



*A fast and accurate FFT-based  
method for pricing early-exercise  
options under Lévy processes*

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# Pricing European options

In general we can write the price of a European option in 'Black-Scholes' form:

$$C(K, T) = \mathbb{E}^T \left[ (S(T) - K)^+ \right] = F\Pi_1 - K\Pi_2$$

where  $F = \mathbb{E}^T [S(T)]$  and:

$$\Pi_1 = \mathbb{S}(S(T) > K) \quad \Pi_2 = \mathbb{P}^T(S(T) > K)$$

Here  $\mathbb{S}$  is the 'stock-price' measure and  $\mathbb{P}^T$  is the T-forward measure.

## *Pricing European options (2)*

The key issue, at least for parametric models, is how to calculate these cumulative probabilities. From Lévy [1925], Gurland [1948] and Gil-Pelaez [1951] we know that if the density of a random variable  $X$  exists, we have:

$$\mathbb{P}(e^X \geq e^x) = \frac{1}{2} + \frac{1}{\pi} \operatorname{Re} \int_0^\infty \frac{\phi(t) \exp(-itx)}{it} dt$$

where  $\phi$  is the characteristic function.

## *Pricing European options (3)*

Heston [1993] was the first to use the results of Gil-Pelaez in option pricing, and found that we can write:

$$\Pi_1 = \mathbb{S}(S(T) \geq K) = \frac{1}{2} + \frac{1}{\pi} \operatorname{Re} \int_0^\infty \frac{\phi(u - i) \exp(-iuk)}{iuF} dt$$

$$\Pi_2 = \mathbb{P}^T(S(T) \geq K) = \frac{1}{2} + \frac{1}{\pi} \operatorname{Re} \int_0^\infty \frac{\phi(u) \exp(-iuk)}{iu} dt$$

with  $\phi$  the characteristic function under the T-forward measure, and  $k = \ln K$ .

## *Pricing European options (4)*

More recently, Carr and Madan [1999] used a technique that dates back to Dubner and Abate [1968] to arrive at a different formulation. Without going into details at the moment, they take the Fourier transform of the damped option price  $c(k) = e^{\alpha k}C(k)$ , and find:

$$\begin{aligned}\hat{c}(v) &= \int_{-\infty}^{\infty} c(k)e^{ivk} dk = \int_{-\infty}^{\infty} e^{ivk} e^{\alpha k} C(k) dk \\ &= \frac{\phi(v - i(\alpha + 1))}{-(v - i\alpha)(v - i(\alpha + 1))}\end{aligned}$$

## *Pricing European options (5)*

A condition for the Fourier transform to exist is that the function is in  $L^1$ , i.e. we must have:

$$\int_{-\infty}^{\infty} |C_{\alpha}(k)| dk < \infty$$

For  $\alpha > 0$  this holds true. When no damping is used, this cannot be true as  $C(k) \rightarrow F$  when  $k \rightarrow -\infty$ . Finally, we can invert the Fourier transform and find:

$$C(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i(v-i\alpha)k} \frac{\phi(v-i(\alpha+1))}{-(v-i\alpha)(v-i(\alpha+1))} dv$$

# *Pricing European options (6)*

Approach is very convenient as:

- ▶ Only one numerical integration is required, integrand decays faster than with Gil-Pelaez;
- ▶  $\alpha$  can be used to control the numerical accuracy, see e.g. Lee [2004] and Lord and Kahl [2006];
- ▶ Characteristic function is known in closed-form for many exponential Lévy models (including the exponentially affine jump-diffusion subclass);
- ▶ The density is typically not known in closed-form, or, when it is, it may involve special functions;

# Pricing Bermudan options

The next step is pricing Bermudan or even American options. Let us introduce some notation:

- ▶ Set of exercise dates  $\mathcal{T} = \{t_1, \dots, t_M\}$ ,  $0 = t_0 \leq t_1$ ;
- ▶ At  $t \in \mathcal{T}$  we may exercise into  $E(t, S(t))$ .

