Numerical Methods for Derivatives Pricing

Uwe Wystup
http://www.mathfinance.de

1 Newton's Method and its Applications

1.1 The Method

Given a differentiable function $f(x)$ we want to determine $\hat{x}$ such that

$$f(\hat{x}) = 0. \tag{1}$$

Starting with x_0 we take the tangent to the curve through the point $(x_0, f(x_0))$ and use its intersection with the x-axis x_1 as a new starting point. We repeat this method until no further changes occur. The recursive relation is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{2}$$

and the result is

$$\lim_{n \to \infty} x_n = \hat{x}. \tag{3}$$

1.2 Problems

Problems can occur due to

1. multiple solutions
2. non convex f, reflection points
3. solutions at extreme values
4. $|f'| = \infty$
5. pathological cases

1.3 Rate of convergence

Considering all these problems, why is Newton's method still so popular? The reason lies in the rate of convergence. Define the error by

$$e_n \stackrel{\Delta}{=} x_n - \hat{x}. \tag{4}$$

From the definition of the Newton iteration, we have

$$\begin{aligned} e_{n+1} &= x_{n+1} - \hat{x} \\ &= x_n - \frac{f(x_n)}{f'(x_n)} - \hat{x} \\ &= e_n - \frac{f(x_n)}{f'(x_n)} \\ &= \frac{e_n f'(x_n) - f(x_n)}{f'(x_n)}. \end{aligned} \tag{5}$$

By Taylor's Theorem, we have

$$0 = f(\hat{x}) = f(x_n - e_n) = f(x_n) - e_n f'(x_n) + \frac{1}{2} e_n^2 f''(\xi_n), \tag{6}$$

where ξ_n is a number between x_n and $\hat{x}$. A rearrangement of this equation yields

$$e_n f'(x_n) - f(x_n) = \frac{1}{2} f''(\xi_n) e_n^2. \tag{7}$$

Putting this in Equation (5) leads to

$$e_{n+1} = \frac{1}{2} \frac{f''(\xi_n)}{f'(x_n)} e_n^2 \approx \frac{1}{2} \frac{f''(\hat{x})}{f'(\hat{x})} e_n^2 = C e_n^2. \tag{8}$$

This equation tells us that e_{n+1} is roughly a constant times e_n. This desirable state of affairs is called **quadratic convergence**.

1.4 Example: Heron's Square Root Finder

The Greek Engineer Heron who lived sometime between 100 B.C. and 100 A.D. had used the recursion

$$x_{n+1} = \frac{1}{2} \left(x_n + \frac{R}{x_n} \right) \tag{9}$$

to finde the square root of R. This is based on Newton's method. Taking $R = 17$ and starting with $x_0 = 4$ we find $x_4 = 4.1231056256176605498214098\,56$ which is correct to 28 figures.

1.5 Example: VanillaVolRetriever

As an appliction we will build a VanillaVolRetriever. Recall the Black-Scholes formula

$$v = \phi e^{-r_d T} [F \mathcal{N}(\phi d_+) - K \mathcal{N}(\phi d_-)], \tag{10}$$

$$F = x e^{(r_d - r_f)T} \text{ “forward” }, \tag{11}$$

$$d_\pm \triangleq \frac{\ln \frac{F}{K} \pm \frac{1}{2}\sigma^2 T}{\sigma \sqrt{T}}. \tag{12}$$

We now look at the function $v(\sigma)$, whose derivative (vega) is

$$v'(\sigma) = x e^{-r_f T} \sqrt{T} n(d_+). \tag{13}$$

The function $\sigma \mapsto v(\sigma)$ is

1. strictly increasing,
2. concave up for $\sigma \in [0, \sqrt{2|\ln F - \ln K|/T})$,
3. concave down for $\sigma \in (\sqrt{2|\ln F - \ln K|/T}, \infty)$

