Effective approximation of FX/EQ options for the hybrid models: Heston and correlated Gaussian interest rates.

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Frankfurt MathFinance Conference

17-18 March 2007

Outline

- Hybrid Heston model with correlated Gaussian interest rates
- Review of analytical approximation methods
- Brief history of Markovian projection to skew and smile models
- Effective approximation of FX/EQ options for the hybrid models via Markovian projection to the Heston model with displaced volatility

Mathematical challenges

- Complexity of modern models
- Sensitivity of the instruments to distant wings of volatility surfaces (wide range of European option strikes)

$$\Downarrow$$

Requirements to European option pricing:

- Effective methods for complicated underlying process
- Good approximation quality for all strikes, capability to reproduce model skew/smile

Example: hybrid correlated Heston/HW model

Exchange rate → *Heston model*

$$
\begin{aligned}
dX &= X(r_1 - r_2)\,dt + X\sqrt{z}\,\lambda \cdot dW^* \\
dz &= \alpha\,(1 - z)\,dt + \sqrt{z}\,\gamma \cdot dW^*, \quad z(0) = 1
\end{aligned}
$$

Domestic and (shifted) foreign IR models → *HW1F models*

$$
\begin{aligned}
dr_1 &= (\zeta_1 - r_1\,a_1)\,dt + \sigma_1 \cdot dW^* \\
dr_2 &= (\zeta_2 - r_2\,a_2 - \sqrt{z}\,\lambda \cdot \sigma_2)\,dt + \sigma_2 \cdot dW^*
\end{aligned}
$$

Remarks:

1. All model parameters are time-dependent (we remove time-argument for better legibility)
2. W^* is a Brownian motion under the domestic risk-neutral measure
3. We use vector volatilities to stress for *arbitrary* correlations.

The above form of Heston model coming from IR's (Andersen-Andreasen (2002))

$$
\begin{aligned}
dX &= X(r_1 - r_2)\,dt + X\sqrt{z}\,\lambda \cdot dW^* \\
dz &= \alpha\,(1 - z)\,dt + \sqrt{z}\,\gamma \cdot dW^*, \quad z(0) = 1
\end{aligned}
$$

is a time-dependent generalization of the initial Heston form

$$
\begin{aligned}
dX &= X(r_1 - r_2)\,dt + X\sqrt{v}\,dU \\
dv &= \kappa(\theta - v)dt + \xi\sqrt{v}\,dV \\
\langle dU\,dV\rangle &= \rho dt
\end{aligned}
$$

Conversion formulas and option pricing recipe can be found at Antonov, Arneguy and Audet (2008)

Challenge

The correlated Heston/HW/HW is not exactly solvable
$\rightarrow$ its affine structure is destroyed by correlations.

Zero correlation case and its uncorrelated volatility displacement modification (both being affine and exactly solvable) was proposed by Andreasen (2006).

Below we present an effective approximation of the correlated standard Heston/HW/HW based on the *Markovian Projection* (MP).

European option on a generic rate process

Rate process (in general non-Markovian)

$$dS(t) = \Sigma(t) \cdot dW(t)$$

with a generic volatility (vector) process $\Sigma(t)$ driven by F independent Brownian motions, $W(t) = \{W_1(t), \cdots, W_F(t)\}$. For example, S can be a swap rate, forward FX-rate, etc.

The process $S(t)$ underlies the European option $E[(S(t) - K)^+]$ maturing at t

$\rightarrow$ for calculations we need only its marginal distribution at t

Approximate analytical methods

- Heuristic methods (Rebonato, Hull & White for LMM swaption pricing):
  represent volatility in a suitable form (e.g., log-normal for the BGM model) by ”freezing” some of the coefficients
- Asymptotic expansion methods (Kawai (2003), Kawai-Jäckel (2006))
- Heat-kernel expansion (Avellaneda et al (2002), Henry-Labordere (2005))
- Markovian projection

Asymptotic expansion methods

Expand $S(t)$ around small volatilities (scaled with ϵ)

$$S(t) = S(0) + \epsilon\, S_1(t) + \epsilon^2\, S_2(t) + \cdots$$

where $S_1(t)$ is a Gaussian variable. PDF is restored from conditional averages $E[S_2(t)\,|\,S_1(t)]$, etc.

Example: log-normal process $dS(t) = \epsilon S(t)\lambda(t)\, dW(t)$ for $S(0) = 1$

$$\begin{aligned} S_1(t) &= \int_0^t dW(\tau)\, \lambda(\tau) \\ S_2(t) &= \int_0^t dW(\tau)\, \lambda(\tau) \int_0^\tau dW(s)\, \lambda(s) \end{aligned}$$

All terms $S_i(t)$ are i-th multiple Ito integrals (with ordered arguments)

Characteristic function expansion around the first gaussian term $S_1(t)$

$$\begin{aligned}
\phi(\xi) &= E\left[\exp\left(\xi\frac{S(t)-1}{\epsilon}\right)\right] \\
&= E\left[e^{\xi S_1(t)}\left(1+\xi\epsilon S_2(t)+O(\epsilon^2)\right)\right] \\
&= E\left[e^{\xi S_1(t)}\left(1+\xi\epsilon E[S_2(t)\,|\,S_1(t)]+O(\epsilon^2)\right)\right] \\
&= E\left[e^{\xi S_1(t)}\right]+\epsilon\,\xi\,E\left[e^{\xi S_1(t)}\,E[S_2(t)\,|\,S_1(t)]\right]+O(\epsilon^2))
\end{aligned}$$

Conditional expectations multiple Ito integrals can be calculated. For example,

$$E[S_2(t)\,|\,S_1(t)] = \frac{1}{2}\left(S_1^2(t)-v(t)\right)$$

where $v(t)$ is the gaussian variable $S_1(t)$ variance, $v(t) = \int_0^t ds\lambda^2(s)$.

