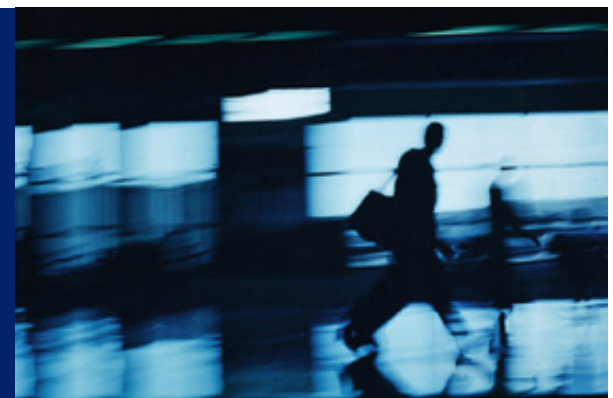


Stochastic Processes for Implied Volatility

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Stochastic Spot Volatility Models (in the risk neutral world)

The asset $(S_t)_{t \geq 0}$ and the spot volatility $(\sigma_t)_{t \geq 0}$ follows the stochastic process

$$\left. \begin{aligned} dS_t &= (r - q)S_t dt + \sigma(S_t, t)S_t dW_t^0 \\ d\sigma_t^2 &= \alpha(S_t, \sigma_t^2, t)dt + \sum_{i=0}^n \theta_i(S_t, \sigma_t^2, t)dW_t^i \end{aligned} \right\} \text{(SSV)}$$

with $\alpha(\cdot, \cdot) : \mathbb{R}^2 \times [0, T] \rightarrow \mathbb{R}$ and $\theta_i(\cdot, \cdot) : \mathbb{R}^2 \times [0, T] \rightarrow \mathbb{R}$ being measurable functions satisfying certain growth and Lipschitz conditions

Stochastic Implied Volatility Model (in the risk neutral world)

There exists a finite collection of M European call options with strike prices K_m and maturities T_m .

Then the asset $(S_t)_{t \geq 0}$ and the implied volatility $(I_t)_{t \geq 0} \in \mathfrak{R}^M$ follows the stochastic process

$$\left. \begin{aligned} dS_t &= (r - q)S_t dt + \sigma(S_t, t)S_t dW_t^0 \\ dI_t^m(S_t, I_t, K_m, T_m) &= u_m(S_t, I_t, t)dt + \gamma_m(S_t, I_t, t)dW_t^0 + \sum_{k=1}^N v_{mk}(S_t, I_t, t)dW_t^k \end{aligned} \right\} \text{(SIV)}$$

with $u(\cdot, \cdot) : \mathfrak{R}^{M+1} \times [0, T] \rightarrow \mathfrak{R}^M$, $\gamma(\cdot, \cdot) : \mathfrak{R}^{M+1} \times [0, T] \rightarrow \mathfrak{R}^M$, $v(\cdot, \cdot) : \mathfrak{R}^{M+1} \times [0, T] \rightarrow \mathfrak{R}^{M \times N}$ and $\sigma(\cdot, \cdot) : \mathfrak{R} \times [0, T] \rightarrow \mathfrak{R}$ satisfying certain growth and Lipschitz conditions

Notation

Price of a European call option with strike K , maturity T and stochastic **spot volatility** σ_t

$$C = C(t, S_t, K, T; \sigma_t)$$

Price of a European call option with strike K_m , maturity T_m and stochastic **implied volatility** I_t^m

$$C^{BS} = C^{BS}(t, S_t, K_m, T_m; I_t^m)$$

Assuming the spot volatility model meets the market prices, we can write for every set $(K_m, T_m)_{m \leq M}$

$$C(t, S_t, K_m, T_m; \sigma_t) = C^{BS}(t, S_t, K_m, T_m; I_t^m)$$

Drift restriction for (SIV)

