A Driftless FX hybrid Markov Functional Model

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Assume that complicated market dynamics can be described by . . .

A Numeraire $N(\vec{x}, t)$ with a complicated functional dependence on simple *driftless* but *hidden* markovian variables $\vec{x}$ whose dynamics is

$$d\vec{x}(t) = \sigma(t) d\vec{W}(t).$$

- $\vec{x}$ do *not* have to have direct financial meaning.
- Numerical cost is low.
- Functional dependence is calibrated to the market.

Good and Bad News

The good news is:

- Flexibility
- Practical

The bad news is:

- Non-trivial interpretation $\Longrightarrow$ a problem in multifactor models…

The Semiparametric Approach

Tries to get the best of both worlds…

- Flexibility $\Longrightarrow$ fit well, say, the smile/skew observed in the market.
- Clear and transparent interpretation of model correlation parameters.

Basic Ingredients

- Want good nonparametric fit to the IR skew/smile on *both* domestic and foreign markets.
- Want to either reuse single-currency 1-F MF calibration results or, at least, calibration metodologies.
- Want a 3-factor model to approximate well the relative independence of domestic, foreign, and FX markets.
- Want to know the “market meaning” of the model correlation parameters.
- Want it to be fast and not numericaly expensive.

What does not have to be in

For the moment…

Prepared to sacrifice FX smile!

- Not intended for PRDCs!
- It is intended for products where IR smiles are important; e.g.
  - callable quanto swaps
  - callable quanto reverse floaters
  - callable quanto range accruals
  - callable quanto spread swaps

No Arbitrage (domestic)

IR tradables are zero-coupon bonds or their linear combinations (all expectations are taken in the T_N-terminal domestic measure).

$$Z(\vec{x}_t, t, T) = N(\vec{x}_t, t) E\left[\left. \frac{1}{N(\vec{x}_T, T)} \right| \mathcal{F}_t \right]. \tag{1}$$

From Tower Law,

$$\begin{aligned} E\left[\left. \frac{Z(\vec{x}_{t_2}, t_2, T)}{N(\vec{x}_{t_2}, t_2)} \right| \mathcal{F}_{t_1} \right] &= E\left[\left. E\left[\left. \frac{1}{N(\vec{x}_T, T)} \right| \mathcal{F}_{t_2} \right] \right| \mathcal{F}_{t_1} \right] \\ &= E\left[\left. \frac{1}{N(\vec{x}_T, T)} \right| \mathcal{F}_{t_1} \right] \\ &= \frac{Z(\vec{x}_{t_1}, t_1, T)}{N(\vec{x}_{t_1}, t_1)}. \end{aligned}$$

So all discounted tradables are martingales ⇒ Arbitrage free!

No Arbitrage (foreign)

Assume that both numeraire and spot FX rate $S(\vec{x}_t, t)$ are deterministic functions of the Markovian state variables. So the foreign bond is given by

$$\tilde{Z}(\vec{x}_t, t, T) = \frac{N(\vec{x}_t, t)}{S(\vec{x}_t, t)} E\left[\left. \frac{S(\vec{x}_T, T)}{N(\vec{x}_T, T)} \right| \mathcal{F}_t \right]. \qquad (2)$$

Foreign bonds are not tradables in the domestic currency, but $S(\vec{x}_t, t)\tilde{Z}(\vec{x}_t, t, T)$ is. From Tower law and (2),

$$\begin{aligned} E\left[\left. \frac{S(\vec{x}_{t_2}, t_2)\tilde{Z}(\vec{x}_{t_2}, t_2, T)}{N(\vec{x}_{t_2}, t_2)} \right| \mathcal{F}_{t_1} \right] &= E\left[E\left[\left. \frac{S(\vec{x}_T, T)}{N(\vec{x}_T, T)} \right| \mathcal{F}_{t_2} \right] \middle| \mathcal{F}_{t_1} \right] \\ &= E\left[\left. \frac{S(\vec{x}_T, T)}{N(\vec{x}_T, T)} \right| \mathcal{F}_{t_1} \right] \\ &= \frac{S(\vec{x}_{t_1}, t_1)\tilde{Z}(\vec{x}_{t_1}, t_1, T)}{N(\vec{x}_{t_1}, t_1)}. \end{aligned}$$

