Efficient Computation of Option Price Sensitivities Using Homogeneity and other Tricks

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1 Introduction

1.1 applications

1. saving time in computing derivatives.
2. robust implementation compared to Greeks via difference quotients.
3. check the quality and consistency of Greeks produced by finite-difference-, tree- or Monte Carlo methods.
4. computation of Greeks for Monte Carlo based values.
5. relationships between Greeks which wouldn't be noticed merely by looking at difference quotients.

1.2 Notation

S	stock price or stock price process
B	cash bond, usually with risk free interest rate r
r	risk free interest rate
q	dividend yield (continuously paid)
σ	volatility of one stock, or volatility matrix of several stocks
ρ	correlation in the two-asset market model
t	date of evaluation ("today")
T	date of maturity
$\tau = T - t$	time to maturity of an option
x	stock price at time t
$f(\cdot)$	payoff function
$v(x, t, \dots)$	value of an option
k	strike of an option
l	level of an option
v_x	partial derivation of v with respect to x (and analogous)

The standard normal distribution and density functions are defined by

$$n(t) \triangleq \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} \tag{1}$$

$$\mathcal{N}(x) \triangleq \int_{-\infty}^{x} n(t)\,dt \tag{2}$$

$$n_2(x,y;\rho) \triangleq \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)}\right) \tag{3}$$

$$\mathcal{N}_2(x,y;\rho) \triangleq \int_{-\infty}^{x}\int_{-\infty}^{y} n_2(u,v;\rho)\,du\,dv \tag{4}$$

See the front office section of `http://www.MathFinance.de` for a source code to compute $\mathcal{N}_2$.

1.3 The Greeks

Delta	Δ	v_x	
Gamma	Γ	v_{xx}	
Theta	Θ	v_t	
Rho	ρ	v_r	in the one-stock model
Rhor	ρ_r	v_r	in the two-stock model
Rhoq	ρ_q	v_q	
Vega	Φ	v_σ	
Kappa	κ	v_ρ	correlation sensitivity (two-stock model)

Greeks, not so commonly used:

Leverage	λ	$\frac{x}{v} v_x$	sometimes Ω, sometimes called "gearing"
Vomma / Volga	Φ'	$v_{\sigma\sigma}$	
Speed		v_{xxx}	
Charm		v_{xt}	
Color		v_{xxt}	
Cross / Vanna		$v_{x\sigma}$	
Forward Delta	Δ^F	v_F	
Driftless Delta	Δ^{dl}	$\Delta e^{q\tau}$	
Dual Theta	DualΘ	v_T	
Strike Delta	Δ^k	v_k	
Strike Gamma	Γ^k	v_{kk}	
Level Delta	Δ^l	v_l	
Level Gamma	Γ^l	v_{ll}	
Beta	β_{12}	$\frac{\sigma_1}{\sigma_2} \rho$	two-stock model

2 Fundamental Properties

2.1 Homogeneity of Time

In most cases the price of the option is not a function of both the current time t **and** the maturity time T, but rather **only** a function of the time to maturity $\tau = T - t$ implying the relations

$$\Theta = v_t = -v_\tau = -v_T = -\text{Dual}\Theta. \tag{5}$$

This relationship extends naturally to the situation of options depending on several intermediate times such as compound or Bermuda options.

2.2 Scale-Invariance of Time

measure time in units other than years in which case interest rates and volatilities, which are normally quoted on an annual basis, must be changed according to the following rules for all $a > 0$.

$$\begin{aligned} \tau &\rightarrow \frac{\tau}{a} \\ r &\rightarrow ar \\ q &\rightarrow aq \\ \sigma &\rightarrow \sqrt{a}\sigma \end{aligned} \tag{6}$$

The option's value must be invariant under this rescaling, i.e.,

$$v(x, \tau, r, q, \sigma, \dots) = v(x, \frac{\tau}{a}, ar, aq, \sqrt{a}\sigma, \dots) \tag{7}$$

We differentiate this equation with respect to a and obtain for $a = 1$

$$0 = \tau\Theta + r\rho + q\rho_q + \frac{1}{2}\sigma\Phi, \tag{8}$$

a general relation between the Greeks *theta, rho, rhoq* and *vega*.

Based on the relation

$$v(x_1,...,x_n,\tau,r,q_1,...,q_n,\sigma_{11},...,\sigma_{nn}) = \\ v(x_1,...,x_n,\frac{\tau}{a},ar,aq_1,...,aq_n,\sqrt{a}\sigma_{11},...,\sqrt{a}\sigma_{nn}) \tag{9}$$

we obtain

Theorem 1 *(scale invariance of time)*

$$0 = \tau\Theta + r\rho + \sum_{i=1}^{n} q_i\rho_{q_i} + \frac{1}{2}\sum_{i,j=1}^{n} \Phi_{ij}\sigma_{ij}, \tag{10}$$

where Φ_{ij} *denotes the differentiation of* v *with respect to* σ_{ij}.