Theorem (Schönbucher 1999): Assuming the market to be arbitrage free, then for every $m = 1, \dots, M$, the risk-neutral drift coefficients $u_m(\cdot, t) : \mathfrak{R}^{M+1} \mapsto \mathfrak{R}$ of the implied volatility of European call options with strike K_m and maturity T_m at time $t \in [0, T)$ are given by

$$I_t^m u_m = \frac{1}{2(T_m - t)} \left((I_t^m)^2 - \sigma_t^2 \right) - \frac{1}{2} d_1 d_2 \left(\sum_{k=1}^N v_{mk}^2 + \gamma_m^2 \right) + \frac{d_2}{\sqrt{T_m - t}} \sigma_t \gamma_m$$

with

$$d_1 = \frac{\ln\left(\frac{S_t}{K_m}\right) + \left(r - q + \frac{1}{2} (I_t^m)^2 (T_m - t)\right)}{I_t^m \sqrt{T_m - t}} \quad \text{and} \quad d_2 = d_1 - I_t^m \sqrt{T_m - t}$$

Drift restriction for (SIV)

Proof of Theorem (Schönbucher 1999): Under the assumption of absence of arbitrage, the option price process $(C^{BS}(t, S_t, T_m, K_m; I_t^m))_{t \geq 0}$ has the drift $rC^{BS}(t, S_t, T_m, K_m; I_t^m)$.

Using Itô's Lemma and the process (SIV), the drift component of dC^{BS} is given by the formula

$$\begin{aligned} rC^{BS} dt = & \frac{\partial C^{BS}}{\partial t} dt + (r - q) \frac{\partial C^{BS}}{\partial S} dt + \frac{1}{2} \sigma_t^2 S_t^2 \frac{\partial^2 C^{BS}}{\partial S^2} dt + u_m \frac{\partial C^{BS}}{\partial I^m} dt \\ & + \frac{1}{2} \left(\sum_{k=1}^N v_{mk}^2 \right) \frac{\partial^2 C^{BS}}{\partial (I^m)^2} dt + \frac{1}{2} \gamma_m^2 \frac{\partial^2 C^{BS}}{\partial (I^m)^2} dt + \gamma_m \sigma_t S_t \frac{\partial^2 C^{BS}}{\partial S \partial I^m} dt. \end{aligned}$$

The result is proved by calculating the derivatives of C^{BS} .

Special cases of drift restriction for (SIV)

- Constant implied volatility i.e. $\gamma_m \equiv u_m \equiv v_{mk} \equiv 0$

$$\Rightarrow I_t^m(K_m, T_m) = \sigma \text{ for every } m = 1, 2, \dots, M \text{ and } t \in [t_0, T_m).$$

- Time-dependent but non-stochastic implied volatility i.e. $\gamma_m \equiv v_{mk} \equiv 0$

$$\Rightarrow u_m = \frac{1}{2(T_m - t)I_t^m} \left((I_t^m)^2 - \sigma_t^2 \right) \text{ for every } m = 1, 2, \dots, M \text{ and } t \in [t_0, T_m).$$

and (SIV) reduces to
$$\frac{dI_t^m}{dt} = \frac{1}{2(T_m - t)I_t^m} \left((I_t^m)^2 - \sigma_t^2 \right)$$

which is solved by
$$(I_t^m)^2 = \frac{1}{(T_m - t)} \int_t^{T_m} \sigma_s^2 ds.$$

Spot volatility in the context of (SIV)

- **Brace et al 2001**: The spot volatility of the underlying asset must equal the implied volatility of an at-the-money option maturing immediately.

$$\lim_{T_m \rightarrow t} I_t^m(S_t, S_t, T_m) = \sigma(S_t, t) = \sigma_t.$$

- **Schönbucher 1999**: Assume the implied volatility together with its volatility parameters γ_m, v_{mk} and the drift of the implied volatility u_m to be given. Then the spot volatility has to be chosen such that

$$\sigma_t = \gamma_m d_2 \sqrt{T_m - t} + \sqrt{(I_t^m)^2 - 2(T_m - t)I_t^m u_m + d_2(T_m - t) \left(\gamma_m^2 d_2 - d_1 \left(\sum_{k=1}^N v_{mk}^2 + \gamma_m^2 \right) \right)}$$

Spot Volatility Models in the context of (SIV)

