the multivariate normal probabilities, we refer to Genz [7]. The Genz method also relies on a Monte Carlo simulation but does so in the context of a chain of unidimensional integrals. For this reason, the closed formula prices of the examples below will also show an error term.

In most cases, we have encountered that the closed formula outperforms the Monte Carlo simulation, though to different degrees depending on several factors. The addition of asset prices to the payoff implies an increase in the number of dimensions of both procedures, although it generally weighs heavier on Monte Carlo simulation. The presence of several time points at which stock prices are observed to compose the claim's payoff greatly increases the dimensionality of the Monte Carlo simulation, degrading thus its performance. Several time points also have an impact on the closed formula alternative, as they give rise to highly correlated random variables. The complexity of the payoff may require the evaluation of a large number of summands in (37), thus worsening the performance of the closed formula while not necessarily changing the Monte Carlo's performance.

Finally, the integrals of the parameter functions r, σ and ς typically have closed form solutions, as their definition is usually done as piece-wise linear functions or functions that have indefinite integrals. Therefore, its calculation has a residual impact on the overall computation time. The same is true, by definition, for integrals of δ, θ and ζ functions.

3.2 Examples

Our market setup for the cases included in this section is described as follows. The *numéraire* asset is chosen to be the asset in which the options pay off. It is the same for all options, and it yields risk free returns at the rate of 5%. We then have 5 currencies with risk free yields, from the first to the last, of 1%, 2%,...,5% respectively. The price of each currency is expressed in terms of the *numéraire* asset (in the natural direction) and they have volatilities, from the first to the last, of 11%, 12%, ..., 15% respectively. The correlation between the currencies' prices is 20% for all combinations. There are also five equity indexes that yield risk free returns, from the first to the last, of 2%, 4%, ...,10% respectively. The price of each index is expressed (in the natural direction) in terms of the currency with the same cardinal as the index. All indexes start with a price of 100. The volatilities of each index, from the first to the last, are 22%, 24%, ..., 30% respectively. The correlation between any two indexes is 60%. The correlation between any combination of index and currency is 10%.

We consider four options: a cliquet on the first index, a best of five indexes, a discrete lookback on the first index, and a Himalaya on the first three indexes. All options have a maturity of one year, $T = 1$, $t_0 = 0$.

The cliquet option has five periods of equal length. Hence, it can be viewed as a portfolio of a vanilla at-the-money (spot) option plus four forward start at-the-money (spot) options. As vanillas and forward start options involve only one condition, the cliquet option is evaluated instantly. In fact, in this case, the general formula (37) reduces to the known closed formula for cliquets.

$$\Phi = \sum_{n=1}^{5} \Phi_n\left(\frac{n}{5}T\right), \quad \Phi_n\left(\frac{n}{5}T\right) = \left(S_1\left(\frac{n}{5}T\right) - S_1\left(\frac{n-1}{5}T\right)\right)^+.$$

The best of five pays off the difference, if positive, between the maximum of the five index values at maturity and 100.

$$\Phi(T) = (\max(S_1(T), \ldots, S_5(T)) - 100)^+ .$$

The discrete lookback pays off the difference, if positive, of the highest stored value of the first index and 100. The index values are stored 12 times during the year at evenly spaced times, starting at 1/12.

$$\Phi(T) = \left(\max\left(S_1\left(\frac{1}{12}T\right), S_1\left(\frac{2}{12}T\right) \ldots, S_1\left(\frac{12}{12}T\right)\right) - 100\right)^+$$

At the end of each period of 1/3 units of time, the Himalaya option pays off the best return of the three first indexes over that period times 100, but only if the best return is positive. The indexes that pay out are not considered for any of the subsequent periods.

$$\Phi = \sum_{n=1}^{3} \Phi_n\left(\frac{n}{3}T\right),$$

$$\Phi_n\left(\frac{n}{3}T\right) = 100 \max\left(0, \eta_{n,1}\frac{S_1\left(\frac{n}{3}T\right)}{S_1\left(\frac{n-1}{3}T\right)}, \eta_{n,2}\frac{S_2\left(\frac{n}{3}T\right)}{S_2\left(\frac{n-1}{3}T\right)}, \eta_{n,3}\frac{S_3\left(\frac{n}{3}T\right)}{S_3\left(\frac{n-1}{3}T\right)}\right),$$

where $\eta_{n,i}$ equals 0 if the asset i has determined the payout of one of the payments at any time $t < \frac{n}{3}T$, and 1 otherwise.

The parameterization of these payoff functions, including the set definition for each of the terms in the payoff summation, is given in the appendix.