2.3 Scale Invariance of Prices

The general idea is that value of securities may be measured in a different unit, just like values of European stocks are now measured in Euro instead of in-currencies. Option contracts usually depend on strikes and barrier levels. Rescaling can have different effects on the value of the option. Essentially we may consider the following types of homogeneity classes. Let $v(x, k)$ be the value function of an option, where x is the spot (or a vector of spots) and k the strike or barrier or a vector of strikes or barriers. Let a be a positive real number.

Definition 1 (homogeneity classes) *We call a value function k-homogeneous of degree n if for all $a > 0$*

$$v(ax, ak) = a^n v(x, k). \tag{11}$$

We call an options whose value function is strike-homogeneous of degree 1 *a* strike-defined option *and similarly an option whose value function is level-homogeneous of degree* 0 *a* level-defined option.

The value function of a European call or put option with strike K is then K-homogeneous of degree 1, a digital option which pays a fixed amount if the stock price is higher than a level L is L-homogeneous of degree 0. The path-independent barrier call option paying $(S-k)^+ I_{\{S>K\}}$ is (k, K)-homogeneous of degree 1. A power call with cap paying $\min(C, ((S-K)^+)^2)$ has a homogeneity structure of $v(aS, aK, a^2C) = a^2 v(S, K, C)$.

2.3.1 Strike-Delta and Strike-Gamma

For a strike-defined value function we have for all $a, b > 0$

$$abv(x,k) \quad = \quad v(abx, abk). \tag{12}$$

We differentiate with respect to a and get for $a = 1$

$$bv(x,k) \quad = \quad bxv_x(bx,bk) + bkv_k(bx,bk). \tag{13}$$

We now differentiate with respect to b get for $b = 1$

$$\begin{aligned} v(x,k) &= xv_x + xv_{xx}x + xv_{xk}k + kv_k + kv_{kx}x + kv_{kk}k & (14)\\ &= x\Delta + x^2\Gamma + 2xkv_{xk} + k\Delta^k + k^2\Gamma^k. & (15)\end{aligned}$$

If we evaluate equation (13) at $b = 1$ we get

$$v = x\Delta + k\Delta^k. \tag{16}$$

We differentiate this equation with respect to k and obtain

$$\begin{aligned} \Delta^k &= xv_{kx} + \Delta^k + k\Gamma^k, & (17)\\ kxv_{kx} &= -k^2\Gamma^k. & (18)\end{aligned}$$

Together with equation (15) we conclude

$$x^2\Gamma = k^2\Gamma^k. \tag{19}$$

2.3.2 Level-Delta and Level-Gamma

For a level-defined value function we have for all $a, b > 0$

$$v(x, l) = v(abx, abl). \tag{20}$$

We differentiate with respect to a and get at $a = 1$

$$0 = v_x(bx, bl)bx + v_l(bx, bl)bl. \tag{21}$$

If we set $b = 1$ we get the relation

$$\Delta x + \Delta^l l = 0. \tag{22}$$

Now we differentiate equation (21) with respect to b and get at $b = 1$

$$0 = v_{xx}x^2 + 2v_{xl}xl + v_{ll}l^2. \tag{23}$$

One the other hand we can differentiate the relation between delta and level-delta with respect to l and get

$$v_{xl}x + l\Gamma^l + \Delta^l = 0. \tag{24}$$

Together with equation (23) we conclude

$$x^2\Gamma + x\Delta = l^2\Gamma^l + l\Delta^l. \tag{25}$$

In general we obtain

Theorem 2 *(price homogeneity)*

$$v = \sum_{i=1}^{n} x_i \Delta_i + \sum_{j=1}^{m} k_j \Delta_j^k \qquad (26)$$

$$\sum_{i,j=1}^{n} x_i x_j \Gamma_{ij} = \sum_{i,j=1}^{m} k_i k_j \Gamma_{ij}^k \qquad (27)$$

for strike-defined options and

$$0 = \sum_{i=1}^{n} x_i \Delta_i + \sum_{j=1}^{m} l_j \Delta_j^l \qquad (28)$$

$$\sum_{i,j=1}^{n} x_i x_j \Gamma_{ij} + \sum_{i=1}^{n} x_i \Delta_i = \sum_{i,j=1}^{m} l_i l_j \Gamma_{ij}^l + \sum_{i=1}^{m} l_i \Delta_i^l \qquad (29)$$

for level-defined options.