Theorem: If the asset $(S_t)_{t \geq 0}$ and the spot volatility $(\sigma_t)_{t \geq 0}$ satisfy under the risk-neutral probability the stochastic differential equation (SSV), then the implied volatility I_t^m satisfies the drift restriction with $N = n$ where

$$\gamma_m(S_t, I_t, t) = \left(\frac{\frac{\partial C}{\partial S} - \frac{\partial C^{BS}}{\partial S}}{\frac{\partial C}{\partial I^m}} \right) \sigma_t S_t + 2 \left(\frac{\frac{\partial C}{\partial \sigma}}{\frac{\partial C}{\partial I^m}} \right) \theta_0(S_t, \sigma_t^2, t)$$

and

$$v_{mk}(S_t, I_t, t) = 2 \left(\frac{\frac{\partial C}{\partial \sigma}}{\frac{\partial C}{\partial I^m}} \right) \theta_k(S_t, \sigma_t^2, t).$$

Spot Volatility Models in the context of (SIV)

Proof of Theorem: In order to extract a stochastic differential equation for the implied volatility we set for every $(K_m, T_m)_{m \leq M}$

$$C(t, S_t, K_m, T_m; \sigma_t) = C^{BS}(t, S_t, K_m, T_m; I_t^m)$$

which has a unique solution $I_t^m = I_t^m(S_t, K_m, T_m; \sigma_t)$ defining the implied volatility.

Since $dC = dC^{BS}$

$$\begin{aligned} \Rightarrow \frac{\partial C^{BS}}{\partial I^m} dI_t^m &= \frac{\partial C}{\partial S} \sigma_t S_t dW_t^0 + \frac{\partial C}{\partial (\sigma^2)} \left(\sum_{i=0}^n \theta_i(S_t, \sigma_t^2, t) dW_t^i \right) + rC dt - \frac{\partial C^{BS}}{\partial S} \sigma_t S_t dW_t^0 \\ &- \left(\frac{\partial C^{BS}}{\partial t} + \frac{\partial C^{BS}}{\partial S} (r - q) S_t + \frac{1}{2} \frac{\partial^2 C^{BS}}{\partial S^2} \sigma_t^2 S_t^2 \right) dt - \frac{1}{2} \frac{\partial^2 C^{BS}}{\partial (I^m)^2} d\langle I_t^m, I_t^m \rangle - \frac{\partial^2 C^{BS}}{\partial I^m \partial S} d\langle I_t^m, S_t \rangle \end{aligned}$$

Spot Volatility Models in the context of (SIV)

Proof of Theorem: Applying Itô's Lemma to $I_t^m = I_t^m(S_t, K_m, T_m; \sigma_t)$ yields

$$\begin{aligned} dI_t^m &= \frac{\partial I_t^m}{\partial t} dt + \frac{\partial I_t^m}{\partial S} (r - q) S_t dt + \frac{\partial I_t^m}{\partial S} \sigma_t S_t dW_t^0 + \frac{\partial I_t^m}{\partial(\sigma^2)} \alpha(S_t, \sigma_t^2, t) dt \\ &+ \frac{\partial I_t^m}{\partial(\sigma^2)} \left(\sum_{i=1}^n \theta_i(S_t, \sigma_t^2, t) dW_t^i \right) + \frac{1}{2} \frac{\partial^2 I_t^m}{\partial S^2} \sigma_t^2 S_t^2 dt + \frac{1}{2} \frac{\partial^2 I_t^m}{\partial(\sigma^2)^2} \left(\sum_{i=1}^n \theta_i^2(S_t, \sigma_t^2, t) \right) dt \\ &+ \frac{\partial^2 I_t^m}{\partial S \partial(\sigma^2)} \theta_0(S_t, \sigma_t^2, t) \sigma_t S_t dt \end{aligned}$$

from which we obtain

$$d\langle I_t^m, I_t^m \rangle = \left[\frac{\partial I_t^m}{\partial S} \sigma_t S_t \right]^2 + \left[\frac{\partial I_t^m}{\partial(\sigma^2)} \right]^2 \left[\sum_{i=1}^n \theta_i^2(S_t, \sigma_t^2, t) \right]$$

$$d\langle I_t^m, S_t \rangle = \frac{\partial I_t^m}{\partial S} \sigma_t^2 S_t^2 + \frac{\partial I_t^m}{\partial(\sigma^2)} \theta_0(S_t, \sigma_t^2, t) \sigma_t$$

