Skew and smile calibration using Markovian projection

Alexandre Antonov and Timur Misirpashaev
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Mathematical challenges

- Complexity of modern models
- Sensitivity of the instruments to distant wings of volatility surfaces (wide range of European option strikes)

$$\Downarrow$$

European option pricing should be fast and accurate; any approximations should reproduce model skew/smile

European option on a generic rate process

Rate process (in general non-Markovian)

$$dS(t) = \Sigma(t) \cdot dW(t)$$

with a generic volatility (vector) process $\Sigma(t)$ driven by F independent Brownian motions, $W(t) = \{W_1(t), \cdots, W_F(t)\}$. For example, S can be a swap rate, discounted FX-rate, etc.

The process $S(t)$ underlies the European option $E[(S(t) - K)^+]$ maturing at t

$\rightarrow$ for calculations we need only its marginal distribution at t

Approximate analytical methods

- Heuristic methods (Rebonato, Hull & White for LMM swaption pricing):
  represent volatility in a suitable form (e.g., log-normal for the BGM model) by "freezing" some of the coefficients

- Asymptotic expansion methods (Kawai):
  expand $S(t)$ around small volatilities (scaled with ϵ)
  $S(t) = S(0) + \epsilon\, S_1(t) + \epsilon^2\, S_2(t) + \cdots$, where $S_1(t)$ is a Gaussian variable. PDF is restored from conditional averages
  $E[S_2(t) \,|\, S_1(t)]$, etc.

- Markovian projection

Markovian projection

Term "Markovian projection" was coined by Piterbarg; idea ascends to Gyöngy (1986) and Dupire (1994).

A complicated, non-Markovian process is replaced by a Markovian one, s.t. both processes have identical one-dimensional marginal distributions. By Gyöngy's lemma, the *Markovian* process $S^*(t)$ satisfying

$$dS^*(t) = \Sigma^*(t, S^*(t)) \cdot dW, \qquad S^*(0) = S(0),$$

with

$$|\Sigma^*(t,s)|^2 = E[|\Sigma(t)|^2 \mid S(t) = s]$$

has the same marginal distributions as $S(t)$ for any t, and therefore can be used to compute the European option,

$$E[(S(t) - K)^+] = E[(S^*(t) - K)^+]$$

Strategy:

- calculate the conditional expectation $E[|\Sigma(t)|^2 \mid S(t) = s]$
- price the option for the local volatility process $S^*(t)$

Techniques:

- Avellaneda et al (2002), Henry-Labordere (2005):
  - heat-kernel approximation and saddlepoint for the expectation
  - approximate relation between local and implied vols for the option price
- Piterbarg (2005), Antonov & Misirpashaev (2006):
  - projection on a displaced diffusion (DD) with time-dependent parameters for the expectation
  - parameter averaging for the option price

Markovian projection on a displaced diffusion (DD)

Capturing the first derivative of effective local vol

$$\Sigma^*(t,s) \simeq (1 + \Delta s\, \beta(t))\, \sigma(t), \qquad \Delta s = s - S(0)$$

The first derivative $\Leftrightarrow$ implied volatility skew

Good approximation accuracy for non-SV models exhibiting a skew

Markovian projection on DD: method

Replace the initial process $dS(t) = \Sigma(t) \cdot dW(t)$ by a DD process,

$$dS^*(t) = (1 + \Delta S^*(t)\beta(t))\,\sigma(t) \cdot dW(t), \qquad S^*(0) = S(0)$$

where $\Delta S^*(t) = S^*(t) - S(0)$, $\sigma(t)$ is an F-component deterministic volatility vector, and $\beta(t)$ is a time-dependent shift (controlling skew).

We look for optimal $\sigma(t)$ and $\beta(t)$, s.t. for any t,

$$|\sigma(t)|^2\,(1 + \Delta s \beta(t))^2 \simeq E[|\Sigma(t)|^2 \mid S(t) = s]$$

holds to the highest order in volatilities.

Projections vs. conditional expectation

Denote the true conditional expectation,

$$|\Xi(t,x)|^2 = E[|\Sigma(t)|^2 \mid S(t) = x]$$

For every fixed t, the conditional expectation can be characterized as a function of state $|\Xi(t,x)|^2$ that minimizes the L_2-distance from the true variance,

$$\chi^2 = E\left[\left(|\Sigma(t)|^2 - |\Xi(t,S(t))|^2\right)^2\right] \to \min.$$

DD case: minimization of

$$\chi^2 = E\left[\left(|\Sigma(t)|^2 - |\sigma(t)|^2\left(1 + \Delta S(t)\beta(t)\right)^2\right)^2\right]$$

for each t gives a condition for optimal DD coefficients $\sigma(t)$ and $\beta(t)$.