Setting  $V(t_M) = E(t_M)$  we find the Bermudan option price via backward induction as  $V(0) = C(0)$  in:

$$\begin{cases} C(t_m, S(t_m)) = e^{-r(t_{m+1}-t_m)} \mathbb{E}_{t_m} [V(t_{m+1}, S(t_{m+1}))] \\ V(t_m, S(t_m)) = \max\{C(t_m, S(t_m)), E(t_m, S(t_m))\} \end{cases}$$

## *Pricing Bermudan options (2)*

Many approaches have appeared in the literature:

- ▶ P(I)DE methods: highly feasible for Black-Scholes, much more complex for general Lévy models;
- ▶ Reiner [2000]: first to note that continuation value of a Bermudan under GBM can be written as a convolution. Only  $M$  timesteps required;
- ▶ Broadie and Yamamoto [2005]: combined Reiner's idea with the Fast-Gauss transform and the double-exponential integration formula to arrive at an  $O(MN)$  method,  $N$  being the no. of grid points.

## *Pricing Bermudan options (3)*

- ▶ Andricopoulos, Widdicks, Duck and Newton [2003] came up with the QUAD method for Black-Scholes. Its computational complexity is  $O(MN^2)$ ;
- ▶ O'Sullivan [2005] showed that the QUAD method is also applicable for exponential Lévy processes, as the density can be recovered via Fourier inversion;
- ▶ Këllezi and Webber [2004] and Maller, Solomon and Szimayer [2006] use tree-based methods;

# *Pricing Bermudan options (4)*

Motivation: to derive a method which is:

- ▶ computationally fast;
- ▶ not restricted to Gaussian-based models;
- ▶ modular in the sense that it works as long as we have a characteristic function available.

# The CONV method

The continuation value can be written as:

$$C(t_m, S(t_m)) = e^{-r(t_{m+1} - t_m)} \int_{-\infty}^{\infty} V(t_{m+1}, y) f(y | S(t_m)) dy$$

**Assumption:**  $f(y|x) = f(y-x)$ , hence:

$$C(t_m, S(t_m)) = e^{-r(t_{m+1} - t_m)} \int_{-\infty}^{\infty} V(t_{m+1}, x + z) f(z) dz$$

Assumption is clearly satisfied in exponential Lévy models, where  $x$  and  $y$  then represent log-asset prices. The assumption means log-returns are independent.

## The CONV method (2)

Like Carr and Madan, we dampen with  $\exp(\alpha x)$  and take the Fourier transform. This yields:

$$e^{r(t_{m+1}-t_m)} \mathcal{F}\left\{e^{\alpha x} C(t_m, x)\right\}(u) = \mathcal{F}\left\{e^{\alpha y} V(t_{m+1}, y)\right\}(u) \cdot \phi(-u + i\alpha)$$

where  $\phi(x + yi) = \int_{-\infty}^{\infty} e^{i(x+yi)z} f(z) dz$ .

Requirements so far:

- ▶ Fourier transform of damped option price exists, similar to the boundary condition in P(1)DEs;
- ▶  $|\phi(-u + i\alpha)| \leq \phi(i\alpha) = \mathbb{E}\left[S(T)^{-\alpha}\right] < \infty$

# *The CONV method (3)*

The algorithm should now be clear:

$$V(t_M, x) = E(t_M, x) \text{ for all } x$$

**For  $m = M-1$  to  $0$**

Dampen  $V(t_{m+1}, y)$  and take its Fourier transform

Multiply with  $\phi(-u+i\alpha)$

Invert and undamp, leading to  $C(t_m, x)$

$$V(t_m, x) = \max \{ E(t_m, x), C(t_m, x) \}$$

**Next  $m$**

Calculate  $C(t_0, x)$  and set  $V(t_0, x) = C(t_0, x)$ .

# *Discretising the CONV method*

Implementation using the FFT, so we require uniform grids for  $u$ ,  $x$  and  $y$  and  $j = 0, \dots, N-1$ :

$$u_j = u_0 + j\Delta u \quad x_j = x_0 + j\Delta x \quad y_j = y_0 + j\Delta x$$

Note –  $x$  represents the log-asset price at  $t_m$ ,  $y$  at  $t_{m+1}$ .  
Further, the Nyquist relation must be satisfied:

$$\Delta u \cdot \Delta x = \frac{2\pi}{N}$$

## *Discretising the CONV method (2)*

**Step 1 – The payoff transform:**

$$\begin{aligned}\mathcal{F}\left\{e^{\alpha y} V(t_{m+1}, y)\right\}(u) &= \int_{-\infty}^{\infty} e^{iuy} e^{\alpha y} V(t_{m+1}, y) dy \\ &\approx \Delta y \sum_{n=0}^{N-1} w_n e^{iu_j y_n} e^{\alpha y_n} V(t_{m+1}, y_n)\end{aligned}$$

can be evaluated using the FFT. We use the trapezoidal rule as this yielded the most stable results, though higher order Newton-Côtes can in principle be used.