Having characteristic function expansion one can calculate option price or the PDF (normalized)

$$P(K) = E\left[\delta\left(S(t) - K\right)\right] = \frac{1}{\epsilon} E\left[\delta\left(\frac{S(t) - 1 - \epsilon\kappa}{\epsilon}\right)\right]$$

where we consider the strike K in the vicinity of ATM, $K = 1 + \epsilon\kappa$. The PDE calculated via the asymptotic expansion methods reads

$$P(1 + \epsilon\kappa) = \frac{1}{\epsilon} \frac{1}{\sqrt{2\pi v(t)}} \, e^{-\frac{\kappa^2}{2\,v(t)}} \left(1 + \frac{\epsilon}{2}\left(\frac{\kappa^3}{v(t)} - 3\kappa\right) + O(\epsilon^2)\right)$$

which coincides with the two terms of the exact log-normal PDF

$$P(K) = \frac{1}{\epsilon} \frac{1}{\sqrt{2\pi v(t)}\,K} \, e^{-\frac{\left(\ln K + \frac{\epsilon^2 v(t))}{2}\right)^2}{2\,\epsilon^2 v(t)}}$$

with substitution $K = 1 + \epsilon\kappa$ (the first term corresponds to gaussian approximation).

Heat-kernel expansion

Heat kernel expansion is asymptotic solution for short times of a forward Kolmogorov equation for a general diffusion model (multi-dimensional) PDF. A lot of effective approximation can be derived using this technique (for example, Hagan-al SABR formulas).

Example: log-normal process $dS(t) = \epsilon S(t)\lambda\, dW(t)$ for $S(0) = 1$ for time-independent volatility

$$P(K) = \frac{1}{\sqrt{2\pi\lambda^2 t}\, K}\, e^{-\frac{\ln^2 K}{2\,\lambda^2 t} - \frac{\ln K}{2}} \left(1 - \frac{\lambda^2}{8} t + O(t^2)\right)$$

(expansion of the exact log-normal PDE for small t with leading asymptotic $\sim t^{-\frac{1}{2}} e^{-\frac{\ln^2 K}{2\,\lambda^2 t}}$).

Main drawback: complications with time-dependent parameters.

Markovian projection

Term "Markovian projection" was coined by Piterbarg; idea ascends to Gyöngy (1986) and Dupire (1994).
A complicated, non-Markovian process $dS(t) = \Sigma(t) \cdot dW(t)$ is replaced by a Markovian one, s.t. both processes have identical one-dimensional marginal distributions. By Gyöngy's lemma, the *Markovian* process $S^*(t)$ satisfying

$$dS^*(t) = \Sigma^*(t, S^*(t)) \cdot dW, \qquad S^*(0) = S(0),$$

with

$$|\Sigma^*(t,s)|^2 = E[|\Sigma(t)|^2 \mid S(t) = s]$$

has the same marginal distributions as $S(t)$ for any t, and therefore can be used to compute the European option,

$$E[(S(t) - K)^+] = E[(S^*(t) - K)^+]$$

Strategy:

- calculate the conditional expectation $E[|\Sigma(t)|^2 \mid S(t) = s]$
- price the option for the local volatility process $S^*(t)$

To avoid complicated *conditional* expectation calculus one postulates plausible mimicking process $S^*(t)$ and calculate its optimal parameters via *non-conditional* expectations. The mimicking process choice depends on characteristics of initial process $S(t)$ (skew, smile).

Examples:

- Piterbarg (2005), Antonov & Misirpashaev (2006a):
  Projection on a displaced diffusion (DD) with time-dependent parameters for the expectation
- Piterbarg (2006), Antonov, Misirpashaev and Piterbarg (2007):
  Projection on a Heston model capturing skew and smile

Markovian projection on a displaced diffusion (DD)

Linear ansatz to capture the first derivative of effective local vol (implied volatility skew)

$$dS^*(t) = \Sigma^*(t, S^*(t)) \cdot dW \simeq (1 + \Delta S^*(t)\beta(t))\, \sigma(t) \cdot dW$$

where $\Delta S^*(t) = S^*(t) - S^*(0)$, $\sigma(t)$ is an F-component deterministic volatility vector, and $\beta(t)$ is a time-dependent shift (controlling skew).

Good approximation accuracy for non-SV models exhibiting a skew

DD results

Optimal DD coefficients in terms of unconditional averages of the initial process $dS(t) = \Sigma(t) \cdot dW(t)$,

$$|\sigma(t)|^2 = E\left[|\Sigma(t)|^2\right]$$

$$\beta(t) = \frac{E\left[|\Sigma(t)|^2 \Delta S(t)\right]}{2\,E\left[|\Sigma(t)|^2\right] E\left[\Delta S^2(t)\right]}$$

Option price can be found from the BS formula after the shift $\beta(t)$ is averaged (Piterbarg).
Closed-form results can be obtained using small volatilities expansions of the above expectations.

Applications of Markovian projection to DD

- Shifted BGM swaption formula
  Piterbarg (2006); Antonov & Misirpashaev (2006b)
- Cross-Currency models (CEV model for FX-rate, gaussian model for interest rates)
  Piterbarg (2006); Antonov & Misirpashaev (2006b)
- LMM Cross-Currency models (CEV for FX-rate, shifted BGM for interest rates)
  Antonov & Misirpashaev (2006a), (2006b);

What to do for a process with a pronounced smile?

The effective mimicking process $S^*(t)$

$$dS^*(t) = \Sigma^*(t, S^*(t)) \cdot dW(t)$$

should have U-shaped local vol.

Due to technical difficulties in MP to non-linear basis we have proposed another solution

$\rightarrow$

project the initial process to the *shifted Heston process*

$$\begin{aligned} dS^*(t) &= (1 + \beta(t)\,\Delta S^*(t))\sqrt{z(t)}\,\sigma_H(t) \cdot dW(t), \qquad & S^*(0) = S(0) \\ dz(t) &= \theta(t)\,(1 - z(t))\,dt + \sqrt{z(t)}\,\sigma_z(t) \cdot dW(t), \qquad & z(0) = 1 \end{aligned}$$

Markovian projection to SV model

Piterbarg (2006), Antonov, Misirpashaev and Piterbarg (2007) developed a systematic approach for direct projection on a stochastic volatility model.

Features:

- Straightforward and universal
- Volatility smile is captured
- Certain features of dynamics are captured (therefore, applications to exotic pricing are possible)

Instead of looking for a local volatility effective model

$$dS^*(t) = \Sigma^*(t, S^*(t)) \cdot dW(t)$$

we consider an SV process

$$dS^*(t) = \sigma_S(t; S^*(t), V^*(t)) \cdot dW(t)$$

where $V^*(t)$ is a stochastic variance.