3 European Options in the Black-Scholes Model

We start with relations among Greeks for European claims in the n-dimensional Black-Scholes model

$$\begin{aligned} dS_i(t) &= S_i(t)[(r-q_i)\,dt + \sigma_i\, dW_i(t)], \quad i=1,\dots,n \quad (30) \\ \mathbf{Cov}(W_i(t), W_j(t)) &= \rho_{ij}t, \quad (31) \end{aligned}$$

where r is the risk-free rate, q_i the dividend rate of asset i or foreign interest rate of exchange rate i, σ_i the volatility of asset i and $(W_1, \dots, W_n)$ a standard Brownian motion (under the risk-neutral measure) with correlation matrix ρ. Let v denote today's value of the payoff $f(S_1(T), \dots, S_n(T))$ at maturity T. Then it is known that v satisfies the *Black-Scholes partial differential equation*

$$0 = -v_\tau - rv + \sum_{i=1}^{n} x_i(r-q_i)v_{x_i} + \frac{1}{2}\sum_{i,j=1}^{n}(\sigma\circ\sigma^T)_{ij}x_ix_jv_{x_ix_j}. \quad (32)$$

3.1 Relations among Greeks Based on the Log-Normal Distribution

The value function v has a representation given by the n-fold integral

$$v = e^{-r\tau} \int f\left(\ldots, S_i(0)e^{\sigma_i\sqrt{\tau}x_i+\mu_i\tau}, \ldots\right) g(\vec{x}, \rho)\, d\vec{x}, \tag{33}$$

where $\mu_i = r - q_i - \frac{1}{2}\sigma_i^2$ and $g(\vec{x}, \rho)$ is the n-variate standard normal density with correlation matrix ρ. Since we do not want to assume differentiability of the payoff f, but we know that the transition density g is differentiable, we define a change the variables $y_i \triangleq S_i(0)e^{\sigma_i\sqrt{\tau}x_i+\mu_i\tau}$, which leads to

$$v = e^{-r\tau} \int f(\ldots, y_i, \ldots) g\left(\frac{\ln \frac{y_i}{S_i(0)} - \mu_i\tau}{\sigma_i\sqrt{\tau}}, \rho\right) \frac{d\vec{y}}{\prod y_i\sigma_i\sqrt{\tau}}. \tag{34}$$

3.1.1 Properties of the Normal Distribution

We collect some properties of the multivariate normal density function g. We suppose that the vector X of n random variables with means zero and unit variances has a nonsingular normal multivariate distribution with probability density function

$$g(x_1, \ldots, x_n; c_{11}, \ldots, c_{nn}) = (2\pi)^{-\frac{1}{2}n} |\mathbf{C}|^{\frac{1}{2}} \exp\left(-\frac{1}{2}\mathbf{x}^T \mathbf{C}\mathbf{x}\right). \tag{35}$$

Here $\mathbf{C}$ is the inverse of the covariance matrix of X, which is denoted by ρ. Then the following identity published by Plackett [1954] can be proved easily by writing the density in terms of its characteristic function.

Theorem 3 *(Plackett's Identity)*

$$\frac{\partial g}{\partial \rho_{ij}} = \frac{\partial^2 g}{\partial x_i \partial x_j}. \tag{36}$$

In the two-dimensional case this reads as

$$\frac{\partial n_2(x, y; \rho)}{\partial \rho} = \frac{\partial^2 n_2(x, y; \rho)}{\partial x \partial y}, \tag{37}$$

which can be extended readily to the corresponding cumulative distribution function, i.e.,

$$\frac{\partial \mathcal{N}_2(x, y; \rho)}{\partial \rho} = \frac{\partial^2 \mathcal{N}_2(x, y; \rho)}{\partial x \partial y} = n_2(x, y; \rho). \tag{38}$$

3.1.2 Correlation Risk and Cross-Gamma

Using the abbreviation $g_{jk} \triangleq \frac{\partial^2 g}{\partial x_j \partial x_k}$ the cross-gamma and correlation risk are

$$\frac{\partial^2 v}{\partial S_j(0)\partial S_k(0)} = e^{-r\tau} \frac{1}{S_j(0)S_k(0)\sigma_j\sigma_k\tau} \int f(\ldots, y_i, \ldots) g_{jk} \frac{d\vec{y}}{\prod y_i\sigma_i\sqrt{\tau}}, \quad (39)$$

$$\frac{\partial v}{\partial \rho_{jk}} = e^{-r\tau} \int f(\ldots, y_i, \ldots) g_{\rho_{jk}} \frac{d\vec{y}}{\prod y_i\sigma_i\sqrt{\tau}}. \quad (40)$$