Calibration Reuse

- Can reuse the single-currency calibration results for the domestic IR market.
- Cannot reuse the single-currency calibration results for the foreign IR market.
- But by defining $\tilde{N}(\vec{x}_t, t) = N(\vec{x}_t, t)/S(\vec{x}_t, t)$, the value of the numeraire in units of the foreign currency, foreign bonds will have the same dependence on $\tilde{N}$,

  $$\tilde{Z}(\vec{x}_t, t, T) \quad = \quad \tilde{N}(\vec{x}_t, t) E \left[\frac{1}{\tilde{N}(\vec{x}_T, T)} \middle| \mathcal{F}_t \right],$$

  as the domestic ones. Added to our semi-parametric form, this allows us to use all single-currency calibration methods (code reuse).

Choice of Semi-parametric Form and Code Reuse

This choice of semi-parametric form,

$$\begin{aligned} N(\vec{x}_t, t) &= N_x(x_t, t), \\ \tilde{N}(\vec{x}_t, t) &= \exp(-y_t) N_z(z_t, t), \end{aligned}$$

ensures not only that domestic zero-coupon bonds (hence all domestic Libor rates and swap rates) depend only on x,

$$Z(\vec{x}_t, t, T) = N_x(x, t) \int \frac{dX}{\sqrt{2\pi v_{xx}}} \frac{\exp(-(X-x)^2/v_{xx}/2)}{N_x(X, T)}, \tag{3}$$

but also that foreign zero-coupon bonds too only depend on z,

$$\tilde{Z}(\vec{x}_t, t, T) = \exp(v_{yy}/2) N_z(z, t) \int \frac{dZ}{\sqrt{2\pi v_{zz}}} \frac{\exp(-(Z-z)^2/v_{zz}/2)}{N_z(Z - v_{yz}, T)}, \tag{4}$$

as in the single currency model. $v_{yz} = \int_t^T d\tau\ \sigma_y(\tau)\sigma_z(\tau)$ in (4) can be interpreted as a *quanto correction*.

FX Calibration and Meaning of Correlations

- Calibrate $\sigma_y(t)$ to reprice all at-the-money FX options.

$$
\begin{aligned}
X[K] &= N(0,0)E\left[\left.\frac{(S(\vec{x}_t,t)-K)^+}{N(\vec{x}_t,t)}\right|\mathcal{F}_t\right] \\
&= N(0,0)\int dxdzI(x,z).
\end{aligned}
$$

- y drives the forward FX for delivery at the terminal date,

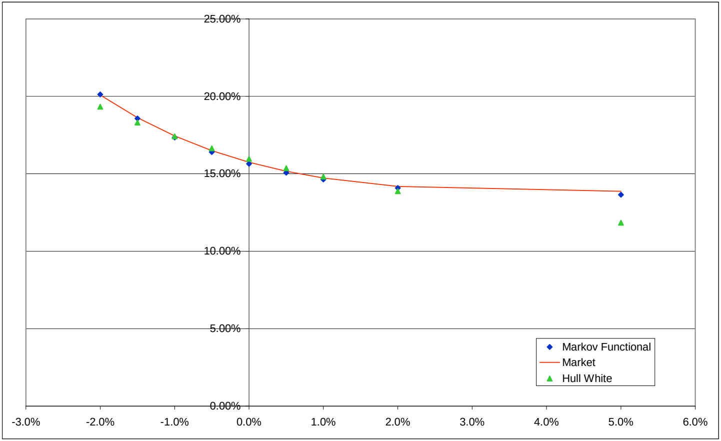
$$
\begin{aligned}
F(t,T_N) &= S(\vec{x}_t,t)\frac{\tilde{Z}(\vec{x}_t,t,T_N)}{Z(\vec{x}_t,t,T_N)} \\
&= E\left[\left.\frac{e^{y_{T_N}}}{N_z(z_{T_N},T_N)}\right|\mathcal{F}_t\right] \\
&\propto e^{y_t}.
\end{aligned}
$$

Case Study: "Normal" Market × "Smiley" Market

- We have chosen the USD and the EUR markets because, on Friday 2nd June 2006, implied swaption volatilities in the former were quite close to those yielded by a normal model, whilst in the latter they showed a pronounced smile at high strikes.
- Compared this model with the market standard Hull-White/Black-Scholes hybrid model.
- Calibrated both models to the appropriate coterminal swap rate in each IR market and to ATM FX options.
- Compared the models prices for two multicallable quanto swaps, the first paying USD LIBOR quantoed into EUR, the second paying EURIBOR plus a spread quantoed into USD.