Theoretical background $\Rightarrow$ multi-dimensional version of Gyöngy's lemma

Choice of process components

The first component is the initial rate, $dS = \Sigma(t) \cdot dW(t)$.
The second component should be related to $|\Sigma(t)|^2$.

We fix a shift function $\beta(t)$ (to be determined later) and write the equation for the rate in the form

$$dS = (1 + \beta(t)\, \Delta S(t))\Lambda(t) \cdot dW(t)$$

where

$$\Lambda(t) = \frac{\Sigma(t)}{1 + \Delta S(t)\beta(t)}$$

The second equation[a] is for the variance $V(t) = |\Lambda(t)|^2$,

$$dV(t) = \mu_V(t)\, dt + \sigma_V(t) \cdot dW(t)$$

[a] The variance SDE can be always written explicitly.

The initial process pair $\{S(t), V(t)\}$

$$\begin{aligned} dS &= (1 + \beta(t)\,\Delta S)\;\Lambda(t) \cdot dW \\ dV &= \mu_V(t)\,dt + \sigma_V(t) \cdot dW \end{aligned}$$

can be mimicked by a Markovian pair $\{S^*(t), V^*(t)\}$ s.t.

$$\begin{aligned} dS^* &= (1 + \beta(t)\,\Delta S^*)\;\sigma_S^*(t; S^*, V^*)\;\cdot dW \\ dV^* &= \mu_V^*(t; S^*, V^*)\,dt + \sigma_V^*(t; S^*, V^*)\;\cdot dW \end{aligned}$$

where

$$\begin{aligned} |\sigma_S^*(t; s, u)|^2 &= E[|\Lambda(t)|^2 \,|\, S(t) = s, V(t) = u] = u \\ |\sigma_V^*(t; s, u)|^2 &= E[|\sigma_V(t)|^2 \,|\, S(t) = s, V(t) = u] \\ \sigma_S^*(t; s, u) \cdot \sigma_V^*(t; s, u) &= E[\Lambda(t) \cdot \sigma_V(t) \,|\, S(t) = s, V(t) = u] \\ \mu_V^*(t; s, u) &= E[\mu_V(t) \,|\, S(t) = s, V(t) = u] \end{aligned}$$

For a fixed skew $\beta(t)$, we look for approximating process in the shifted Heston model

$$\begin{aligned} dS^* &= (1 + \beta(t)\,\Delta S^*)\sqrt{V^*}\frac{\sigma_H(t)}{|\sigma_H(t)|} \cdot dW \\ dV^* &= \left(V^* \left(\left(\log|\sigma_H(t)|^2\right)' - \theta(t)\right) + \theta(t)\,|\sigma_H(t)|^2\right) dt \\ &+ |\sigma_H(t)|\,\sqrt{V^*}\,\sigma_z(t) \cdot dW \end{aligned}$$

with variance $V(t) = z(t)\,|\sigma_H(t)|^2$. This determines the ansatz:

$$\begin{aligned} \mu^*_\Lambda(t; s, v) &= v\,\left(\left(\log|\sigma_H(t)|^2\right)' - \theta(t)\right) + \theta(t)\,|\sigma_H(t)|^2, \\ |\sigma^*_V(t; s, v)|^2 &= v\,|\sigma_H(t)|^2\,|\sigma_z(t)|^2, \\ \sigma^*_S(t; s, v)\cdot\sigma^*_V(t; s, v) &= v\,\sigma_z(t)\cdot\sigma_H(t). \end{aligned}$$

Finally, we find the optimal $\beta(t)$ which minimizes the distance to the projected model

The optimal parameters of the shifted Heston model (expressed via unconditional averages and shift $\beta(t)$)

$$
\begin{aligned}
|\sigma_H(t)|^2 &= E[V(t)] \\
\theta(t) &= (\log E[V(t)])' - \frac{1}{2}\left(\log E[\delta V^2(t)]\right)' + \frac{E[|\sigma_V(t)|^2}{2\,E[\delta V^2(t)]} \\
|\sigma_z(t)|^2 &= \frac{E[V(t)|\sigma_V(t)|^2]}{E[V^2(t)]E[V(t)]} \\
\rho(t) &= \frac{E[V(t)\Lambda(t)\cdot\sigma_V(t)]}{\sqrt{E[V^2(t)]E[V(t)|\sigma_V(t)|^2]}}
\end{aligned}
$$

where $\delta V(t) = V(t) - E[V(t)]$.

Optimal skew function $\rightarrow$ choose $\beta(t)$ s.t. it minimizes projection defects.

Example: projecting a basket of Heston models

A weighted sum of shifted Heston models (Heston basket) was approximated by a *single effective* shifted Heston model in Antonov, Misirpashaev and Piterbarg (2007) giving:

- approximation of the basket index european option (calibration) possibility
- approximation of exotic options written on the basket index (the MP to Heston model keeps ceratin dynamics)

Hybrid Heston/HW model example

Exchange rate → *Heston model*

$$
\begin{aligned}
dX &= X(r_1 - r_2)\,dt + X\sqrt{z}\,\lambda \cdot dW^* \\
dz &= \alpha\,(1 - z)\,dt + \sqrt{z}\,\gamma \cdot dW^*, \quad z(0) = 1
\end{aligned}
$$

Domestic and (shifted) foreign IR models → *HW1F models*

$$
\begin{aligned}
dr_1 &= (\zeta_1 - r_1\,a_1)\,dt + \sigma_1 \cdot dW^* \\
dr_2 &= (\zeta_2 - r_2\,a_2 - \sqrt{z}\,\lambda \cdot \sigma_2)\,dt + \sigma_2 \cdot dW^*
\end{aligned}
$$

Remarks:

1. All model parameters are time-dependent (we remove time-argument for better legibility)
2. W^* is a Brownian motion under the domestic risk-neutral measure
3. We use vector volatilities to stress for *arbitrary* correlations.