To assess the performance of the closed formula, we benchmark the results against a Monte Carlo experiment. The results are shown in Table 2. The Table shows a price estimate and a 99% confidence error bound expressed in percentage of the price estimate. The pricing routines were allowed to run for 10 seconds and for five minutes.

Table 2: Tests results.

	Cliquet		Best of 5		Lookback		Himalaya	
calculation time = 10"								
MC	18.27	0.53%	19.16	0.43%	13.50	1.03%	174.46	0.57%
CF	18.33	—	19.16	0.29%	13.47	1.24%	173.90	0.05%
calculation time = 5'								
MC	18.33	0.10%	19.16	0.08%	13.50	0.19%	173.97	0.10%
CF	18.33	—	19.15	0.05%	13.51	0.23%	173.93	0.01%

The results show that the closed formula is superior in all cases but the lookback. The cliquet case just shows that the general formula is able to produce the already known formulas, namely for vanilla options, exchange options, forward starts, digitals and others of European style that constitute unidimensional problems. The best-of-5 is an example with low correlation between random variables, in this case between different stocks, and only one time horizon, the maturity date. The closed formula increases the precision by a factor of 1.45(=0.427%/0.294%). Hence, considering the rate of convergence of the Monte Carlo, the closed formula is $2.11(=1.45^2)$ times faster. In the Himalaya case, the performance is even more extreme with the precision increasing by a factor of 10.52(=0.571%/0.543%) or, equivalently, $111(=10.52^2)$ times faster. The Himalaya is a case in which the closed formula performs particularly well. Even though it requires the evaluation of 63 cumulative probability functions, they are of low dimensionality, 6.9 on average, while the Monte Carlo engine needs to account for a 9 dimensional problem (3 stocks observed at 3 time horizons). In the Lookback case, the dimensionality was 12 for both methods and required the evaluation of 13 cumulative probability functions.

The Lookback result came as a suprise as the closed formula performed worse than in the Monte Carlo simulation. To figure out what was causing the poor performance, we applied two variations to the initial problem. We first diminished the number of observation points to 4 to test if the dimensionality constituted a problem. Then we enlarged the time between two observations from 1 month to 3 months. The results for 5 minute simulations are listed in Table 3.

Table 3: Lookback results.

	Observations							
	12				4			
	MC		CF		MC		CF	
$\Delta t = 1/12$	13.50	0.19%	13.51	0.23%	6.85	0.09%	6.85	0.10%
$\Delta t = 3/12$	20.99	0.20%	20.99	0.14%	11.33	0.10%	11.34	0.06%

These results lead us to conclude that the closed formula does not provide better performance when the time between observations is small and starts to perform better the larger the time between observations. Small intervals between observations give rise to highly correlated random variables, the asset prices at each observation moment. Such cases are known to carry convergence problems for numerical procedures, and thus it is not suprising that the multivariate normal numerical procedure performance shows degradation. What is suprising though is that it shows worst results than the Monte Carlo simulation, which also suffers from the same effect as it is also a numerical procedure.

4 Conclusion and Future Research

The results above produce a closed formula that generalizes a large class of multivariate European style options, ranging from the plain vanilla to mountain range options. It does so in a generalized Black-Scholes model, with time dependent parameters, able to cope with an arbitrary number of currency zones. It introduces the concept of abstract assets as an intermediate random variable that allows the formula to cover variations like geometric averages, baskets, asset prices expressed in foreign currencies, and forward start features. In fact, abstract assets are a useful generalization of the asset concept and should be considered as a replacement of plain assets in Monte Carlo engines.

The closed formula performs better than the alternative Monte Carlo simulations in most cases, improving performance by more than 100 times in the most extreme. However, for problems with highly correlated random variables the performance was worse than Monte Carlo's. The examples show that even when the closed formula requires the evaluation of a large number of cumulative probability functions, it still outperforms Monte Carlo.

As a byproduct of the definition of the closed formula, a language for option payoff definition arises. This language acquires extreme importance in industrial systems as it enables the decoupling of the payoff definition function from the pricing function. Hence, the pricing function is able to price any option as long as its payoff is expressable in terms of the language.

Future research should focus on including other features of options for which there are closed formulas, namely continuous barrier and lookback features. The problem with including barrier options in the general formula above is that it requires the knowledge of the joint distribution of a Brownian motion with time dependent drift and its running maximum. The results on Brownian motion with constant drift are applicable neither to models with time dependent parameters nor to abstract asset's dynamics.