# *Discretising the CONV method (3)*

## **Step 2 – Convolution**

Define  $\hat{v}(u) = \mathcal{F} \left\{ e^{\alpha y} V(t_{m+1}, y) \right\}(u)$  and calculate:

$$\hat{v}(u_j) \cdot \phi(-u_j + i\alpha)$$

for  $j = 0, \dots, N-1$ .

# Discretising the CONV method (4)

## Step 3 – Inverting the Fourier transform

Now we have to approximate:

$$\begin{aligned} e^{\alpha x} C(t_m, X_p) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\alpha x_p - iuy} \hat{v}(u_j) \phi(-u_j + i\alpha) du \\ &\approx \frac{\Delta u}{2\pi} \sum_{j=0}^{N-1} e^{-iu_j x_p} \hat{v}(u_j) \phi(-u_j + i\alpha) \end{aligned}$$

which can again be evaluated using the FFT.

Note – the left-rectangle rule is used here. The decay of the cf however dictates the convergence.

## *Discretising the CONV method (5)*

**Step 4 – To exercise, or not to exercise?**

Calculate:

$$\max \left\{ C(t_m, x_p), E(t_m, x_p) \right\}$$

and continue with Step 1 if any steps are left.

Clearly this method is  $O(MN \log N)$ , a lower complexity than the QUAD method, but a higher one than Broadie and Yamamoto's algorithm. However, the CONV method is generally applicable.

# *Error analysis of the CONV method*

One can rederive this discretised CONV formula by a Fourier series expansion of the continuation value. This reveals that:

- ▶ Only moment restriction on  $\alpha$  is necessary ( $L^1$ -integrability is replaced with  $L^1$ -summability);
- ▶ If  $\phi$  decays faster than a polynomial, the discretised CONV formula converges as  $O(1/N^2)$  for continuous payoff functions;
- ▶ If  $\phi$  decays as  $x^{-\beta}$ , the order is  $O(1/N^{\min(1+\beta, 2)})$  for continuous payoff functions.

# Dealing with discontinuities

We consider two discretisations:

- ▶ **Discretisation I:**

$x = y$  throughout, and  $\ln S(0)$  lies on the grid;

- ▶ **Discretisation II:**

At each time  $t_m$  we can place  $d_m$  on the  $x$ -grid. Specifically, for Bermudan options we:

- 1) Estimate  $d_m$  in  $C(t_m, d_m) = E(t_m, d_m)$ ;
- 2) Place  $d_m$  on the  $x$ -grid and recalculate  $C(t_m)$ ;
- 3) Re-evaluate exercise decision and continue.

# Pricing American options

We can approximate an American as a Bermudan with many exercise opportunities, or we can use extrapolation techniques. To this end we assume the Bermudan price  $V(\Delta t)$ , with  $\Delta t$  the timestep between two consecutive exercise moments, can be written as:

$$V(\Delta t) = V(0) + \sum_{i=1}^{\infty} a_i (\Delta t)^{\gamma_i}$$

American option price

We only know from Howison [2005] that  $\gamma_1 = 1$  in the Black-Scholes model, but the assumption that  $\gamma_i = i$  seems to work fine for all considered models.