Invoking Plackett's identity (36) saying that $g_{\rho_{jk}} = g_{jk}$ leads to

Theorem 4 *(cross-gamma-correlation-risk relationship)*

$$\frac{\partial v}{\partial \rho_{jk}} = S_j(0)S_k(0)\sigma_j\sigma_k\tau \frac{\partial^2 v}{\partial S_j(0)\partial S_k(0)}. \quad (41)$$

3.1.3 Interest Rate Risk and Delta

A similar computation yields

Theorem 5 *(delta-rho relationship)*

$$\frac{\partial v}{\partial q_j} = -S_j(0)\tau \frac{\partial v}{\partial S_j(0)}, \tag{42}$$

$$\frac{\partial v}{\partial r} = -\tau \left(v - \sum_{j=1}^{n} S_j(0) \frac{\partial v}{\partial S_j(0)} \right). \tag{43}$$

3.1.4 Volatility Risk and Gamma

Theorem 6 *(gamma-vega relationship)*

$$\sigma_j \frac{\partial v}{\partial \sigma_j} = \sum_{k=1}^{n} \rho_{jk}\sigma_j\sigma_k S_j(0)S_k(0)\tau \frac{\partial^2 v}{\partial S_j(0)\partial S_k(0)}. \tag{44}$$

4 The One-Dimensional Case

4.1 Results for European Claims in the Black-Scholes Model

We list several relations for European options.

$$0 = \tau\Theta + r\rho + q\rho_q + \frac{1}{2}\sigma\Phi \quad \text{scale invariance of time} \tag{45}$$

$$v = x\Delta + k\Delta^k \quad \text{price homogeneity and strikes} \tag{46}$$

$$x^2\Gamma = k^2\Gamma^k \quad \text{price homogeneity and strikes} \tag{47}$$

$$x\Delta = -l\Delta^l \quad \text{price homogeneity and levels} \tag{48}$$

$$x^2\Gamma + x\Delta = l^2\Gamma^l + l\Delta^l \quad \text{price homogeneity and levels} \tag{49}$$

$$\rho = -\tau(v - x\Delta) \quad \text{delta-rho relationship} \tag{50}$$

$$\rho + \rho_q = -\tau v \quad \text{rates symmetry} \tag{51}$$

$$rv = \Theta + (r - q)x\Delta + \frac{1}{2}\sigma^2 x^2\Gamma \quad \text{Black-Scholes PDE} \tag{52}$$

$$qv = \Theta + (q - r)k\Delta^k + \frac{1}{2}\sigma^2 k^2\Gamma^k \quad \text{dual Black-Scholes (strike)} \tag{53}$$

$$rv = \Theta + (q - r + \sigma^2)l\Delta^l + \frac{1}{2}\sigma^2 l^2\Gamma^l \quad \text{dual Black-Scholes (level)} \tag{54}$$

$$\rho_q = -\tau x\Delta \quad \text{delta-rho relationship} \tag{55}$$

$$\rho = -\tau k\Delta^k \quad \text{combination of (55) and (46)} \tag{56}$$

$$\Phi = \sigma\tau x^2\Gamma \quad \text{gamma-vega relationship} \tag{57}$$

Theorem 7 *If the price and two Greeks g_1, g_2 of a European option are given with*

$$g_1 \in G_1 = \{\Delta, \Delta^k, \Delta^l, \rho, \rho_q\}, \tag{58}$$
$$g_2 \in G_2 = \{\Gamma, \Gamma^k, \Gamma^l, \Phi, \Theta\}, \tag{59}$$

then all the other Greeks ($\in G_1 \cup G_2$) can be calculated. Furthermore, if Θ and another Greek from G_2 is given, it is also possible, to determine all other Greeks.

4.2 Options Depending on the Final Time Value and the Running Extremum

applies to barrier options, one-touch options and lookback options.

4.2.1 The Pricing Formula

We denote the running extremum by

$$M_T \triangleq \eta \min_{0 \le t \le T} [\eta S_t], \tag{60}$$

$$\eta = +1 \text{ for minimum }, \eta = -1 \text{ for maximum.} \tag{61}$$

The payoff is a function $f(M_T, S_T)$ and its value at time zero is computed via

$$v = e^{-r\tau} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(S_0 e^{\sigma y}, S_0 e^{\sigma x}) g(y, x)\, dy dx, \tag{62}$$

where the joint density of the random vector

$$\left(W(T) + \theta_{-} T, \eta \min_{0 \le t \le T} [\eta (W(t) + \theta_{-} t)] \right) \tag{63}$$