Due to correlations between Heston and IR's the model is not *affine*, and cannot be solved exactly

$$\Downarrow$$

<u>Two ways to handle the calibration to FX-options</u>

1. Come up with an approximation
2. Use Andreasen (2006) trick: modify Heston with a volatility displacement *to preserve its affine structure*

$$\begin{aligned} dX &= X(r_1 - r_2)\,dt + X(\sqrt{z}\,\lambda + \nu)\,\cdot dW^* \\ dz &= \alpha\,(1-z)\,dt + \sqrt{z}\,\gamma\,\cdot dW^*, \quad z(0) = 1 \end{aligned}$$

– the volatility displacement vector ν is correlated with IR's, $\nu \cdot \sigma_i \neq 0$, and not correlated with Heston vol and vol-of-vol, $\nu \cdot \lambda = 0$, $\nu \cdot \gamma = 0$

– Heston vol and vol-of-vol are not correlated with IR's, $\nu \cdot \sigma_i = 0$, $\nu \cdot \sigma_i = 0$

Below we keep our initial hybrid model structure are make an approximation for FX-option

$\rightarrow$

use Markovian projection to Heston model with orthogonal displaced volatility (Heston DV).

Reasons of using initial Heston/HW/HW structure w.r.t. to the Andreasen scheme:

- The initial structure has *standard* parameters meaning
- The Andreasen scheme cannot correlate IR's with SV. Also, a presence of the IR correlated displacement ν in the Heston diffusion term can eventually lead to extreme parameters (vol-of-vol and Heston correlation) for big $|\nu|$ or to small *effective* correlations between FX and IR's for small $|\nu|$.

FX-option for Heston/HW model.

A european options price with strike K and maturity T under risk neutral-measure

$$\mathbb{E}\left[\frac{(X(T)-K)^+}{N_1(T)}\right]$$

where domestic savings-account is $N_1(t) = \exp\left(\int_0^t ds\, r_1(s)\right)$.
Represent it in a domestic T-forward measure associated with a domestic zero bond $P_1(t,T)$ maturing at T,

$$\mathbb{E}\left[\frac{(X(T)-K)^+}{N_1(T)}\right] = P_1(0,T)\,\mathbb{E}_T\left[(X(T)-K)^+\right]$$

Introduce the forward FX-rate $S(t)$, a martingale process under the domestic T-forward measure

$$S(t) = \frac{X(t)\, P_2(t,T)}{P_1(t,T)}$$

simplifying the options price

$$\mathbb{E}\left[\frac{(X(T)-K)^+}{N_1(T)}\right] = P(0,T)\,\mathbb{E}_T\left[(S(T)-K)^+\right]$$

Introduce domestic and foreign zero bond volatilities $\Sigma_i(t,T)$

$$\frac{dP_i(t,T)}{P_i(t,T)} = \cdots - \Sigma_i(t,T)\cdot dW(t)$$

where

$$\Sigma_i(t,T) = \sigma_i(t)\int_t^T d\tau e^{-\int_t^\tau a_i(s)\,ds}$$

SDE for the forward FX-rate

$$dS(t) = S(t)\left(\sqrt{z_T(t)}\,\lambda(t) + \Sigma_1(t,T) - \Sigma_2(t,T)\right)\cdot dW_T$$

where W_T is T-forward brownian motion, related with risk-neutral one as

$$dW_T(t) = dW^*(t) + \Sigma_1(t,T)\,dt$$

Process z_T is the stochastic variance in the T-forward measure satisfying SDE

$$\begin{aligned} dz_T(t) &= \left(\alpha(t)\,(1 - z_T(t)) - \Sigma_1(t,T)\cdot\gamma(t)\,\sqrt{z_T(t)}\right)\,dt \\ &+ \sqrt{z_T(t)}\,\gamma(t)\,\cdot dW_T(t), \quad z(0) = 1 \end{aligned}$$

Denote:

– vector volatility displacement $\eta(t) \equiv \Sigma_1(t,T) - \Sigma_2(t,T)$

– risk-neutral/ T-forward term $\varepsilon(t) \equiv -\Sigma_1(t,T) \cdot \gamma(t)$

– $\tilde{z}(t) \equiv z_T(t)$

Omit:

– subscript T from average operator and Brownian motion

– omit time-argument t (all parameters are time-dependent)

Goal → evaluation of average $\mathbb{E}\left[(S(T) - K)^+\right]$ for the forward process

$$\begin{aligned} dS &= S\left(\sqrt{\tilde{z}}\,\lambda + \eta\right) \cdot dW \\ d\tilde{z} &= \left(\alpha\,(1 - \tilde{z}) + \varepsilon\,\sqrt{\tilde{z}}\right) dt + \sqrt{\tilde{z}}\,\gamma\ \cdot dW, \quad \tilde{z}(0) = 1 \end{aligned}$$

for *correlated* volatility vector λ, volatility displacement η and vol-of-vol vector γ.

One can apply MP to Heston model to calculate a european option price, but the accuracy will be *poor*.

Reason. The variance process

$$V(t) = \left| \sqrt{\tilde{z}(t)}\, \lambda(t) + \eta(t) \right|^2$$

has a floor. But the effective variance process $z(t)\,|\sigma_H(t)|^2$ of the approximating Heston model spans from zero to infinity.

Solution. Elaborate the MP target process having a positive floor for stochastic variance $\Rightarrow$ Heston model with displaced variance

Shifted Heston model with displaced volatility (Heston DV)

$$\begin{aligned} dS^*(t) &= (1 + \beta(t)\,\Delta S^*(t))(\sqrt{z(t)}\,\sigma_H(t) + \sigma_D(t))\cdot dW(t), \\ dz(t) &= \theta(t)\,(1 - z(t))\,dt + \sqrt{z(t)}\,\sigma_z(t)\cdot dW(t) \end{aligned}$$

The only difference is the previous Heston model is in the volatility displacement vector $\sigma_D(t)$.

Important requirement to the volatility displacement to preserve analytical solvability (affine structure) of the Heston DV

$\Rightarrow$

the vector $\sigma_D(t)$ should be *perpendicular* to both rate and SV volatility vectors, $\sigma_H(t)$ and $\sigma_z(t)$.

MP to Heston DV

MP to Heston DV gets an extra parameter for optimization, variance displacement

$$V_D(t) = |\sigma_D(t)|^2$$

This guarantees the variance positive floor and gives an extra degree of freedom for parameters fit.