# Greeks

Conveniently, differentiation is exact in Fourier space. We illustrate this with the delta. First, define:

$$A(u) = e^{r(t_1-t_0)} \mathcal{F} \left\{ e^{\alpha x} V(t_0, x) \right\}(u)$$

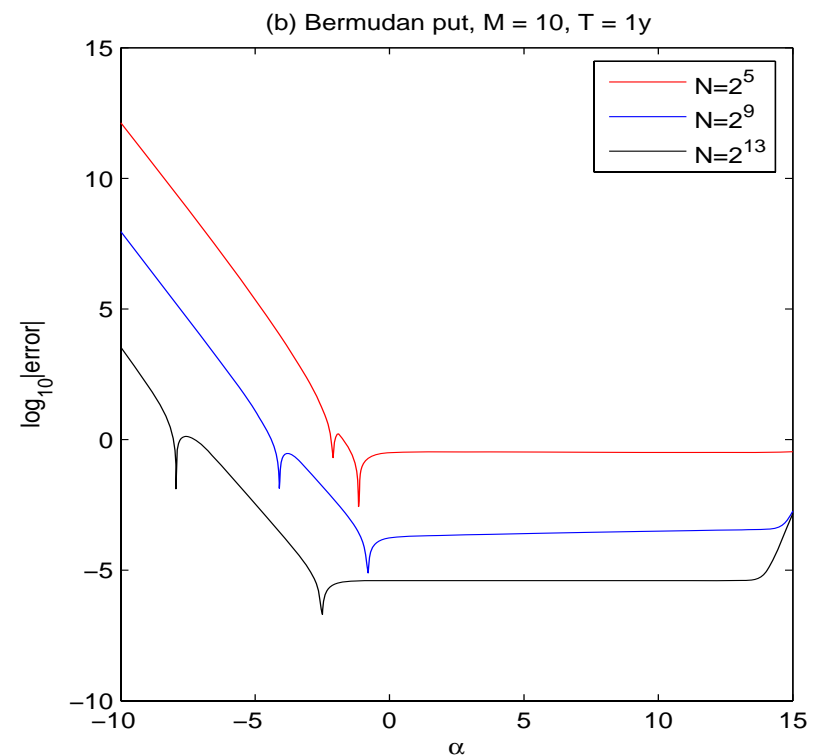
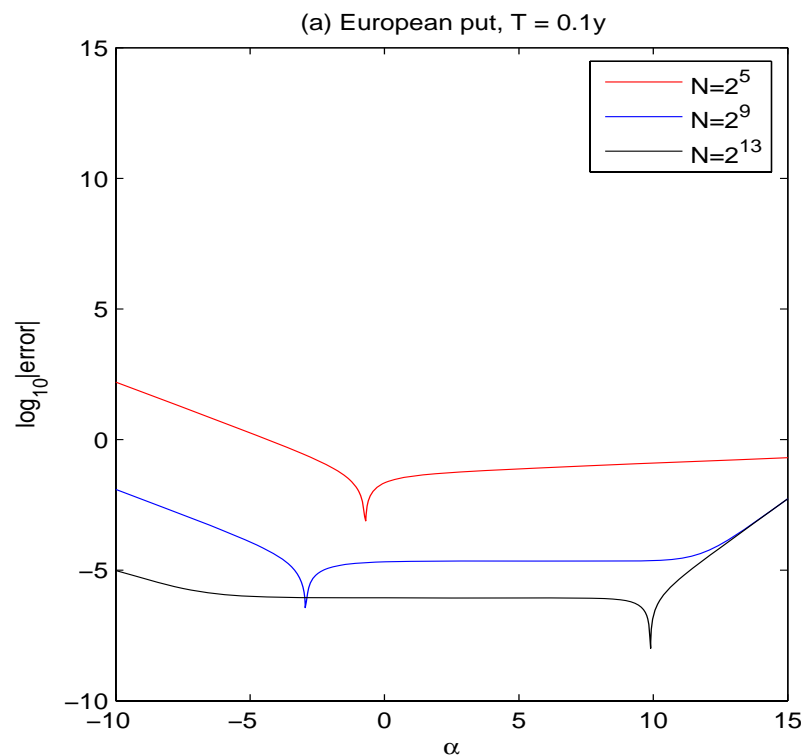
Then the delta can be found as:

$$\frac{\partial V}{\partial S} = \frac{e^{-\alpha x - r(t_1-t_0)}}{S} \left[ \mathcal{F}^{-1} \left\{ -iuA(u) \right\}(x) - \alpha \mathcal{F}^{-1} \left\{ A(u) \right\}(x) \right]$$

requiring just one additional FFT at the final timestep.

