

Introduction to Mathematical Finance

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<http://www.mathfinance.de>

Contents

1	Introduction	2
2	History	2
3	Financial Markets	3
4	Quantitative Issues	3
5	Choosing a Model	4
5.1	Black & Scholes Model	4
5.2	Shortcomings and Improvements	4
5.3	Term Structure Models	4
6	Pricing Techniques - Financial Engineering	4
7	Current Issues	5
8	Contents of the Lecture “Computational Finance”	6
8.1	Newton’s Method and a Vanilla Vol Retriever	7
8.2	Option Price Sensitivities, Numerical Differentiation, Characteristic Functions, Residual Calculus, Implicit Differentiation	7
8.3	Numerical Integration: Midpoint Rule, Trapezoidal Rule, Simpson’s Rule, Gaussian Quadrature and their Application to Spread Options	7
8.4	Heston’s Stochastic Volatility Model: Riccati Differential Equation, Inversion Theorem, Complex Numbers	7
8.5	Fast Fourier Transform: Pricing Vanilla Options in the Variance Gamma Model	7
8.6	Principal Component Analysis: Pricing Path-Independent Multi-Asset Options, Correlation Matrices, Hedging Correlation Risk in Foreign Exchange Markets	7
8.7	Entropy-based Option Pricing, Lagrange Multipliers	7
8.8	Binomial Trees and American Options	7
8.9	Partial Differential Equations: Implicit, Explicit and Crank-Nicholson method. Dupire’s Implied Trees	7
8.10	Monte Carlo Simulation - Variance Reduction Techniques, Greeks in Monte Carlo	7
9	Literature	7

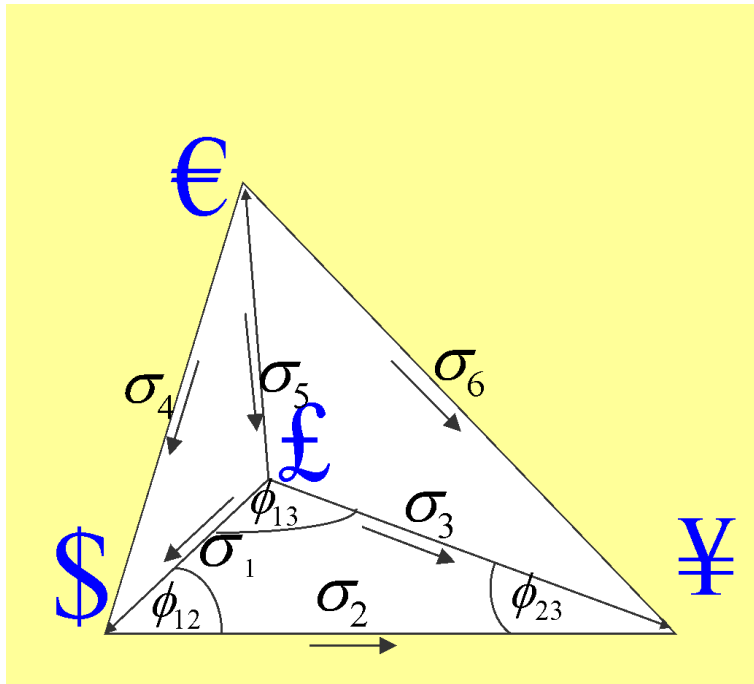


Figure 1: Correlation Tetrahedron for Foreign Exchange Market

1 Introduction

Mathematical Finance has become a discipline of its own ever since the turnover in the options markets have increased worldwide and pricing techniques for financial products have become highly sophisticated.

Figure (1) shows how the volatilities of a foreign exchange market with four currencies determine the correlation coefficients of all 15 combinations of the six currency pairs.