As above, for the Heston case, consider the initial process pair $\{S(t), V(t)\}$

$$\begin{aligned} dS &= (1 + \beta(t)\,\Delta S)\;\Lambda(t) \cdot dW \\ dV &= \mu_V(t)\,dt + \sigma_V(t) \cdot dW \end{aligned}$$

We look for approximating Markovian pair $\{S^*(t), V^*(t)\}$

$$
\begin{aligned}
dS^* &= (1 + \beta(t)\,\Delta S^*)\ \sigma_S^*(t; S^*, V^*)\ \cdot dW \\
dV^* &= \mu_V^*(t; S^*, V^*)\,dt + \sigma_V^*(t; S^*, V^*)\ \cdot dW
\end{aligned}
$$

using a *modified* ansatz corresponding to the Heston DV model

$$
\begin{aligned}
\mu_V^*(t; s, v) &= (v - V_D(t))\ \left(\left(\ln|\sigma_H(t)|^2\right)' - \theta(t)\right) \\
&+ V_D'(t) + \theta(t)\,|\sigma_H(t)|^2 \\
|\sigma_V^*(t; s, v)|^2 &= (v - V_D(t))\ |\sigma_H(t)|^2\,|\sigma_z(t)|^2 \\
\sigma_S^*(t; s, v)\cdot\sigma_V^*(t; s, v) &= (v - V_D(t))\ \sigma_z(t)\cdot\sigma_H(t)
\end{aligned}
$$

MP to Heston DV: optimal parameters

This gives following optimal Heston DV parameters

$$
\begin{aligned}
|\sigma_H(t)|^2 &= \mathbb{E}[V(t)] - V_D(t) \\
\theta(t) &= \left(\ln|\sigma_H(t)|^2\right)' - \frac{1}{2}\left(\ln \operatorname{Var}[V(t)]\right)' + \frac{\mathbb{E}[|\sigma_V(t)|^2]}{2\operatorname{Var}[V(t)]} \\
|\sigma_z(t)|^2 &= \frac{\mathbb{E}[(V(t) - V_D(t))|\sigma_V(t)|^2]}{\mathbb{E}[(V(t) - V_D(t))^2]\mathbb{E}[V(t) - V_D(t)]} \\
\rho(t) &= \frac{\mathbb{E}[(V(t) - V_D(t))\Lambda(t)\cdot\sigma_V(t)]}{\sqrt{\mathbb{E}[(V(t) - V_D(t))^2]\mathbb{E}[(V(t) - V_D(t))|\sigma_V(t)|^2]}}
\end{aligned}
$$

Note that if we set the displacement to zero, $V_D(t) = 0$, the Heston case will be restored.

MP to Heston DV: optimal volatility displacement

Set effective covariance matrix

$$\begin{pmatrix} \mathbb{E}[|\sigma_V(t)|^4] & \mathbb{E}[|\sigma_V(t)|^2\,\Lambda(t)\cdot\sigma_V(t)] \\ \mathbb{E}[|\sigma_V(t)|^2\,\Lambda(t)\cdot\sigma_V(t)] & \mathbb{E}[(\Lambda(t)\cdot\sigma_V)^2] \end{pmatrix}$$

and compute eigen vector $\{e_1(t), e_2(t)\}$ corresponding to the biggest eigen value. The main direction vector

$$X_D(t) = e_1(t)\,|\sigma_V(t)|^2 + e_2(t)\,\Lambda(t)\cdot\sigma_V(t)$$

leads to the optimal volatility displacement

$$V_D(t) = \mathbb{E}[V(t)] - \frac{\mathbb{E}[X_D(t)]\mathrm{Var}[V(t)]}{\mathbb{E}[X_D(t)V(t)] - \mathbb{E}[X_D(t)]\mathbb{E}[V(t)]}$$

Details can be found at Antonov, Arneguy and Audet (2008)

Application to Heston/HW FX-option

The generic initial process

$$\begin{aligned} dS &= (1+\beta(t)\,\Delta S)\;\Lambda(t)\cdot dW \\ dV &= \mu_V(t)\,dt + \sigma_V(t)\cdot dW \end{aligned}$$

concretizes into

$$\begin{aligned} dS &= S\left(\sqrt{\tilde{z}}\,\lambda(t)+\eta(t)\right)\cdot dW \\ d\tilde{z} &= \left(\alpha(t)\,(1-\tilde{z})+\varepsilon(t)\,\sqrt{\tilde{z}}\right)\,dt+\sqrt{\tilde{z}}\,\gamma(t)\;\cdot dW \end{aligned}$$

after identification

- The skew $\beta = 1$ to preserve log-normality $(S(0) = 1)$
- Stochastic diffusion coefficient becomes $\Lambda(t) = \sqrt{\tilde{z}}\,\lambda(t) + \eta(t)$
- The stochastic variance $V(t) = |\Lambda(t)|^2$ and its SDE can be easily calculated

The stochastic variance

$$V(t) = |\Lambda(t)|^2 = z_T(t)|\lambda(t)|^2 + 2\sqrt{z_T(t)}\lambda(t) \cdot \eta(t) + |\eta(t)|^2$$

SDE looks as follows

$$dV(t) = \cdots + \sigma_V(t) \cdot dW = \cdots + \Lambda(t) \cdot \lambda(t)\ \gamma(t) \cdot dW$$

giving[a]

$$\sigma_V(t) = \Lambda(t) \cdot \lambda(t)\ \gamma(t)$$

The last step. Substitute above $\Lambda(t)$, $V(t)$ and $\sigma_V(t)$ into the MP final formulas to derive the Heston DV optimal parameters via the shifted CIR process moments, $\mathbb{E}[z_T(t)]$, $\mathbb{E}[\sqrt{z_T(t)}]$, $\mathbb{E}[z_T^{\frac{3}{2}}(t)]$ and $\mathbb{E}[z_T^2(t)]$. Technical details and computation of $z_T(t)$ moments can be found at Antonov, Arneguy and Audet (2008)

[a]Note that the final formulas do not include the variance drift μ_V.