# Choice of damping coefficient

Recently Lord and Kahl [2006] related the problem of  $\alpha$  in the Carr-Madan framework to saddle point methods. We opt for  $\alpha = 0$  as the payoff-transform is not generally known. Illustration for puts in VG:

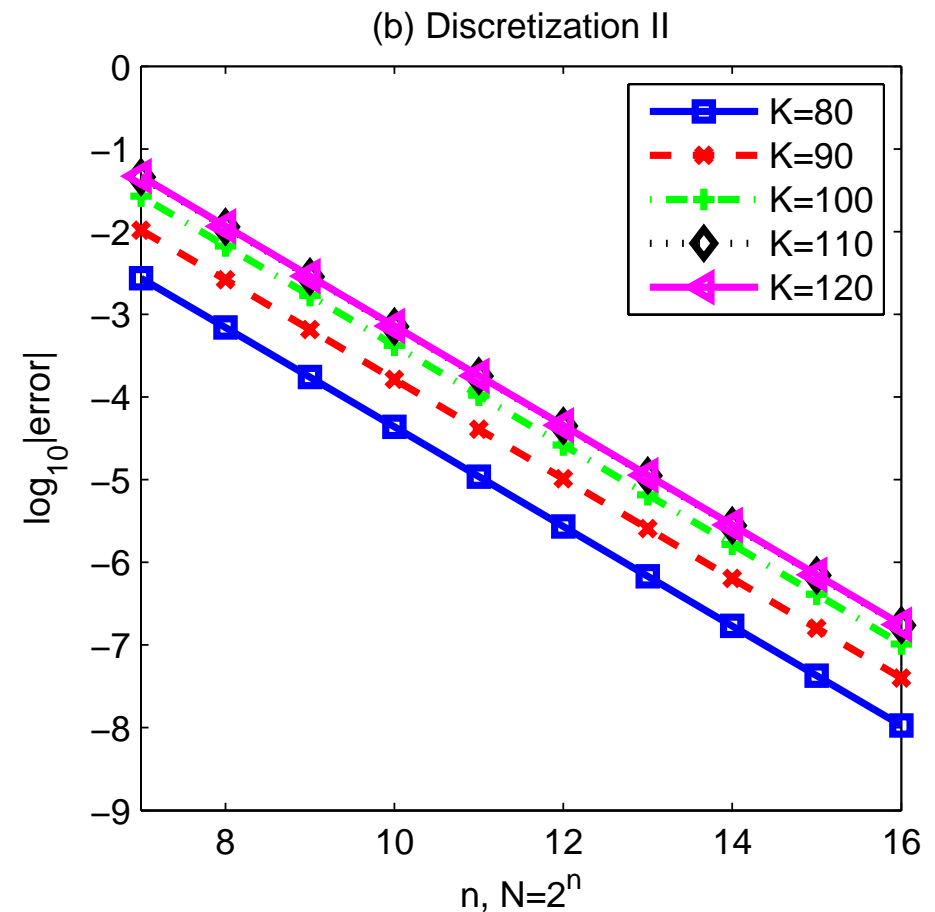
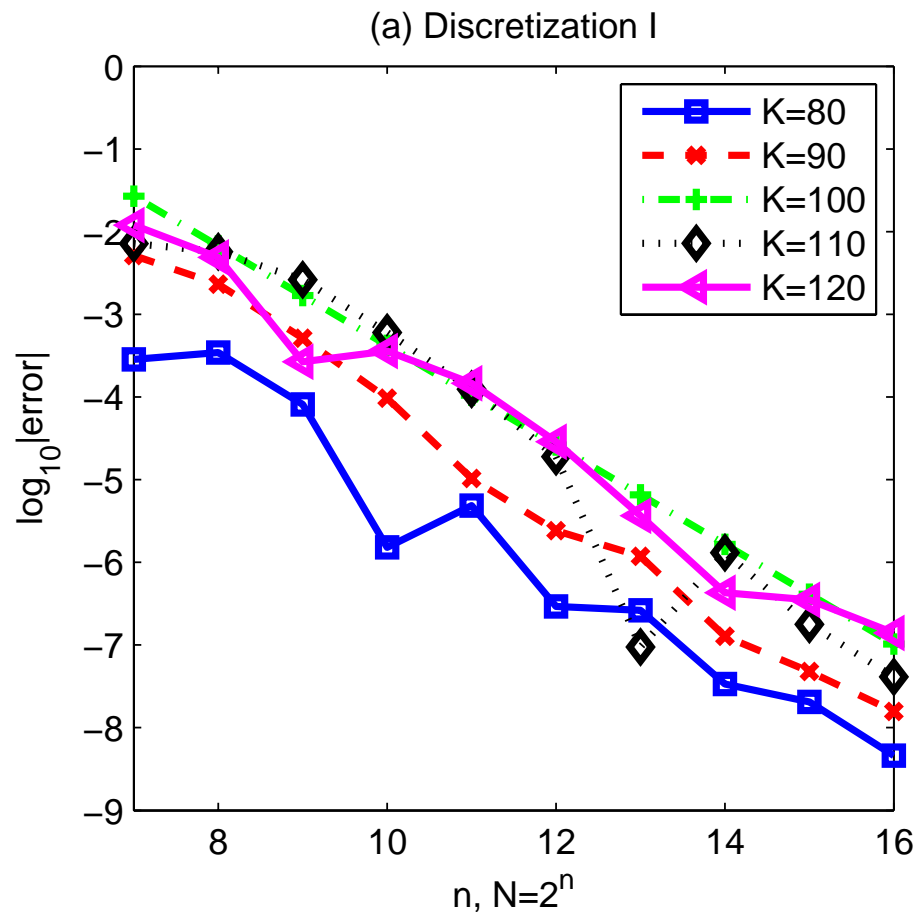