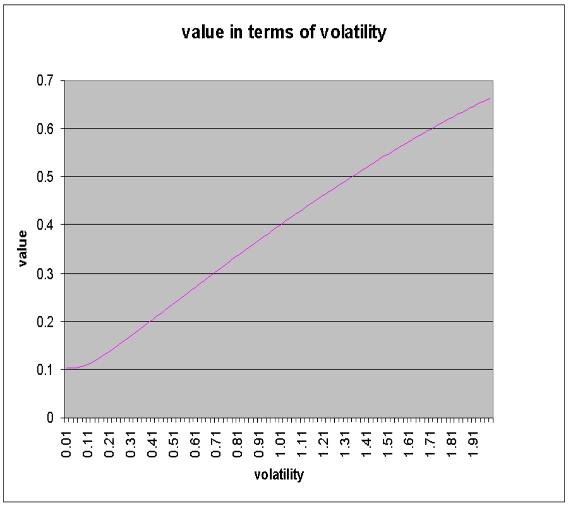

Figure 1: Value of a European call in terms of volatility with parameters $x = 1$, $K = 0.9$, $T = 1$, $r_d = 6\%$, $r_f = 5\%$. The saddle point is at $\sigma = 48\%$.

and also satisfies

$$\begin{aligned} v(0) &= [\phi(xe^{-r_f T} - Ke^{-r_d T})]^+, & (14)\\ v(\infty, \phi = 1) &= xe^{-r_f T}, & (15)\\ v(\sigma = \infty, \phi = -1) &= Ke^{-r_d T}, & (16)\\ v'(0) &= xe^{-r_f T}\sqrt{T}/\sqrt{2\pi}\mathbb{I}_{\{F=K\}}, & (17) \end{aligned}$$

In particular the mapping $\sigma \mapsto v(\sigma)$ is invertible. However, the starting guess for employing Newton's method should be chosen with care, because the mapping $\sigma \mapsto v(\sigma)$ has a saddle point at

$$\left(\sqrt{\frac{2}{T}|\ln\frac{F}{K}|}, \phi e^{-r_d T}\left\{F\mathcal{N}\left(\phi\sqrt{2T[\ln\frac{F}{K}]^+}\right) - K\mathcal{N}\left(\phi\sqrt{2T[\ln\frac{K}{F}]^+}\right)\right\}\right), \tag{18}$$

as illustrated in Figure (1). To ensure convergence of Newton's method, we are advised to use initial guesses for σ on the same side of the saddle point as the desired implied volatility. The danger is that a large initial guess could lead to a negative successive guess for σ. Therefore one should start with small initial guesses at or below the saddle point. For at-the-money options, the saddle point is degenerate for a zero volatility and small volatilities serve as good initial guesses.

1.6 Visual Basic Source Code

```
Function VanillaVolRetriever(spot As Double, rd As Double, _
rf As Double, strike As Double, T As Double, _
type As Integer, GivenValue As Double) As Double
```

```
Dim func As Double
Dim dfunc As Double
Dim maxit As Integer 'maximum number of iterations
Dim j As Integer
Dim s As Double
'first check if a volatility exists, otherwise set result to zero
If GivenValue < Application.Max _
   (0, type * (spot * Exp(-rf * T) - strike * Exp(-rd * T))) Or _
   (type = 1 And GivenValue > spot * Exp(-rf * T)) Or _
   (type = -1 And GivenValue > strike * Exp(-rd * T)) Then
   VanillaVolRetriever = 0
Else
   ' there exists a volatility yielding the given value,
   ' now use Newton's method:
   ' the mapping vol to value has a saddle point.
   ' First compute this saddle point:
   saddle = Sqr(2 / T * Abs(Log(spot / strike) + (rd - rf) * T))
   If saddle > 0 Then
      VanillaVolRetriever = saddle * 0.9
   Else
      VanillaVolRetriever = 0.1
   End If
   maxit = 100
   For j = 1 To maxit Step 1
      func = Vanilla(spot, strike, VanillaVolRetriever, _
      rd, rf, T, type, value) - GivenValue
      dfunc = Vanilla(spot, strike, VanillaVolRetriever, _
      rd, rf, T, type, vega)
      VanillaVolRetriever = VanillaVolRetriever - func / dfunc
      If VanillaVolRetriever <= 0 Then VanillaVolRetriever = 0.01
      If Abs(func / dfunc) <= 0.0000001 Then j = maxit
   Next j
End If
End Function
```

References

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