Numerical results

Important example → a hybrid equity-interest rate setup with *correlated* driving factors (special case of the general FX hybrid when the foreign IR's become deterministic and have dividend yield sense)

	Equity Heston info
spot, $X(0)$	100 %
vol of rate, $\lvert\lambda\rvert$	25 %
correlation, ρ	-40 %
SV mean-reversion, α	25 %
vol-of-vol, $\lvert\gamma\rvert$	250 %

where the Heston correlation is that of two vol vectors, $\rho = \frac{\lambda \cdot \gamma}{|\lambda|\,|\gamma|}$.

This setup in terms of the Heston variance $dv = \kappa(\theta - v)dt + \xi\sqrt{v}\,dV$ corresponds to $\kappa = 25\%$, $\theta = v_0 = 0.0625$ and $\xi = 62.5\%$.

	domestic IR HW info
yield	5 %
vol, $\|\sigma_1\|$	1 %
mean-reversion, a_1	5 %

The foreign interest rates are deterministic with 2% yield, or, in equity terms, the continuous dividend yield is 2%.

Correlations

Correlations between the Brownian motions driving the equity $X(t)$, the stochastic volatility multiplier $z(t)$ and the domestic short rate $r_1(t)$

$$\begin{array}{lcccc} \text{Corr between IR and FX-rate} & \rightarrow & \dfrac{\lambda \cdot \sigma_1}{|\lambda|\,|\sigma_1|} & = & 30\% \\ \text{Corr between IR and SV} & \rightarrow & \dfrac{\gamma \cdot \sigma_1}{|\gamma|\,|\sigma_1|} & = & 15\% \end{array}$$

which results in following correlation matrix

	EQ	SV	IR
EQ	1	-0.4	0.3
SV	-0.4	1	0.15
IR	0.3	0.15	1

We compare Black implied volatilities of european option prices for a large set of maturities and strikes between[a]

- target implied volatilities ("Sim vol") calculated using 50,000 low-discrepancy Monte Carlo paths (their standard deviation ("*std.dev.*") is estimated for 50 independent runs)
- approximate implied volatilities ("Heston DV") calculated by the MP to the displaced volatility generalization of the Heston model
- approximate implied volatilities ("Heston") calculated by the MP to the standard Heston model

[a]The strikes are presented in percentage of forward values, 100% strike corresponds to ATM options.

T (*Year*)	Strike	Sim vol (*std.dev.*)	MP analytic vol		MP analytic vol error	
			Heston DV	Heston	Heston DV	Heston
1	86.07	24.45 (0.06)	24.49	24.50	**0.04**	0.04
1	92.77	22.25 (0.05)	22.27	22.26	**0.02**	0.02
1	100.00	20.36 (0.05)	20.32	20.30	**-0.04**	-0.06
1	107.79	19.42 (0.05)	19.34	19.32	**-0.08**	-0.10
1	116.18	19.67 (0.06)	19.64	19.63	**-0.03**	-0.04
3	77.12	22.61 (0.08)	22.65	22.65	**0.03**	0.04
3	87.82	20.05 (0.08)	20.05	20.03	**0.01**	-0.02
3	100.00	17.95 (0.09)	17.91	17.80	**-0.04**	-0.15
3	113.87	17.23 (0.13)	17.14	17.02	**-0.09**	-0.21
3	129.67	18.02 (0.18)	17.92	17.88	**-0.09**	-0.14
5	71.50	21.89 (0.06)	21.94	21.95	**0.06**	0.06
5	84.56	19.43 (0.05)	19.45	19.37	**0.02**	-0.06
5	100.00	17.49 (0.06)	17.44	17.21	**-0.05**	-0.28
5	118.26	16.83 (0.08)	16.72	16.46	**-0.11**	-0.37
5	139.85	17.55 (0.12)	17.42	17.30	**-0.13**	-0.25
10	62.23	21.55 (0.07)	21.61	21.57	**0.06**	0.02
10	78.89	19.52 (0.07)	19.51	19.26	**0.00**	-0.26
10	100.00	18.01 (0.08)	17.91	17.33	**-0.10**	-0.69
10	126.77	17.41 (0.11)	17.22	16.53	**-0.19**	-0.88
10	160.70	17.75 (0.16)	17.51	17.08	**-0.24**	-0.67