is given by

$$g(x, y) = -\eta e^{\theta_{-} x - \frac{1}{2}\theta_{-}^2 T} \frac{2(2y - x)}{T\sqrt{2\pi T}} \exp\left\{ -\frac{(2y - x)^2}{2T} \right\}, \tag{64}$$

$$\eta y \le \min(S_0, \eta x).$$

4.2.2 Foreign-Domestic Symmetry

Since $\frac{\partial g}{\partial r} = -\frac{\partial g}{\partial q}$, we obtain

$$v_r + v_q = -Tv. \tag{65}$$

4.2.3 Remaining Relations

The rho-delta-relationships as in Theorem 5 do not hold, and the vega-gamma relationship generalizes to take the form

$$(r-q)[Txv_x + v_q] + \frac{T}{2}\sigma^2 x^2 v_{xx} = \frac{1}{2}\sigma v_\sigma, \tag{66}$$

which agrees with Theorem 6 if the cost of carry $r - q$ equals zero.

5 A European Claim in the Two-Dimensional Black-Scholes Model

5.1 Relations among the Greeks

We specialize the relationships among the Greeks found in n dimensions. Some results are

$$\begin{aligned}
0 &= \rho_{q_1} + S_1(0)\tau\Delta_1, & (67)\\
0 &= \rho_{q_2} + S_2(0)\tau\Delta_2, & (68)\\
0 &= q_1\rho_{q_1} + q_2\rho_{q_2} + \frac{1}{2}\sigma_1\Phi_1 + \frac{1}{2}\sigma_2\Phi_2 + r\rho_r + \tau\Theta, & (69)\\
0 &= \Theta - rv + (r-q_1)S_1(0)\Delta_1 + (r-q_2)S_2(0)\Delta_2 \\
&\quad + \frac{1}{2}\sigma_1^2 S_1(0)^2\Gamma_{11} + \rho\sigma_1\sigma_2 S_1(0)S_2(0)\Gamma_{12} + \frac{1}{2}\sigma_2^2 S_2(0)^2\Gamma_{22}, & (70)\\
\kappa &= \sigma_1\sigma_2\tau S_1(0)S_2(0)\Gamma_{12}, & (71)\\
0 &= \rho\kappa - \sigma_1\Phi_1 + \sigma_1^2\tau S_1(0)^2\Gamma_{11}, & (72)\\
0 &= \rho\kappa - \sigma_2\Phi_2 + \sigma_2^2\tau S_2(0)^2\Gamma_{22}, & (73)\\
0 &= \sigma_1\Phi_1 - \sigma_2\Phi_2 - \sigma_1^2\tau S_1(0)^2\Gamma_{11} + \sigma_2^2\tau S_2(0)^2\Gamma_{22}, & (74)\\
\rho_r &= -\tau\left(v - S_1(0)\Delta_1 - S_2(0)\Delta_2\right), & (75)\\
0 &= \tau v + \rho_{q_1} + \rho_{q_2} + \rho_r. & (76)
\end{aligned}$$

5.2 European Options on the Minimum/Maximum of Two Assets

We consider the payoff

$$[\phi\,(\eta\min(\eta S_1(T), \eta S_2(T)) - K)]^+ \,. \tag{77}$$

This is a European put or call on the minimum ($\eta = +1$) or maximum ($\eta = -1$) of the two assets $S_1(T)$ and $S_2(T)$ with strike K. As usual, the binary variable ϕ takes the value $+1$ for a call and -1 for a put. Its value function has been published in Stulz [1982] and can be written as

$$\begin{aligned} & v(t, S_1(t), S_2(t), K, T, q_1, q_2, r, \sigma_1, \sigma_2, \rho, \phi, \eta) \qquad (78) \\ = \;& \phi\left[S_1(t)e^{-q_1\tau}\mathcal{N}_2(\phi d_1, \eta d_3; \phi\eta\rho_1)\right. \\ & + S_2(t)e^{-q_2\tau}\mathcal{N}_2(\phi d_2, \eta d_4; \phi\eta\rho_2) \\ & \left. - Ke^{-r\tau}\left(\frac{1-\phi\eta}{2} + \phi\mathcal{N}_2(\eta(d_1 - \sigma_1\sqrt{\tau}), \eta(d_2 - \sigma_2\sqrt{\tau}); \rho)\right)\right], \end{aligned}$$

$$\sigma^2 \triangleq \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2, \tag{79}$$

$$\rho_1 \triangleq \frac{\rho\sigma_2 - \sigma_1}{\sigma}, \tag{80}$$

$$\rho_2 \triangleq \frac{\rho\sigma_1 - \sigma_2}{\sigma}, \tag{81}$$

$$d_1 \triangleq \frac{\ln(S_1(t)/K) + (r - q_1 + \frac{1}{2}\sigma_1^2)\tau}{\sigma_1\sqrt{\tau}}, \tag{82}$$