# *Numerical results*

- ▶ Variety of results in Black-Scholes, VG and CGMY
- ▶ Reference values from literature or from CONV method with  $2^{20}$  gridpoints
- ▶ Univariate results generated in Matlab, on an Intel Xeon CPU 5160, 3 GHz with 2GB RAM
- ▶ Multivariate results generated in C, on an Intel Pentium IV 2667, 1 GHz with 8GB RAM

# Discretisation I vs. II

Placing the discontinuity (here the strike) on the grid ensures smooth convergence:



# Bermudan put – Black-Scholes

10-times exercisable Bermudan put with  $T = 1y$

$S = 100$ ,  $K = 110$ ,  $r = 10\%$ ,  $\sigma = 25\%$

TABLE 3

*CPU time, error and convergence rate pricing a 10-times exercisable Bermudan put under T1-GBM;  $K = 110$ ,  $T = 1$  and  $V_{ref}(0, S_0) = 11.98745352$ ,*

$(N = 2^n)$ $n$	Discretisation I			Discretisation II		
	time(sec)	error	conv.	time(sec)	error	conv.
7	0.0003	9.09e-3	-	0.006	-3.31e-2	-
8	0.0004	-1.29e-3	7.1	0.007	-8.53e-3	3.9
9	0.0007	1.80e-6	717.7	0.008	-2.13e-3	4.0
10	0.0019	2.71e-5	0.06	0.011	-5.33e-4	4.0
11	0.0041	-9.31e-6	2.9	0.018	-1.33e-4	4.0
12	0.0082	-1.31e-5	0.71	0.029	-3.33e-5	4.0

# Bermudan put – Variance Gamma

10-times exercisable Bermudan put with  $T = 1$  y

$S = 100$ ,  $K = 110$ ,  $r = 10\%$ ,  $\sigma = 12\%$ ,  $\theta = -14\%$ ,  $\nu = 20\%$

TABLE 4

*CPU time, error and convergence rate pricing a 10-times exercisable Bermudan put under T2-VG;  $K = 110, T = 1$  with reference value  $V_{ref}(0, S_0) = 9.040646119$ .*

$(N = 2^n)$ $n$	Discretisation I			Discretisation II		
	time(sec)	error	conv.	time(sec)	error	conv.
7	0.0003	8.45e-2	-	0.006	9.53e-2	-
8	0.0005	9.02e-3	9.4	0.007	1.09e-2	8.7
9	0.0010	-1.70e-4	53.1	0.008	2.50e-3	4.4
10	0.0019	-2.04e-4	0.8	0.010	6.51e-4	3.8
11	0.0047	-4.28e-5	4.8	0.016	1.65e-4	4.0
12	0.0094	-1.11e-5	3.9	0.026	4.15e-5	4.0