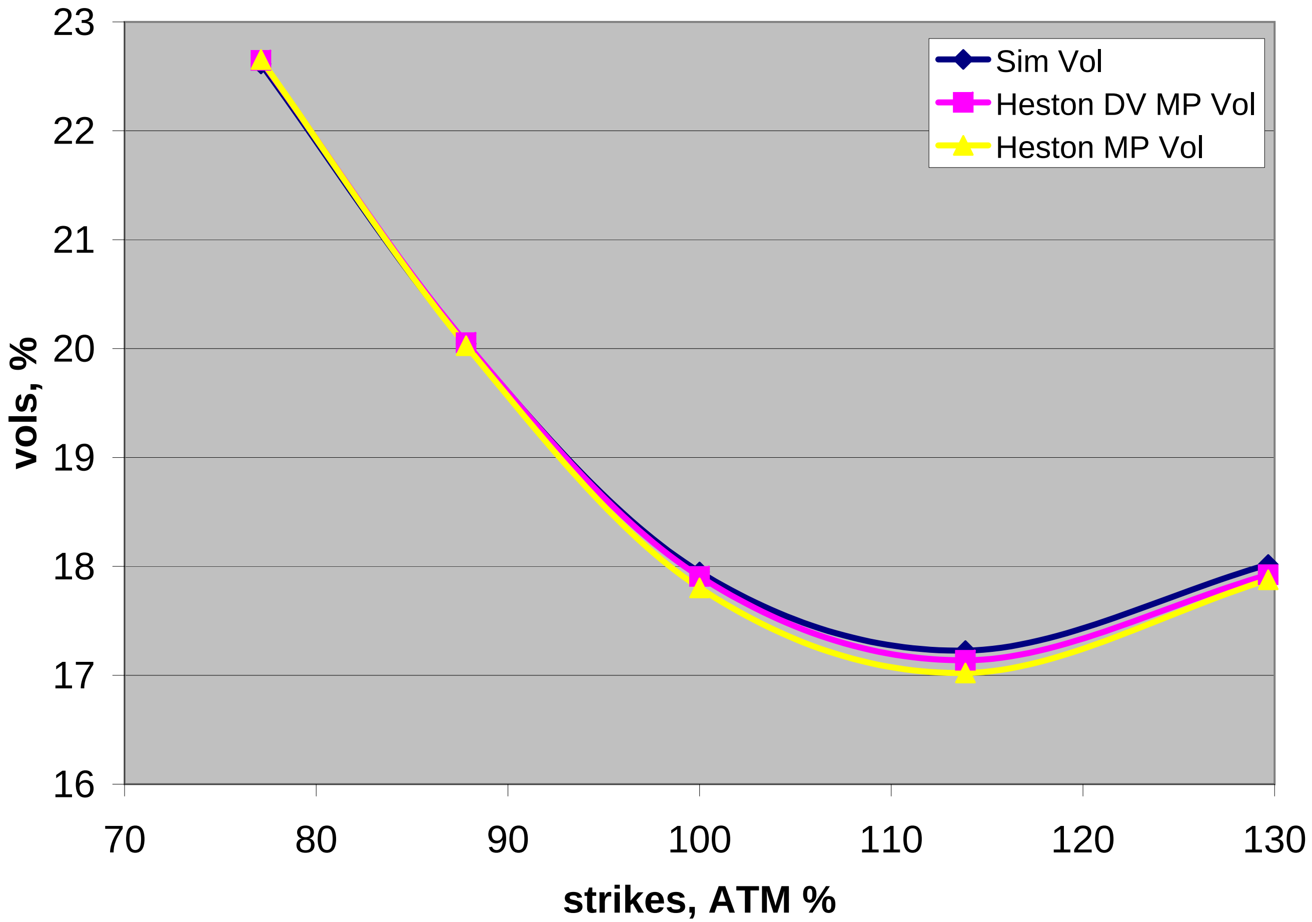

Figure 1: Heston/HW option implied volatilities for 3Y maturity

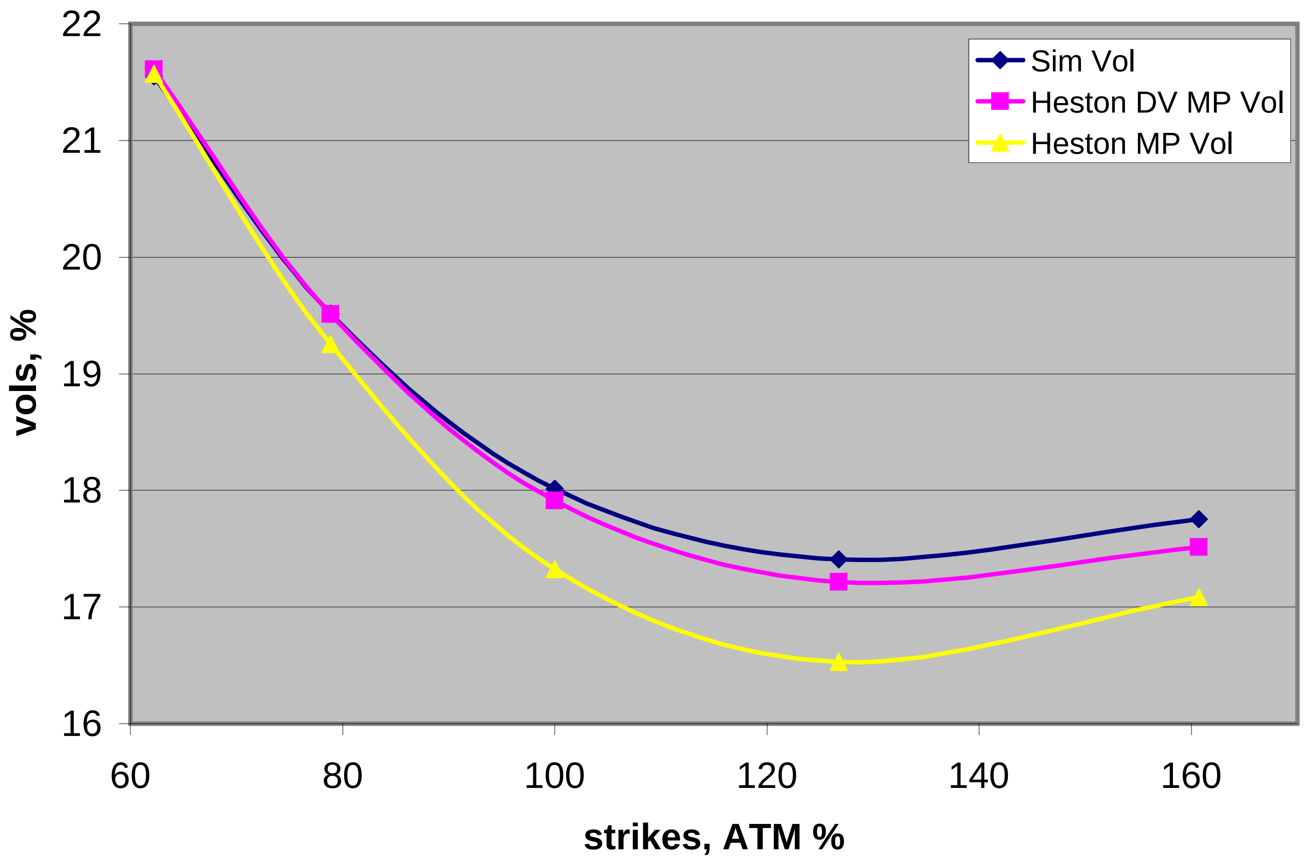

Figure 2: Heston/HW option implied volatilities for 10Y maturity

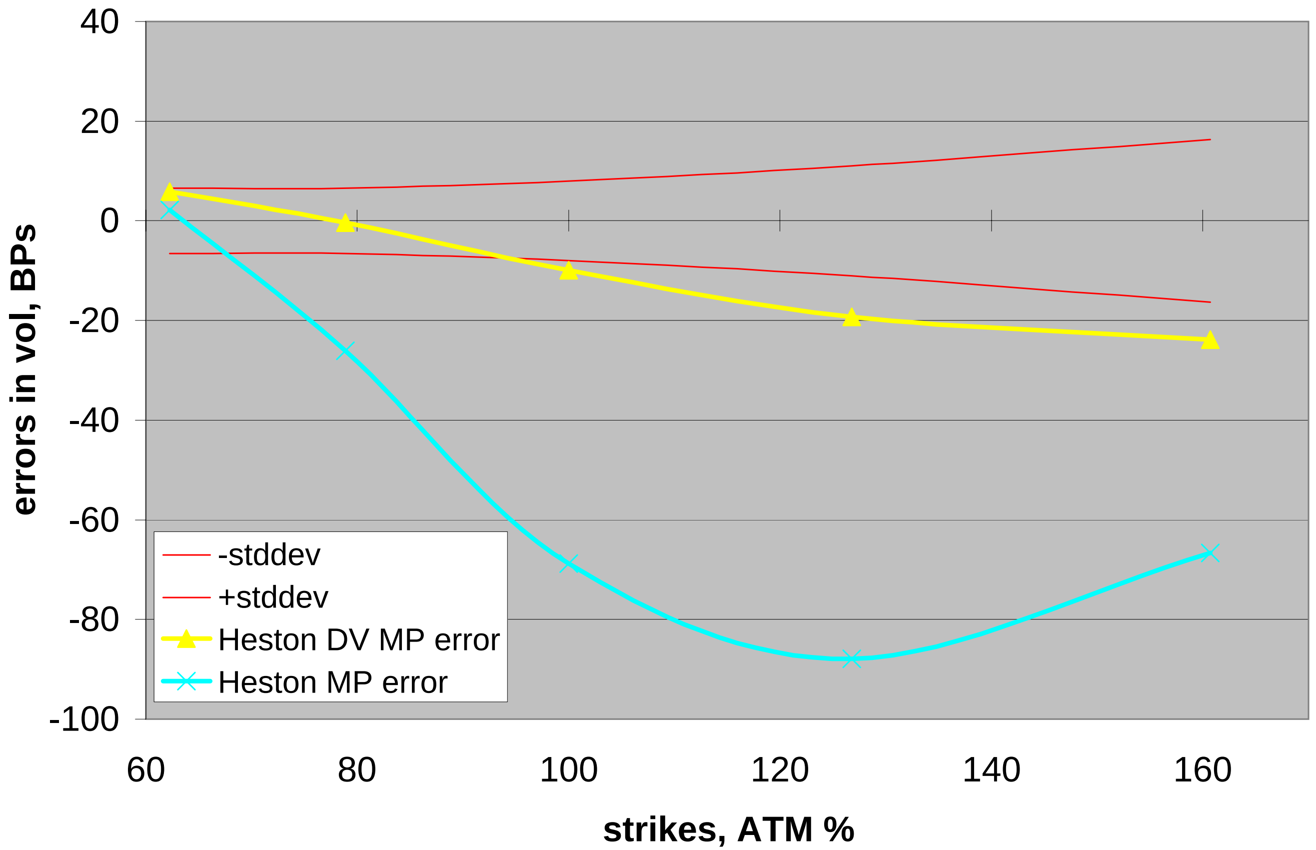

Figure 3: Heston/HW option implied volatility *errors* for 10Y maturity

Observations:

- Analytics captures both skew and smile.
- Excellent approximation quality for the Heston DV Markovian projection
- Degradation of the approximation quality of the pure Heston Markovian projection for large maturities

Summary

We presented

- a review of the modern analytical methods including Markovian projection to the displaced diffusion and the Heston model
- new results for the Markovian projection to the displaced volatility generalization of the Heston model
- important application to FX- or equity-option approximation for the correlated Heston/HW model
- numerical confirmation of the approximation quality

References

[1] L. Andersen and J. Andreasen (2002) "Volatile volatilities", *Risk*, Dec, 163–168.

[2] J. Andreasen (2006) "Closed Form Pricing of FX Options under Stochastic Rates and Volatility", Global Derivatives Conference, ICBI

[3] A. Antonov and T. Misirpashaev (2006a) "Markovian Projection onto a Displaced Diffusion: Generic Formulas with Applications," available at SSRN: http://ssrn.com/abstract=937860.

[4] A. Antonov and T. Misirpashaev (2006b) "Efficient Calibration to FX Options by Markovian Projection in Cross-Currency LIBOR Market Models," available at SSRN:

http://ssrn.com/abstract=936087

[5] A. Antonov, T. Misirpashaev and V. Piterbarg (2007) "Markovian Projection Onto a Heston Model," available at SSRN: http://ssrn.com/abstract=997001