$$d_2 \triangleq \frac{\ln(S_2(t)/K) + (r - q_2 + \frac{1}{2}\sigma_2^2)\tau}{\sigma_2\sqrt{\tau}}, \tag{83}$$

$$d_3 \triangleq \frac{\ln(S_2(t)/S_1(t)) + (q_1 - q_2 - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}, \tag{84}$$

$$d_4 \triangleq \frac{\ln(S_1(t)/S_2(t)) + (q_2 - q_1 - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}. \tag{85}$$

5.2.1 Greeks

Delta. Space homogeneity implies that

$$v = S_1(t)\frac{\partial v}{\partial S_1(t)} + S_2(t)\frac{\partial v}{\partial S_2(t)} + K\frac{\partial v}{\partial K}. \tag{86}$$

read off the deltas:

$$\frac{\partial v}{\partial S_1(t)} = \phi e^{-q_1\tau}\mathcal{N}_2(\phi d_1, \eta d_3; \phi\eta\rho_1), \tag{87}$$

$$\frac{\partial v}{\partial S_2(t)} = \phi e^{-q_2\tau}\mathcal{N}_2(\phi d_2, \eta d_4; \phi\eta\rho_2), \tag{88}$$

$$\frac{\partial v}{\partial K} = -\phi e^{-r\tau}\left(\frac{1-\phi\eta}{2} + \phi\mathcal{N}_2(\eta(d_1 - \sigma_1\sqrt{\tau}), \eta(d_2 - \sigma_2\sqrt{\tau}); \rho)\right). \tag{89}$$

Gamma. We use the identities

$$\frac{\partial}{\partial x}\mathcal{N}_2(x,y;\rho) = n(x)\mathcal{N}\left(\frac{y-\rho x}{\sqrt{1-\rho^2}}\right), \tag{90}$$

$$\frac{\partial}{\partial y}\mathcal{N}_2(x,y;\rho) = n(y)\mathcal{N}\left(\frac{x-\rho y}{\sqrt{1-\rho^2}}\right), \tag{91}$$

and obtain

$$\begin{aligned}\frac{\partial^2 v}{\partial(S_1(t))^2} = \frac{\phi e^{-q_1\tau}}{S_1(t)\sqrt{\tau}}\left[\frac{\phi}{\sigma_1}n(d_1)\mathcal{N}\left(\eta\sigma\frac{d_3-d_1\rho_1}{\sigma_2\sqrt{1-\rho^2}}\right)\right. \\ \left. -\frac{\eta}{\sigma}n(d_3)\mathcal{N}\left(\phi\sigma\frac{d_1-d_3\rho_1}{\sigma_2\sqrt{1-\rho^2}}\right)\right],\end{aligned} \tag{92}$$

$$\begin{aligned}\frac{\partial^2 v}{\partial(S_2(t))^2} = \frac{\phi e^{-q_2\tau}}{S_2(t)\sqrt{\tau}}\left[\frac{\phi}{\sigma_2}n(d_2)\mathcal{N}\left(\eta\sigma\frac{d_4-d_2\rho_2}{\sigma_1\sqrt{1-\rho^2}}\right)\right. \\ \left. -\frac{\eta}{\sigma}n(d_4)\mathcal{N}\left(\phi\sigma\frac{d_2-d_4\rho_2}{\sigma_1\sqrt{1-\rho^2}}\right)\right],\end{aligned} \tag{93}$$

$$\frac{\partial^2 v}{\partial S_1(t)\partial S_2(t)} = \frac{\phi\eta e^{-q_1\tau}}{S_2(t)\sigma\sqrt{\tau}}n(d_3)\mathcal{N}\left(\phi\sigma\frac{d_1-d_3\rho_1}{\sigma_2\sqrt{1-\rho^2}}\right). \tag{94}$$

Kappa. The sensitivity with respect to correlation is directly related to the cross-gamma

$$\frac{\partial v}{\partial \rho} = \sigma_1 \sigma_2 \tau S_1(t) S_2(t) \frac{\partial^2 v}{\partial S_1(t) \partial S_2(t)}. \tag{95}$$