Appendix

The payoff parameterization of the options considered in the examples section above follows expression (19) and uses that same notation. The C_i set definition follows expression (28) and also uses its notation. For each term in the payoff (19), it is still required to select if the set C_i or its complement, $\overline{C_i}$, determines the payment.

The index of the *numéraire* asset is $I_. = 0$. As the examples do not include payments of currency prices, we index the underlying equity indexes with $I_. = 1, \ldots, 5$.

Cliquet Option

This option payoff is composed of 10 terms. The terms follow a structure that can be summarized by iterating $t = 0.2, 0.4, 0.6, 0.8, 1$. In this case the sets C_i have only one condition for all terms, i.e., $m_i = 1$.

Term $i = 1, 3, 5, 7, 9$ *(strike payment for each* t*)*

c_i	I_i	t_i	T_i	Set Complement Flag
-1	1	$t - 0.2$	t	*false*

Set C_i

I_{l_u}	t_{l_u}	I_{l_d}	t_{l_d}	h_l
1	$t - 0.2$	1	t	1

Term $i = 2, 4, 6, 8, 10$ *(index price reception for each* t*)*

c_i	I_i	t_i	T_i	Set Complement Flag
1	1	t	t	*false*

Set C_i

I_{l_u}	t_{l_u}	I_{l_d}	t_{l_d}	h_l
1	$t - 0.2$	1	t	1

Best of 5

This option payoff is composed of 6 terms, five for the reception of each of the five possible maximum index prices at maturity, and one payment of the exercise price 100.

The terms for each of the five index payments follow the following rule. Let $a_i = 1, \ldots, 5$ and let $b_{i,1}, \ldots, b_{i,4}$ be the elements of the set $\{1, 2, 3, 4, 5\} \setminus a_i$.

The terms for reception of each of the five index prices is parameterized by

Term $i = 1, \ldots, 5$ *(index price reception)*

c_i	I_i	t_i	T_i	Set Complement Flag
1	a_i	1	1	*false*

Set C_i

I_{l_u}	t_{l_u}	I_{l_d}	t_{l_d}	h_l
0	0	a_i	1	1/100
$b_{i,1}$	1	a_i	1	1
$b_{i,2}$	1	a_i	1	1
$b_{i,3}$	1	a_i	1	1
$b_{i,4}$	1	a_i	1	1

The term of the strike payment is parameterized as

Term i=6 (strike payment)

c_i	I_i	t_i	T_i	Set Complement Flag
-100	0	1	1	*true*

Set C_6

I_{l_u}	t_{l_u}	I_{l_d}	t_{l_d}	h_l
a_1	1	0	0	100
a_2	1	0	0	100
a_3	1	0	0	100
a_4	1	0	0	100
a_5	1	0	0	100

Discrete Lookback

This option payoff is composed of 13 terms, 12 for the reception of each of the 12 possible maximum values of the index prices during the life of the option, and one payment of the exercise price 100.

The terms for each of the 12 index payments follow the following rule. Let $u_i = 1/12,\ 2/12, \ldots, 12/12$ and let $v_{i,1}, \ldots, v_{i,11}$ be the elements of the set $\{1/12, 2/12, \ldots, 12/12\} \setminus u_i$.

The terms for reception of each of the 12 possible maximum index prices is parameterized by

Term $i = 1, \ldots, 12$ (index price reception)

c_i	I_i	t_i	T_i	Set Complement Flag
1	1	u_i	1	*false*

Set C_i

I_{l_u}	t_{l_u}	I_{l_d}	t_{l_d}	h_l
0	0	1	u_i	1/100
1	$v_{i,1}$	1	u_i	1
1	$v_{i,2}$	1	u_i	1
		$\vdots$		
1	$v_{i,11}$	1	u_i	1

The term of the strike payment is parameterized as

Term i=13 (strike payment)

c_i	I_i	t_i	T_i	Set Complement Flag
-100	0	0	1	*true*

Set C_{13}

I_{l_u}	t_{l_u}	I_{l_d}	t_{l_d}	h_l
1	u_1	0	0	100
1	u_2	0	0	100
		$\vdots$		
1	u_{12}	0	0	100

Himalaya

This option has payments at three distinct times. For each of the periods that end at these payment dates, only the period return matters and not the accumulated return since inception. We represent each of these returns with an abstract asset price of the form

$$\mathbb{S}_j = S_{j,\frac{n}{3}T} / S_{j,\frac{n-1}{3}T},$$

the ratio of two versions of the same price process frozen at different moments in time. The numerator version, frozen at the end and the denominator at the beginning of the reference return period. As we have 3 assets and 3 return periods, we have 9 abstract assets for all combinations of both.