[6] A. Antonov, M. Arneguy and N. Audet (2008) Markovian projection to a Displaced Volatility Heston model. NumeriX working paper, forthcoming at SSRN

[7] M. Avellaneda, D. Boyer-Olson, J. Busca, and P. Friz (2002) "Reconstructing volatility," Risk, October, 87–91.

[8] B. Dupire (1997) "A Unified Theory of Volatility," Banque Paribas working paper, reprinted in "Derivatives Pricing: the Classic Collection," edited by P. Carr, Risk Books, London, 2004.

[9] I. Gyöngy (1986) "Mimicking the One-Dimensional Marginal

Distributions of Processes Having an Ito Differential" *Probability Theory and Related Fields*, 71, 501–516.

[10] P. Henry-Labordere (2005) "A General Asymptotic Implied Volatility for Stochastic Volatility Models," available at SSRN: http://ssrn.com/abstract=698601

[11] A. Kawai (2003) "A New Approximate Swaption Formula in the Libor Market Model: an Asymptotic Expansion Approach," *Appl. Math. Finance*, 10, 49–74.

[12] A. Kawai, P. Jäckel (2006) "An asymptotic FX option formula in the cross currency Libor market model," working paper

[13] V. Piterbarg (2005a) "Stochastic volatility model with time-dependent skew," *Applied Mathematical Finance*, 12(2),147–185.

[14] V. Piterbarg (2005b) "A Multi-currency Model with FX

Volatility Skew," available at SSRN:
http://ssrn.com/abstract=685084.

[15] V. Piterbarg (2006a) "Markovian Projection Method for Volatility Calibration," available at SSRN: http://ssrn.com/abstract=906473.

[16] V. Piterbarg (2006b) "Modern Approaches to Stochastic Volatility Calibration," Proceedings of the WBS 3rd Fixed Income Conference, Amsterdam.

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