DD results

Optimal DD coefficients in terms of unconditional averages,

$$|\sigma(t)|^2 = E\left[|\Sigma(t)|^2\right]$$

$$\beta(t) = \frac{E\left[|\Sigma(t)|^2 \Delta S(t)\right]}{2\,E\left[|\Sigma(t)|^2\right] E\left[\Delta S^2(t)\right]}$$

Option price can be found from the BS formula after the shift $\beta(t)$ is averaged (Piterbarg).

Closed-form results can be obtained using small volatilities expansions in the case of *separable* processes.

Separable processes

The rate process satisfies

$$dS(t) = \sum_n X_n(t) a_n(t) \cdot dW(t),$$

where $a_n(t)$ are deterministic vector functions with scalar components $a_{n,\nu}(t)$ ($\nu = 1, \ldots, F$), and $X_n(t)$ are stochastic processes obeying SDEs of the form

$$dX_n(t) = \mu_n(t, \{X_k(t)\})dt + \sigma_n(t, \{X_k(t)\}) \cdot dW(t).$$

We assume the drift terms μ_n are small in the sense that they are of the second or higher order in volatilities,

$$\mu_n = O(\sigma_+^2)$$

where

$$\sigma_+ = \max\{\sigma_{n,\nu}(t, \{X_k(t)\}), a_{n,\nu}(t)\}$$

Optimal DD coefficients for a separable process

$$\sigma(t) = \sum_n X_n(0) a_n(t) + O(\sigma_+^3),$$

$$\beta(t) = \frac{\sum_n (a_n(t) \cdot \sigma(t)) \int_0^t (\sigma_n(\tau, \{X_k(0)\}) \cdot \sigma(\tau))\, d\tau}{|\sigma(t)|^2 \int_0^t |\sigma(\tau)|^2 d\tau} + O(\sigma_+^2).$$

Many challenging calibration problems (not involving SV) can be easily resolved by separable models approximations.

Technical details and multiple applications can be found in Antonov & Misirpashaev (2006b).

Applications

- Shifted BGM swaption formula
  Kawai (2003) by asymptotic expansion method; Piterbarg (2006); Antonov & Misirpashaev (2006b)

- Cross-Currency models (CEV model for FX-rate, gaussian model for interest rates)
  Piterbarg (2006); Antonov & Misirpashaev (2006b)

- LMM Cross-Currency models (CEV for FX-rate, shifted BGM for interest rates)
  Antonov & Misirpashaev (2006a), (2006b); Kawai & Jäckel (2006) by asymptotic expansion method

Example: projecting a basket of DD models

Initial process is a weighted sum

$$S(t) = \sum_i w_i \, S_i(t)$$

of DD models

$$dS_i(t) = (1 + \Delta S_i(t)\beta_i) \, \lambda_i \cdot dW(t)$$

For simplicity we take time-independent parameters, although the approximation works for time-dependent one.

Representing the DD basket as a separable process, we obtain effective volatility and skew parameter.

Basket of DD models as a separable process

$$dS(t) = \sum_i w_i \, dS_i(t)$$

$$\Downarrow$$

$$dS(t) = \sum_n X_n(t) a_n(t) = \left(\sum_i w_i \, \lambda_i + \Delta S_i(t) \, w_i \, \beta_i \, \lambda_i \right) \cdot dW(t)$$

This immediately gives the optimal DD parameters

$$\sigma(t) = \sigma \simeq \sum_i w_i \, \lambda_i$$

$$\beta(t) = \beta \simeq \frac{\sum_i w_i \beta_i (\lambda_i \cdot \sigma)^2}{|\sigma|^4}$$

Numerical experiments: MP to DD

Basket of 5 assets with the following parameters of DD

	Asset 1	Asset 2	Asset 3	Asset 4	Asset 5
vol of rate (%)	14	15	16	17	18
beta skew (%)	30	40	50	60	70

- Initial values $S_i(0) = 1$ and weights $w_i = 1/5$ for all i
- Correlations between rates: $\frac{\lambda_i \cdot \lambda_j}{|\lambda_i|\,|\lambda_j|} = 70\%$ for all i, j