We shall index the abstract assets that represent the first period return on the first 3 assets by $a_{1,i} = 1, 2, 3$. The returns of the second period returns are indexed as $a_{2,i} = 4, 5, 6$. Finally, the returns of the third period are indexed by $a_{3,i} = 7, 8, 9$, with $i = 1, 2, 3$. We also have $b_{j,i,1}, b_{j,i,2}$, the elements of the set $\{a_{j,1}, a_{j,2}, a_{j,3}\} \setminus a_{j,i}$.

For the payments at the end of the first period, at $t = T/3$ we have 3 terms in the payoff function.

Term $i = 1, 2, 3$ (index price return reception, $t = T/3$)

c_i	I_i	t_i	T_i	Set Complement Flag
100	$a_{1,i}$	1	1	*false*

Set C_i

I_{l_u}	t_{l_u}	I_{l_d}	t_{l_d}	h_l
$a_{1,i}$	0	$a_{1,i}$	1	1
$b_{1,i,1}$	1	$a_{1,i}$	1	1
$b_{1,i,2}$	1	$a_{1,i}$	1	1

For the payments at the end of the first period, at $t = 2T/3$ we have 12 terms in the payoff function. For each of the 3 possible return payments there are 4 terms, all with the same asset payment and the 4 sets C_i below.

Term i, with $i = 1, 2, 3$ (index price return reception, $t = 2T/3$)

c_i	I_i	t_i	T_i	Set Complement Flag
100	$a_{2,i}$	2	2	*false*

Sets C_i, with g_1, g_2 the elements of all the possible permutations of the elements of the set $\{1, 2\}$

I_{l_u}	t_{l_u}	I_{l_d}	t_{l_d}	h_l
$a_{2,i}$	1	$a_{2,i}$	2	1
$b_{2,i,1}$	2	$a_{2,i}$	2	1
$b_{2,i,2}$	2	$a_{2,i}$	2	1
$a_{1,i}$	1	$b_{1,i,1}$	1	1
$a_{1,i}$	1	$b_{1,i,2}$	1	1

I_{l_u}	t_{l_u}	I_{l_d}	t_{l_d}	h_l
$a_{2,i}$	1	$a_{2,i}$	2	1
$b_{2,i,1}$	2	$a_{2,i}$	2	1
$b_{2,i,2}$	2	$a_{2,i}$	2	1
b_{1,i,g_1}	1	$a_{1,i}$	1	1
$a_{1,i}$	1	b_{1,i,g_2}	1	1

I_{l_u}	t_{l_u}	I_{l_d}	t_{l_d}	h_l
$a_{2,i}$	1	$a_{2,i}$	2	1
$b_{2,i,1}$	2	$a_{2,i}$	2	1
$b_{2,i,2}$	2	$a_{2,i}$	2	1
$b_{1,i,1}$	1	$a_{1,i}$	1	1
$b_{1,i,2}$	1	$a_{1,i}$	1	1
$a_{1,i}$	1	$a_{1,i}$	0	1

For the payments at the end of the first period, at $t = 3T/3$ we have 48 terms in the payoff function. For each of the 3 possible return payments there are 16 terms, all with the same asset payment and the 16 sets C_i below.

Term i, with $i = 1, 2, 3$ (index price return reception, $t = 3T/3$)

c_i	I_i	t_i	T_i	Set Complement Flag
100	$a_{3,i}$	3	3	*false*

Sets C_i, with g_1, g_2 the elements of all the possible permutations of the *elements of the set* $\{1,2\}$*, and* h_1, h_2 *also the elements of a similar permutation.*

I_{l_u}	t_{l_u}	I_{l_d}	t_{l_d}	h_l
$a_{3,i}$	1	$a_{3,i}$	2	1
$b_{3,i,1}$	2	$a_{3,i}$	2	1
$b_{3,i,2}$	2	$a_{3,i}$	2	1
$a_{1,i}$	1	$b_{1,i,1}$	1	1
$a_{1,i}$	1	$b_{1,i,2}$	1	1
$a_{2,i}$	2	$b_{2,i,1}$	2	1
$a_{2,i}$	2	$b_{2,i,2}$	2	1

I_{l_u}	t_{l_u}	I_{l_d}	t_{l_d}	h_l
$a_{3,i}$	1	$a_{3,i}$	2	1
$b_{3,i,1}$	2	$a_{3,i}$	2	1
$b_{3,i,2}$	2	$a_{3,i}$	2	1
$a_{1,i}$	1	$b_{1,i,1}$	1	1
$a_{1,i}$	1	$b_{1,i,2}$	1	1
$a_{2,i}$	2	b_{2,i,g_1}	2	1
b_{2,i,g_2}	2	$a_{2,i}$	2	1