Vega. We refer to (72) and (73) to get the following formulas for the vegas,

$$\begin{aligned}
\frac{\partial v}{\partial \sigma_1} &= \frac{\rho v_\rho + \sigma_1^2 \tau (S_1(t))^2 v_{S_1(t) S_1(t)}}{\sigma_1} && (96)\\
&= S_1(t) e^{-q_1 \tau} \sqrt{\tau} \left[\rho_1 \phi \eta n(d_3) \mathcal{N}\left(\phi \sigma \frac{d_1 - d_3 \rho_1}{\sigma_2 \sqrt{1-\rho^2}} \right) \right. \\
&\quad \left. + n(d_1) \mathcal{N}\left(\eta \sigma \frac{d_3 - d_1 \rho_1}{\sigma_2 \sqrt{1-\rho^2}} \right) \right], && (97)\\
\frac{\partial v}{\partial \sigma_2} &= \frac{\rho v_\rho + \sigma_2^2 \tau (S_2(t))^2 v_{S_2(t) S_2(t)}}{\sigma_2} && (98)\\
&= S_2(t) e^{-q_2 \tau} \sqrt{\tau} \left[\rho_2 \phi \eta n(d_4) \mathcal{N}\left(\phi \sigma \frac{d_2 - d_4 \rho_2}{\sigma_1 \sqrt{1-\rho^2}} \right) \right. \\
&\quad \left. + n(d_2) \mathcal{N}\left(\eta \sigma \frac{d_4 - d_2 \rho_2}{\sigma_1 \sqrt{1-\rho^2}} \right) \right]. && (99)
\end{aligned}$$

Rho. Looking at (67), (68) and (75) the rhos are given by

$$\frac{\partial v}{\partial q_1} = -S_1(t)\tau\frac{\partial v}{\partial S_1(t)}, \tag{100}$$

$$\frac{\partial v}{\partial q_2} = -S_2(t)\tau\frac{\partial v}{\partial S_2(t)}, \tag{101}$$

$$\frac{\partial v}{\partial r} = -K\tau\frac{\partial v}{\partial K}. \tag{102}$$

Theta. Among the various ways to compute theta one may use the one based on (69).

$$\frac{\partial v}{\partial t} = -\frac{1}{\tau}\left[q_1 v_{q_1} + q_2 v_{q_2} + r v_r + \frac{\sigma_1}{2} v_{\sigma_1} + \frac{\sigma_2}{2} v_{\sigma_2}\right]. \tag{103}$$

6 Heston's Stochastic Volatility Model

$$\begin{aligned} dS_t &= S_t\left[\mu\, dt + \sqrt{v(t)} dW_t^{(1)}\right], && (104)\\ dv_t &= \kappa(\theta - v_t)\, dt + \sigma\sqrt{v(t)} dW_t^{(2)}, && (105)\\ \mathbf{Cov}\left[dW_t^{(1)}, dW_t^{(2)}\right] &= \rho\, dt, && (106)\\ \Lambda(S, v, t) &= \lambda v. && (107) \end{aligned}$$

Heston provides a closed-form solution for European vanilla options paying

$$[\phi\,(S_T - K)]^+ . \qquad (108)$$

As usual, the binary variable ϕ takes the value $+1$ for a call and -1 for a put, K the strike in units of the domestic currency

6.1 Abbreviations

$$a \triangleq \kappa\theta \tag{109}$$

$$u_1 \triangleq \frac{1}{2} \tag{110}$$

$$u_2 \triangleq -\frac{1}{2} \tag{111}$$

$$b_1 \triangleq \kappa + \lambda - \sigma\rho \tag{112}$$

$$b_2 \triangleq \kappa + \lambda \tag{113}$$

$$d_j \triangleq \sqrt{(\rho\sigma\varphi i - b_j)^2 - \sigma^2(2u_j\varphi i - \varphi^2)} \tag{114}$$

$$g_j \triangleq \frac{b_j - \rho\sigma\varphi i + d_j}{b_j - \rho\sigma\varphi i - d_j} \tag{115}$$

$$\tau \triangleq T - t \tag{116}$$

$$D_j(\tau, \varphi) \triangleq \frac{b_j - \rho\sigma\varphi i + d_j}{\sigma^2}\left[\frac{1 - e^{d_j\tau}}{1 - g_j e^{d_j\tau}}\right] \tag{117}$$

$$\begin{aligned} C_j(\tau, \varphi) \triangleq\ & (r - q)\varphi i\tau \\ & + \frac{a}{\sigma^2}\left\{(b_j - \rho\sigma\varphi i + d)\tau - 2\ln\left[\frac{1 - g_j e^{d_j\tau}}{1 - e^{d_j\tau}}\right]\right\} \end{aligned} \tag{118}$$

$$f_j(x, v, t, \varphi) \triangleq e^{C_j(\tau,\varphi) + D_j(\tau,\varphi)v + i\varphi x} \tag{119}$$