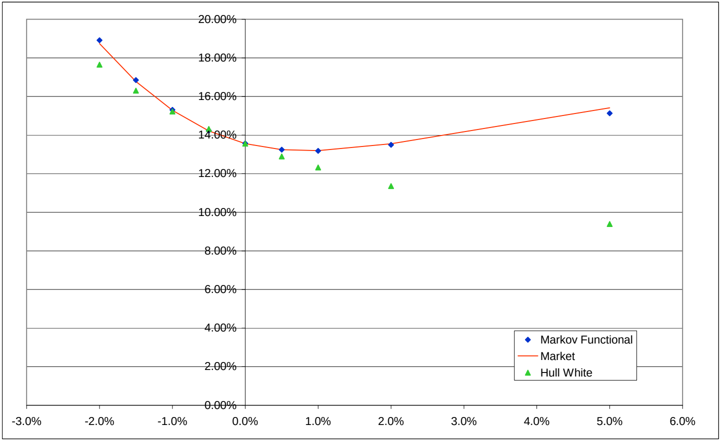
USD Implied Swaption Volatilities

Black vols implied by the prices of 5y/5y USD European swaptions as a function of offset from ATM strike.

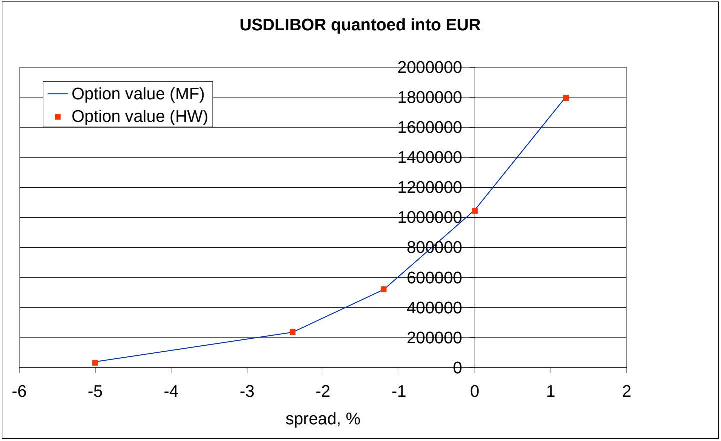


EUR Implied Swaption Volatilities

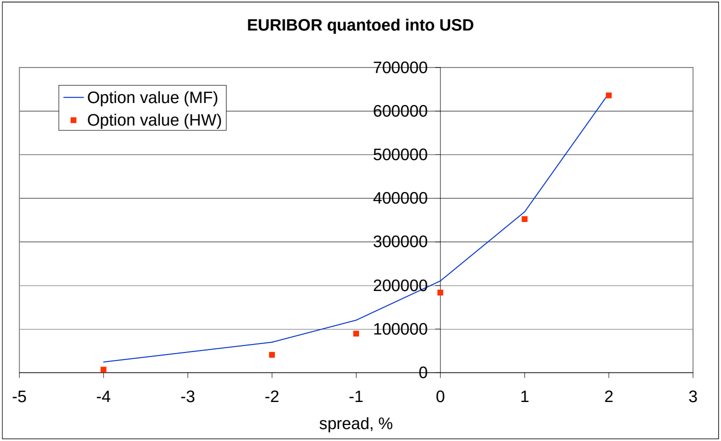
Black vols implied by the prices of 5y/5y EUR European swaptions as a function of offset from ATM strike.



USD LIBOR multicallable swap quantoed into EUR



EURIBOR multicallable swap quantoed into USD



Conclusions:

1. Can develop a zero-drift cross-currency Markov Functional model.
2. Easy to implement and computationally cheap.
3. The state variables of the model have a simple financial interpretation.
4. quanto interest rate products can have a significant dependence on IR smiles.

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