The optimal values are

$$|\sigma| = 13.95\%$$

$$\beta = 63.24\%$$

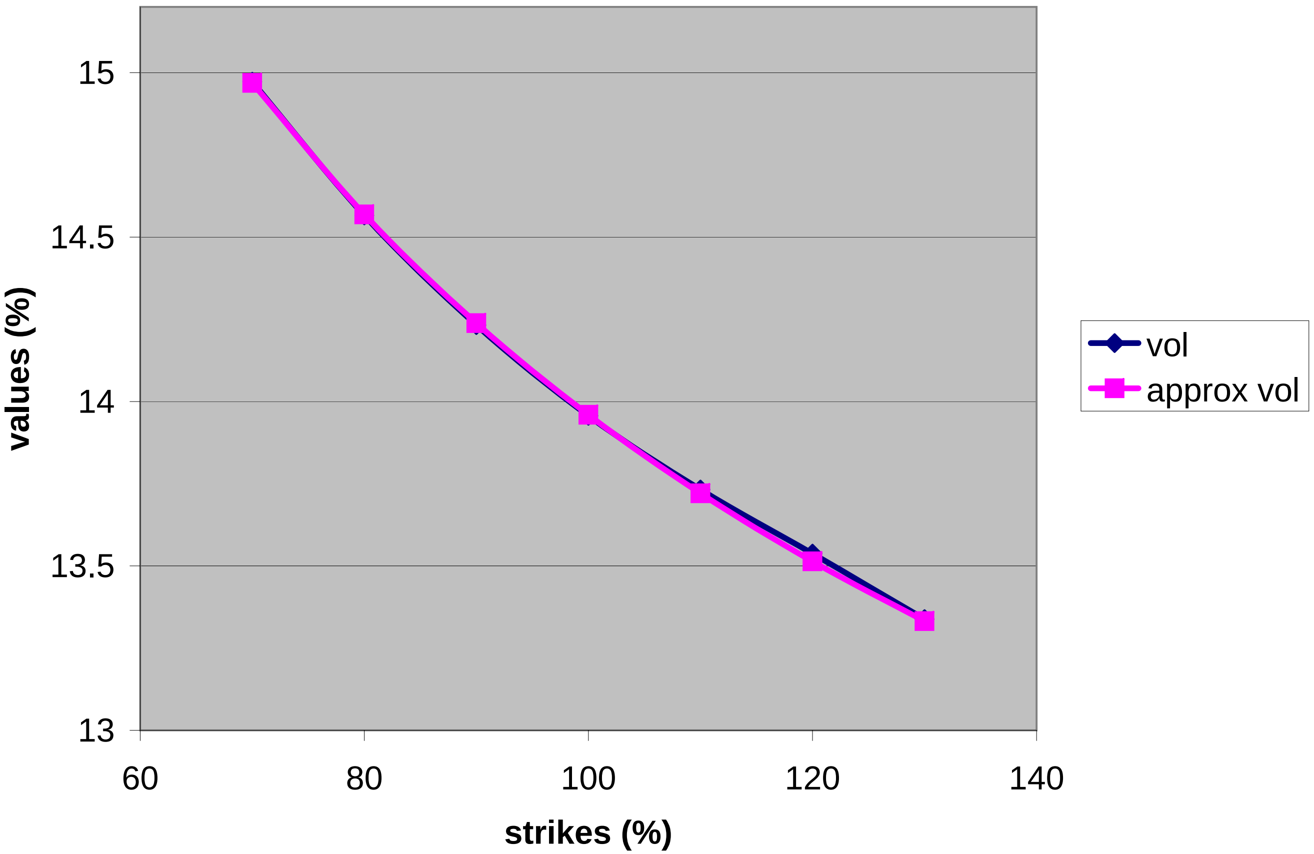

Figure 1: DD basket option implied volatilities for 1Y maturity

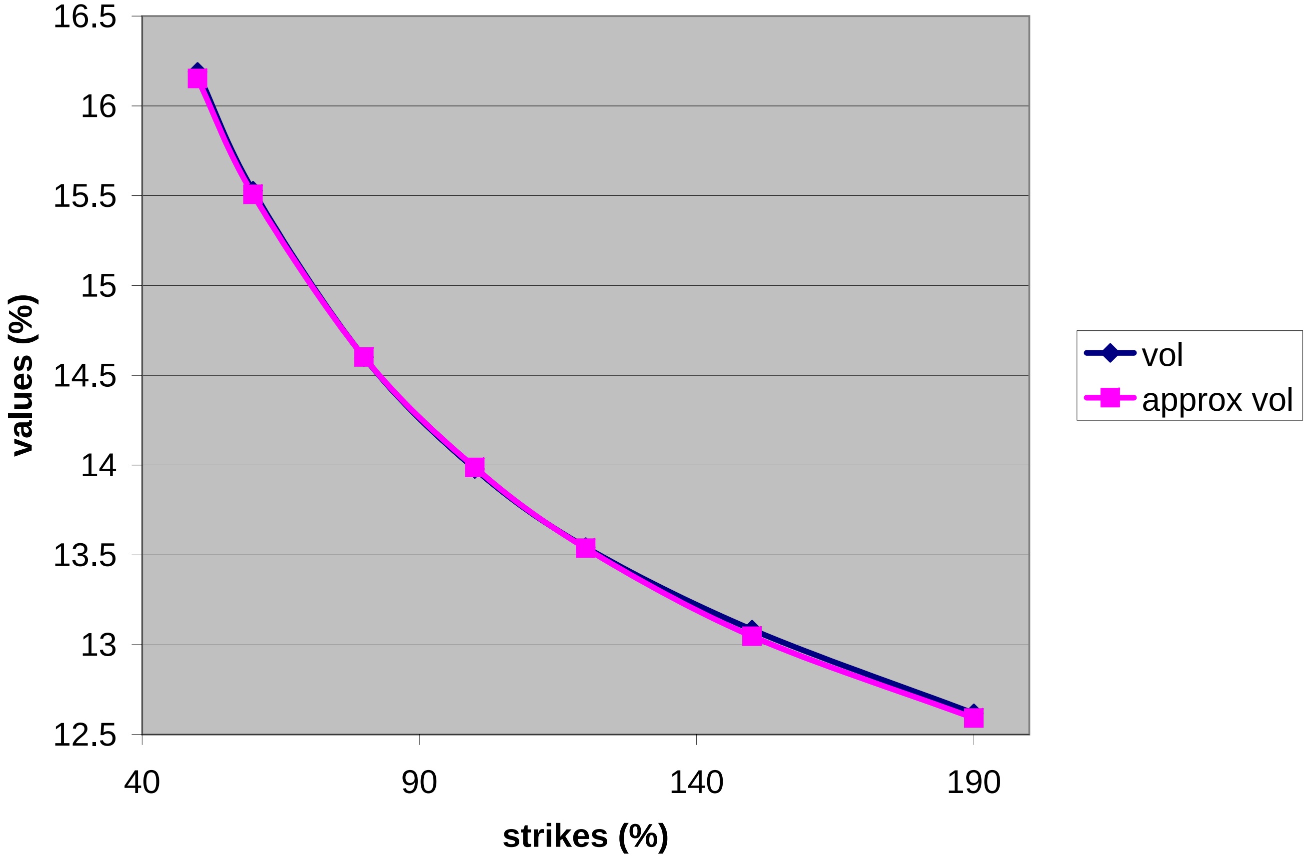

Figure 2: DD basket option implied volatilities for 5Y maturity

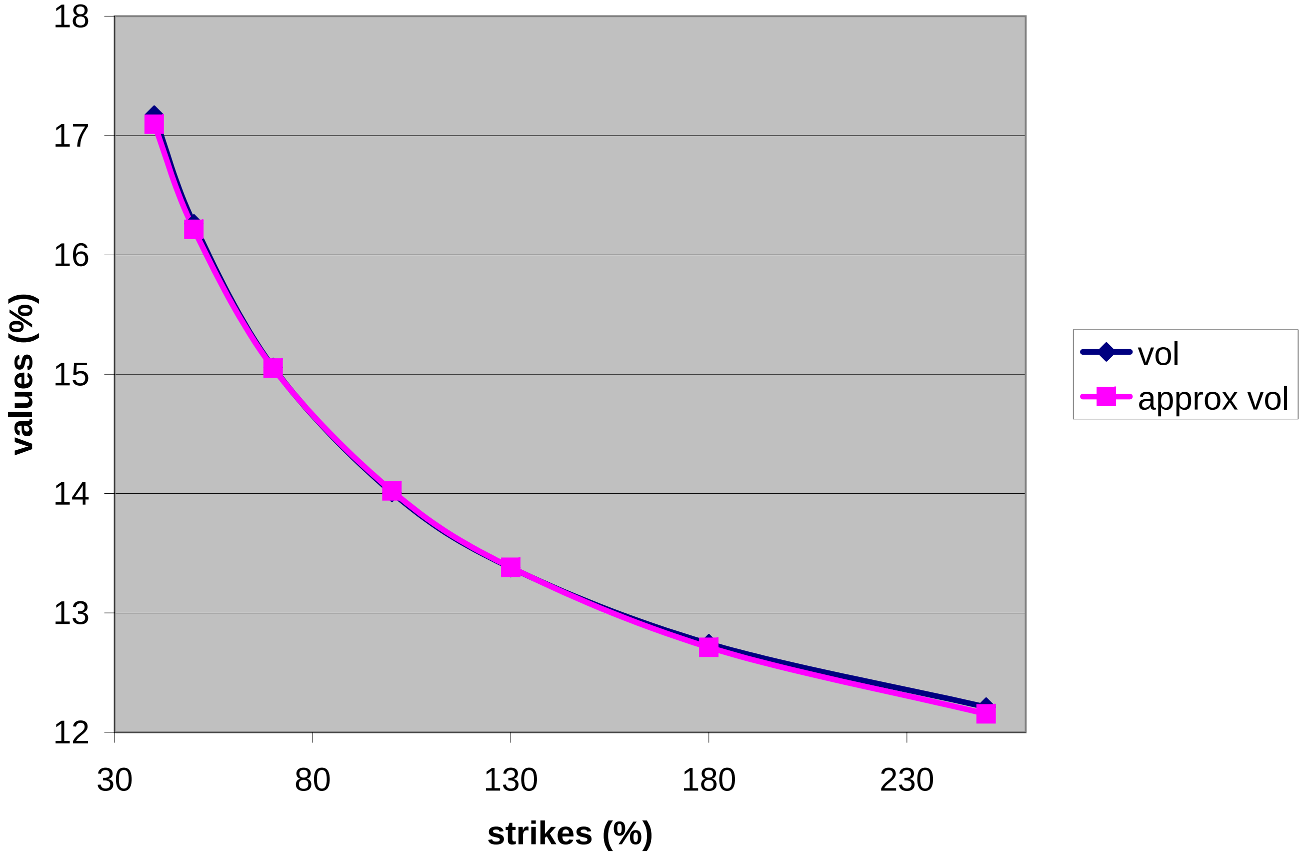

Figure 3: DD basket option implied volatilities for 10Y maturity

Summary for the projection on DD

- Advantages
  - Straightforward and universal method
  - Volatility skew is captured
  - Good accuracy for typical cases
- Drawbacks
  - Volatility smile (convexity) is not matched
  - Accuracy gets worse for highly heterogeneous cases

What to do for a process with a pronounced smile?