2 History

- 17th century - The Netherlands: put options on tulip bulbs
- 1637: tulip bulb crash: put option sellers cannot meet their buying obligations. This leads to a bad reputation of options in continental Europe.
- 18th century - London: problem - no legal frame for default of counterparties
- 1930: legal frame
- 1970: significance increases
- 1973: Chicago Board Options Exchange (CBOE)
- 1990: Eurex (formerly known as Deutsche Terminbörse)

On the academic side we can report

- 1900: Bachelier's dissertation "Théorie de la Spéculation" taking Brownian motion as a market model
- 1973: Geometric Brownian motion - Fischer Black and Myron Scholes' partial differential equation approach using a risk-free portfolio of bond, option and stock.
- 1981: Harrison - Pliska: Martingale Theory
- 1990s: Mathematical Finance/Computational Finance established as a discipline. Many educational programs start in the US and in the UK.

3 Financial Markets

Financial Markets trade a whole variety of basic securities among which most prominently we mention

1. Foreign Exchange
2. Commodities
3. Precious Metals
4. Equities
5. Bonds
6. Swaps
7. Energy
8. Power
9. Bandwidth
10. Weather

Markets trade the spot, forwards and all kind of derivatives. Whereas modeling the spot rate requires modeling a single scalar, modeling forward prices requires modeling an entire curve over time. Trading derivatives furthermore requires assumptions about market completeness and arbitrage possibilities. Each market has its peculiarities such as liquidity, taxation, seasonality, tradability, convenience yield, calendar conventions, etc.

4 Quantitative Issues

The issues of mathematical finance in investment banking are mainly

1. Pricing of Derivatives
2. Hedging Strategies for Derivatives
3. Risk Management of Portfolios
4. Portfolio Optimization
5. Model Choice and Calibration

5 Choosing a Model

5.1 Black & Scholes Model

The *geometric Brownian motion* model for a stock or exchange rate has been very popular among researchers and traders for the last two decades.

$$dS_t = S_t[(r_d - r_f)dt + \sigma dW_t] \quad (1)$$

Its popularity is due to the fact that in many cases the normal distribution allows explicit computation of derivatives prices if arbitrage is excluded via the *risk-neutral valuation formula*

$$v(0, S_0) = e^{-r_d T} \mathbb{E}[f(S_T)], \quad (2)$$

where f is the payoff of a derivative at maturity T and $v(0, S_0)$ is its value today if the spot is at S_0 . Numerous examples can be found on <http://www.mathfinance.de>.

5.2 Shortcomings and Improvements

The Black-Scholes model is not suitable for many markets and has been criticized a lot for its constant volatility assumption and the thin tails of the spot return distribution. To overcome the first one has introduced volatility to be itself a stochastic process. A prominent example is that of Heston [4]. To get the distribution right there is now a lot of literature coming up on using *generalized Levy processes* or *fractional Brownian motion* as a model. Other approaches are *Entropy based* such as in [3].

5.3 Term Structure Models

Modeling an entire curve such as the *term structure of interest rates* is by far more complicated than just modeling a scalar. The unifying framework is the model by Heath, Jarrow and Morton, which is a general model for the forward rate. Different choices of coefficients produce models for the short rate such as the one by Cox, Ingersoll and Ross [1]

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v(t)}dW_t. \quad (3)$$

6 Pricing Techniques - Financial Engineering

The main tools for pricing and hedging are

1. Closed-form solutions in terms of known functions such as cumulative normal, gamma, chi-square,...
2. Numerical Integration
3. Binomial and Trinomial Trees
4. Partial Differential Equations and their solution via finite difference methods.