# American put – Black-Scholes

Reference value from a PDE with a very fine grid

3-point extrapolation on Bermudans with  $M = 128, 64$  and  $32$

TABLE 5

*CPU time and errors for an American put under T1-GBM, with:  $K = 110, T = 1$ ,  $V_{ref}(0, S(0)) = 12.169417$*

$(N = 2^n)$ $n$	P(N/2)			Richardson		
	time(sec)	error	conv.	time(sec)	error	conv.
7	0.025	-6.34e-2	–	0.011	-4.88e-2	–
8	0.055	-2.34e-3	2.7	0.020	8.77e-3	5.6
9	0.130	-9.49e-3	2.5	0.038	2.24e-3	3.9
10	0.346	-4.19e-3	2.3	0.078	5.53e-4	4.1
11	1.18	-1.95e-3	2.1	0.181	1.29e-4	4.3
12	3.98	-9.40e-4	2.1	0.436	2.30e-5	5.6

# American put – Black-Scholes (2)

Accurate Greeks are easily obtained at virtually no extra cost:

TABLE 9

*Values of hedge parameters for an American put under T1-GBM;  $K = 110, T = 0.1$*

$(N = 2^d)$	American put:	
$d$	$\Delta_{ref} = -0.62052$	$\Gamma_{ref} = 0.0284400$
7	-0.62170	0.028498
8	-0.62035	0.028687
9	-0.62050	0.028464
10	-0.62053	0.028463
11	-0.62054	0.028463
12	-0.62055	0.028463

# American put – VG/CGMY

VG example uses same parameters as before

CGMY test sets have  $Y = 0.5$  (Almendral and Oosterlee [2007]) and  $Y = 1.0102$  (Wang, Wan and Forsyth [2006])

TABLE 6  
*CPU time and errors for American puts under VG and CGMY*

$(N = 2^n)$	T2-VG		T3-CGMY		T4-CGMY	
	$K = 110, T = 1$		$K = 1, T = 1$		$K = 98, T = 0.25$	
	$V_{ref}(0, S(0)) = 10.0000$		$V_{ref}(0, S(0)) = 0.112152$		$V_{ref}(0, S(0)) = 9.225439$	
$n$	time(sec)	error	time(sec)	error	time(sec)	error
7	0.073	-3.49e-1	0.062	-6.35e-3	0.074	-1.89e-1
8	0.096	4.13e-2	0.097	1.38e-4	0.093	2.93e-2
9	0.115	1.37e-2	0.116	1.16e-4	0.118	-1.30e-3
10	0.157	-6.17e-3	0.160	1.10e-5	0.162	-3.97e-4
11	0.270	6.03e-3	0.275	1.18e-5	0.278	2.89e-4
12	0.466	1.31e-3	0.482	-2.35e-6	0.483	9.59e-5

# Multivariate Black-Scholes

4D basket put option,  $T = 1y$ , Bermudan has  $M = 10$

$S = 40$ ,  $K = 40$ ,  $r = 6\%$ ,  $q = 4\%$ ,  $\sigma = 20\%$ ,  $\rho = 25\%$

TABLE 7

*CPU time and prices for multi-asset European and 10-times exercisable Bermudan basket put options under T5-GBM,  $K = 40$ ,  $T = 1$*

European			10-times exerc. Bermudan	
$N$	result	time (sec)	result	time (sec)
$16^4$	1.6428	0.02	1.7721	0.18
$32^4$	1.6537	0.51	1.7390	3.40
$64^4$	1.6539	9.5	1.7394	65.7
$128^4$	1.6538	202.4	1.7393	1526.3

# Conclusions

- ▶ We presented an FFT-based method for pricing early exercise options, certain path-dependent options (barriers, Asians, cliquets) can also be valued along the same lines
- ▶ The method is  $O(MN \log N)$
- ▶ Very flexible w.r.t. the choice of asset process, unnecessary to develop bespoke schemes
- ▶ Work underway in various directions – extensions to stochastic volatility, incorporating non-uniform grids and improvements for multiple dimensions (work by C.C.W. Leentvaar from Delft)