Spot volatility models in the context of (SIV)

Proof of Theorem: Furthermore, we know by the implicit function theorem that

$$\frac{\partial I_t^m}{\partial S} = \left(\frac{\frac{\partial C}{\partial S} - \frac{\partial C^{BS}}{\partial S}}{\frac{\partial C^{BS}}{\partial I^m}} \right) \quad \text{and} \quad \frac{\partial I_t^m}{\partial(\sigma^2)} = 2 \left(\frac{\frac{\partial C}{\partial \sigma}}{\frac{\partial C^{BS}}{\partial I^m}} \right) .$$

This, in turn with calculating all the derivatives of C^{BS} yields the desired result.

Special case of (SSV) – Hull and White model (risk neutral world)

The asset $(S_t)_{t \geq 0}$ and the spot volatility $(\sigma_t)_{t \geq 0}$ follows the stochastic process

$$dS_t = (r - q)S_t dt + \sigma_t S_t dW_t^0$$

$$d\sigma_t^2 = \alpha(\sigma_t^2, t)dt + \tilde{\theta}_1(\sigma_t^2, t)dW_t^1$$

Hull/White
model

with $\alpha(\cdot, \cdot) : \mathbb{R} \times [0, T] \rightarrow \mathbb{R}$ and $\tilde{\theta}_1(\cdot, \cdot) : \mathbb{R} \times [0, T] \rightarrow \mathbb{R}$ being measurable functions satisfying certain growth and Lipschitz conditions

Symmetry of smile in context of (SSV)

Theorem: In the Hull White model, the smile is a symmetric function of the variable

$$x = \ln\left(\frac{K}{S_t e^{(r-q)(T-t)}}\right).$$

Proof of Theorem: The solution to the Hull/White model is given by the formula

$$C(t, S_t, K, T; \sigma_t) = S_t e^{-q(T-t)} \int_0^{\infty} g(x, u) p(u; \sigma_t) du$$