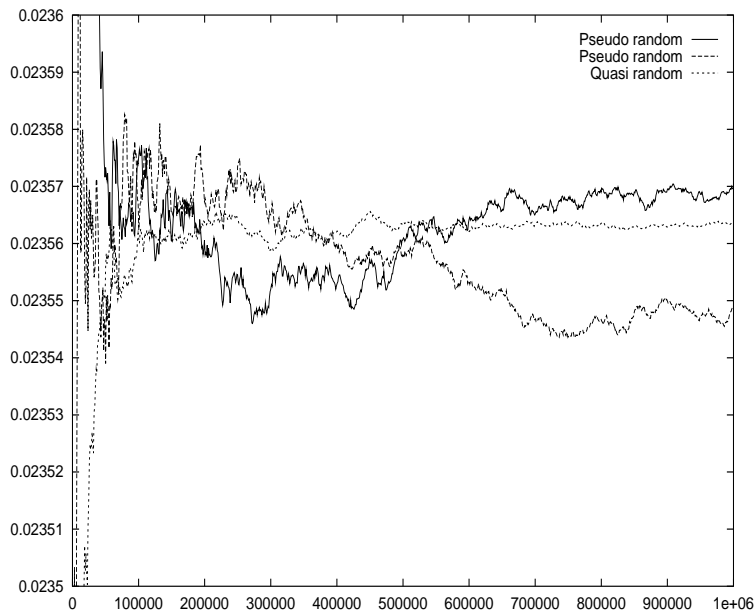


Figure 2: Fixed strike lookback put, 22 fixings: how in a quasi random number based Monte Carlo simulation the values converge to the approximate analytic value

5. Monte Carlo Simulations. Here in particular the usage of *quasi-random numbers*, which produces much better convergence as shown in Figure (2).
6. Minimization Routines for Model Calibration. Figure (3) shows the best fit to the given one year volatility smile.

Many algorithms can be found in [2].

7 Current Issues

Currently researchers, traders, risk managers and quantitative analysts are concerned with

1. Finding the right model, lots about Levy processes, fractional Brownian motion
2. Volatility modeling
3. Correlation risk modeling in large portfolios
4. Pricing techniques for higher-dimensional and path-dependent payoffs in more complex models
5. Forward curve modeling, specially in the energy markets
6. modeling liquidity and default risk

market volatility vs. best fit in Heston's model

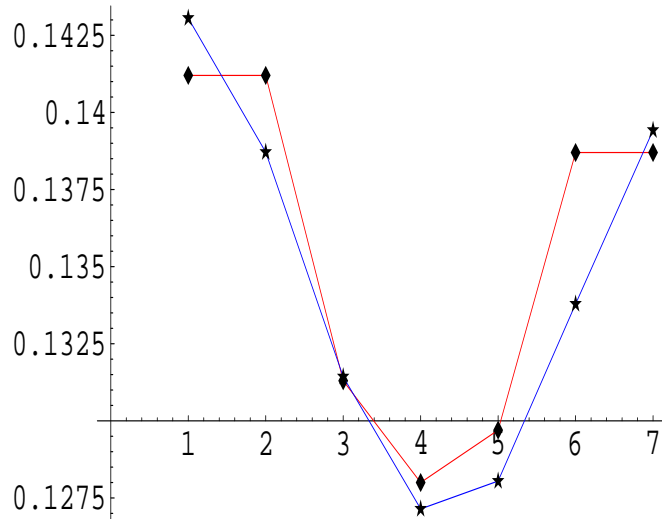


Figure 3: Best least square fit for one year maturity

8 Contents of the Lecture “Computational Finance”

I intend to provide the audience with insights and skills which are relevant in the area of pricing and hedging derivatives. I will explain some numerical methods and show how they are applied to practical problems. I will also show how the theory can be put into practice by showing some source code in Visual Basic, C/C++, Mathematica, Fortran and Java Script. Students are encouraged to work through problems provided in order to develop their own computational and programming skills. In detail I will present the following sessions.

- 8.1 Newton's Method and a Vanilla Vol Retriever
- 8.2 Option Price Sensitivities, Numerical Differentiation, Characteristic Functions, Residual Calculus, Implicit Differentiation
- 8.3 Numerical Integration: Midpoint Rule, Trapezoidal Rule, Simpson's Rule, Gaussian Quadrature and their Application to Spread Options
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9 Literature

See <http://www.mathfinance.de> for an extensive list of related literature. For computational finance the book by Seydel [6] is particularly recommended for beginners and a German speaking audience. The book by W. Shaw [7] is an excellent source if one plans to use Mathematica. Wilmott's volumes [9] are very PDE oriented. The number of books appearing nowadays is immense. However, the best way to learn computational finance is to *do it oneself*.

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