I_{l_u}	t_{l_u}	I_{l_d}	t_{l_d}	h_l
$a_{3,i}$	1	$a_{3,i}$	2	1
$b_{3,i,1}$	2	$a_{3,i}$	2	1
$b_{3,i,2}$	2	$a_{3,i}$	2	1
$a_{1,i}$	1	b_{1,i,h_1}	1	1
b_{1,i,h_2}	1	$a_{1,i}$	1	1
$a_{2,i}$	2	$b_{2,i,1}$	2	1
$a_{2,i}$	2	$b_{2,i,2}$	2	1

I_{l_u}	t_{l_u}	I_{l_d}	t_{l_d}	h_l
$a_{3,i}$	1	$a_{3,i}$	2	1
$b_{3,i,1}$	2	$a_{3,i}$	2	1
$b_{3,i,2}$	2	$a_{3,i}$	2	1
$a_{1,i}$	1	b_{1,i,h_1}	1	1
b_{1,i,h_2}	1	$a_{1,i}$	1	1
$a_{2,i}$	2	b_{2,i,g_1}	2	1
b_{2,i,g_2}	2	$a_{2,i}$	2	1

I_{l_u}	t_{l_u}	I_{l_d}	t_{l_d}	h_l
$a_{3,i}$	1	$a_{3,i}$	2	1
$b_{3,i,1}$	2	$a_{3,i}$	2	1
$b_{3,i,2}$	2	$a_{3,i}$	2	1
$a_{1,i}$	1	$b_{1,i,1}$	1	1
$a_{1,i}$	1	$b_{1,i,2}$	1	1
$b_{2,i,1}$	2	$a_{2,i}$	2	1
$b_{2,i,2}$	2	$a_{2,i}$	2	1
$a_{2,i}$	2	$a_{2,i}$	1	1

I_{l_u}	t_{l_u}	I_{l_d}	t_{l_d}	h_l
$a_{3,i}$	1	$a_{3,i}$	2	1
$b_{3,i,1}$	2	$a_{3,i}$	2	1
$b_{3,i,2}$	2	$a_{3,i}$	2	1
$a_{1,i}$	1	b_{1,i,g_1}	1	1
b_{1,i,g_2}	1	$a_{1,i}$	1	1
$b_{2,i,1}$	2	$a_{2,i}$	2	1
$b_{2,i,2}$	2	$a_{2,i}$	2	1
$a_{2,i}$	2	$a_{2,i}$	1	1

I_{l_u}	t_{l_u}	I_{l_d}	t_{l_d}	h_l
$a_{3,i}$	1	$a_{3,i}$	2	1
$b_{3,i,1}$	2	$a_{3,i}$	2	1
$b_{3,i,2}$	2	$a_{3,i}$	2	1
$b_{1,i,1}$	1	$a_{1,i}$	1	1
$b_{1,i,2}$	1	$a_{1,i}$	1	1
$a_{1,i}$	1	$a_{1,i}$	0	1
$a_{2,i}$	2	$b_{2,i,1}$	2	1
$a_{2,i}$	2	$b_{2,i,2}$	2	1

I_{l_u}	t_{l_u}	I_{l_d}	t_{l_d}	h_l
$a_{3,i}$	1	$a_{3,i}$	2	1
$b_{3,i,1}$	2	$a_{3,i}$	2	1
$b_{3,i,2}$	2	$a_{3,i}$	2	1
$b_{1,i,1}$	1	$a_{1,i}$	1	1
$b_{1,i,2}$	1	$a_{1,i}$	1	1
$a_{1,i}$	1	$a_{1,i}$	0	1
$a_{2,i}$	2	b_{2,i,g_1}	2	1
b_{2,i,g_2}	2	$a_{2,i}$	2	1

I_{l_u}	t_{l_u}	I_{l_d}	t_{l_d}	h_l
$a_{3,i}$	1	$a_{3,i}$	2	1
$b_{3,i,1}$	2	$a_{3,i}$	2	1
$b_{3,i,2}$	2	$a_{3,i}$	2	1
$b_{1,i,1}$	1	$a_{1,i}$	1	1
$b_{1,i,2}$	1	$a_{1,i}$	1	1
$a_{1,i}$	1	$a_{1,i}$	0	1
$b_{2,i,1}$	2	$a_{2,i}$	2	1
$b_{2,i,2}$	2	$a_{2,i}$	2	1
$a_{2,i}$	2	$a_{2,i}$	1	1

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