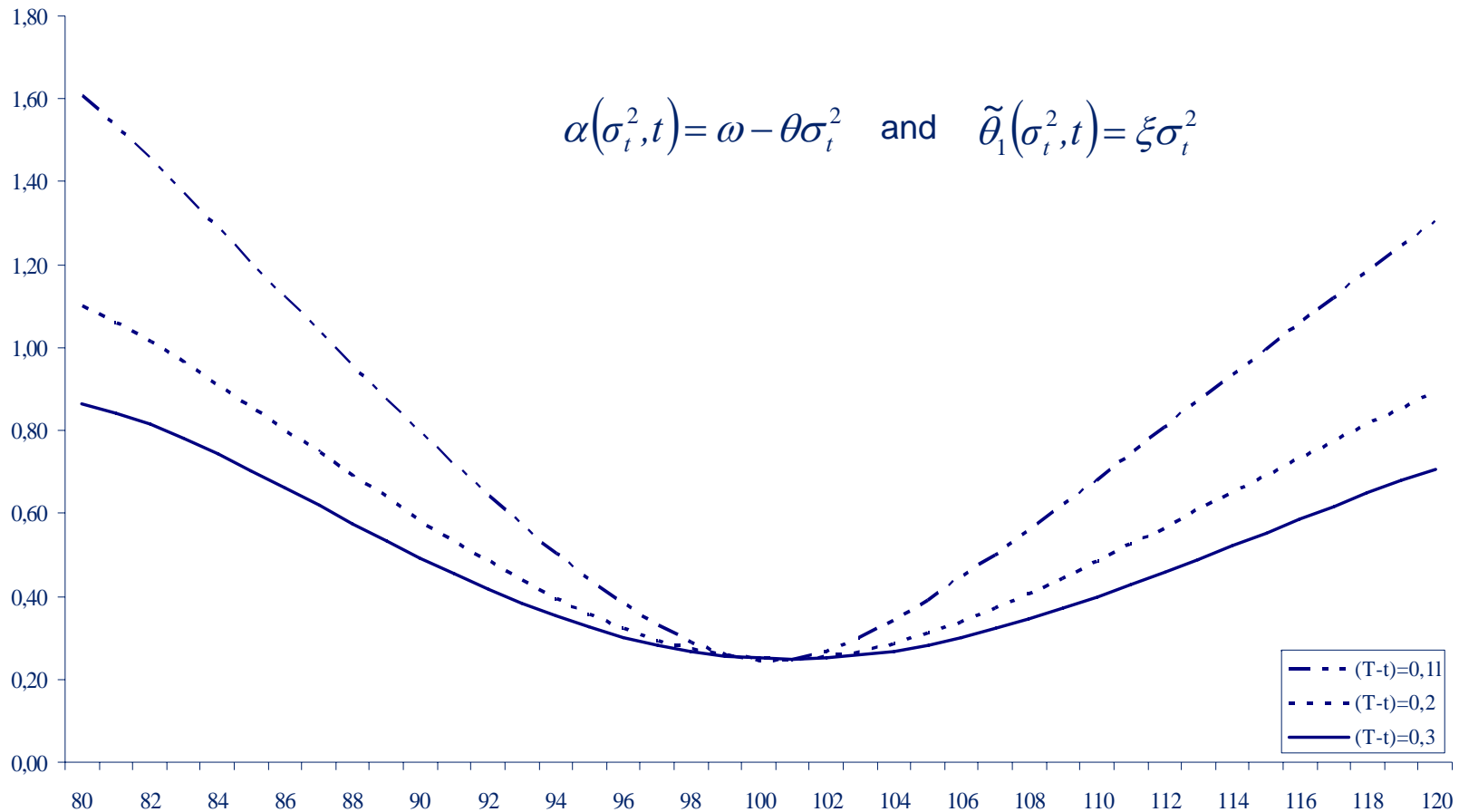
where

$$g(x, u) = N\left(\frac{-x}{\sqrt{u}} + \frac{1}{2}\sqrt{u}\right) - e^x N\left(\frac{-x}{\sqrt{u}} - \frac{1}{2}\sqrt{u}\right).$$

For the proof we use the fact that this function has the property

$$g(-x, U) = e^{-x} g(x, U) + 1 - e^{-x}.$$

Symmetry of smile in context of (SSV)



Implied volatilities for different maturities assuming $S = 100, \sigma = 0.2, \omega = 0.1, \theta = 4, \xi = 0.1, r = 0.03$ and $q = 0$.

Symmetry of smile in context of (SIV)

Theorem: Set $x_m = \ln\left(\frac{K_m}{S_t e^{(r-q)(T_m-t)}}\right)$.

Assume

- $u_m \equiv 0, \gamma_m \equiv 0,$
- $(S_t, K_m) \mapsto \sum_{k=1}^N v_{mk}^2(S_t, t)$ is independent of I_t and a symmetric function in $x = (x_m)_{0 \leq m \leq M}$.