$$P_j(x, v, \tau, y) \triangleq \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \Re\left[\frac{e^{-i\varphi y} f_j(x, v, \tau, \varphi)}{i\varphi}\right] d\varphi \tag{120}$$

$$p_j(x, v, \tau, y) \triangleq \frac{1}{\pi}\int_0^\infty \Re\left[e^{-i\varphi y} f_j(x, v, \tau, \varphi)\right] d\varphi \tag{121}$$

$$P_+(\phi) \triangleq \frac{1 - \phi}{2} + \phi P_1(\ln S_t, v_t, \tau, \ln K) \tag{122}$$

$$P_-(\phi) \triangleq \frac{1 - \phi}{2} + \phi P_2(\ln S_t, v_t, \tau, \ln K) \tag{123}$$

This notation is motivated by the fact that the numbers P_j are the cumulative distribution functions (in the variable y) of the log-spot price after time τ starting at x for some drift μ. The numbers p_j are the respective densities.

6.2 Value

The value function for European vanilla options is given by

$$V = \phi \left[e^{-q\tau} S_t P_+(\phi) - K e^{-r\tau} P_-(\phi) \right] \tag{124}$$

The value function takes the form of the Black-Scholes formula for vanilla options. The probabilities $P_\pm(\phi)$ correspond to $\mathcal{N}(\phi d_\pm)$ in the constant volatility case.

6.3 Greeks

Spot delta.

$$\Delta \triangleq \frac{\partial V}{\partial S_t} = \phi e^{-q\tau} P_+(\phi) \tag{125}$$

Dual delta.

$$\Delta^K \triangleq \frac{\partial V}{\partial K} = -\phi e^{-r\tau} P_-(\phi) \tag{126}$$

Gamma.

$$\Gamma \triangleq \frac{\partial \Delta}{\partial S_t} = \frac{\partial \Delta}{\partial x}\frac{\partial x}{\partial S_t} = \frac{e^{-q\tau}}{S_t} p_1(\ln S_t, v_t, \tau, \ln K) \tag{127}$$

Dual Gamma.

$$\Gamma^K \triangleq \frac{\partial \Delta^K}{\partial K} = \frac{\partial \Delta^K}{\partial y}\frac{\partial y}{\partial K} = \frac{e^{-r\tau}}{K} p_1(\ln S_t, v_t, \tau, \ln K) \tag{128}$$

Rho. Rho is connected to delta via equations (56) and (55).

$$\frac{\partial V}{\partial r} = \phi K e^{-r\tau} \tau P_-(\phi), \tag{129}$$

$$\frac{\partial V}{\partial q} = -\phi S_t e^{-q\tau} \tau P_+(\phi). \tag{130}$$

Theta. Theta can be computed using the partial differential equation for the Heston vanilla option

$$V_t + (r-q)SV_S + \frac{1}{2}\sigma v V_{vv} + \frac{1}{2} v S^2 V_{SS} + \rho \sigma v S V_{vS} - qV$$
$$+[\kappa(\theta - v) - \lambda]V_v = 0, \tag{131}$$

where the derivatives with respect to initial variance v must be evaluated numerically.

7 Summary

- Understand homogeneity-based methods to compute analytical formulas of Greeks for analytically known value functions of options in a one-and higher-dimensional market
- Restricting the view to the Black-Scholes model there are numerous further relations between various Greeks which are of fundamental interest
- Saving computation time for the mathematician who has to differentiate complicated formulas as well as for the computer, because analytical results for Greeks are usually faster to evaluate than finite differences involving at least twice the computation of the option's value
- Knowing how the Greeks are related among each other can speed up finite-difference-, tree-, or Monte Carlo-based computation of Greeks or lead at least to a quality check
- Many of the results are valid beyond the Black-Scholes model
- Most remarkably some relations of the Greeks are based on properties of the normal distribution refreshing the active interplay between mathematics and financial markets.

		$Greeks \in G_1$					$Greeks \in G_2$				
equation	v	Δ	Δ^k	Δ^l	ρ	ρ_q	Γ	Γ^k	Γ^l	Φ	Θ
(45)					X	X				X	X
(46)	O	O	O								
(47)							O	O			
(48)		O		O							
(49)		X		X			X		X		
(50)	O	O			O						
(51)	O				O	O					
(52)	X	X					X				X
(53)	X		X					X			X
(54)	X			X					X		X
(56)			O		O						
(57)							O			O	
(55)		O				O					

Table 1. Overview over all relations among Greeks. With X or O we denote that the marked Greek appears in the relation. The relations marked with X show that there is a relation between Greeks of G_1 and G_2 and the O shows, that this relation concerns only the Greeks of one set.

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