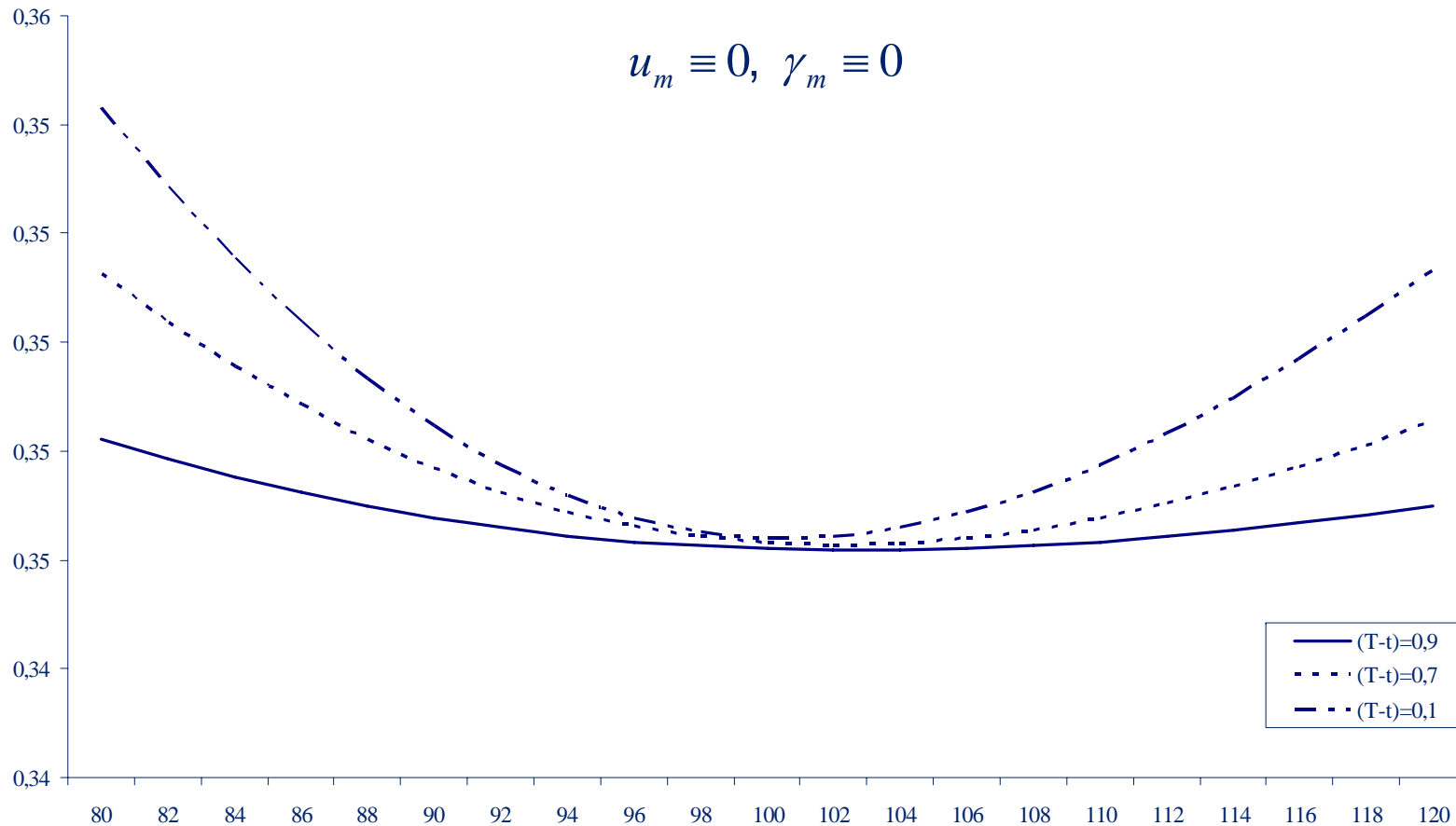
Then the dependence of the implied volatility on x is symmetric.

Proof of Theorem: Under the assumptions the drift restriction can be rewritten as

$$(I_t^m)^4 \left[1 + (T_m - t)^2 \sum_{k=1}^N v_{mk}^2 \right] - (I_t^m)^2 \sigma_t^2 + \frac{1}{2} x_m^2 \sum_{k=1}^N v_{mk}^2 = 0, \text{ which can be solved}$$

according to $(I_t^m)^2$.

Symmetry of smile in context of (SIV)



Implied volatilities for different maturities assuming $S = 100, \sigma = 0.2, v_m \equiv v = 0.1, r = 0.03$ and $q = 0$.

Literature

- **A. Brace, B Goldys, F. Klebaner, R. Womersley:** Market Model of Stochastic Implied Volatility with Application to the BGM Model, *Working Paper S01-1, Departement of Statistics, University New South Wales, 2001.*
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