The effective mimicking process $S^*(t)$

$$dS^*(t) = \Sigma^*(t, S^*(t)) \cdot dW(t)$$

should have U-shaped local vol.

Due to technical difficulties in MP to non-linear basis we propose another solution $\rightarrow$
project the initial process to the *shifted Heston process*

$$\begin{aligned} dS^*(t) &= (1 + \beta(t)\,\Delta S^*(t))\sqrt{z(t)}\,\sigma_H(t) \cdot dW(t), \qquad & S^*(0) = S(0) \\ dz(t) &= \theta(t)\,(1 - z(t))\,dt + \sqrt{z(t)}\,\sigma_z(t) \cdot dW(t), \qquad & z(0) = 1 \end{aligned}$$

Markovian projection on an SV model

Piterbarg (2006) gave an example of the projection onto a displaced Heston model for the spread of interest rates, using an auxiliary equivalent local volatility model. Here we develop a systematic approach for direct projection on a stochastic volatility model.

Advantages:

- Straightforward and universal (will be developed for any separable process)
- Volatility smile is captured
- Certain features of dynamics are captured (therefore, applications to pricing and not just calibration are possible)

Instead of looking for a local volatility effective model

$$dS^*(t) = \Sigma^*(t, S^*(t)) \cdot dW(t)$$

we consider an SV process

$$dS^*(t) = \sigma_S(t; S^*(t), V^*(t)) \cdot dW(t)$$

where $V^*(t)$ is a stochastic variance.

Theoretical background $\Rightarrow$ multi-dimensional version of Gyöngy's lemma

Markovian projection with multiple components

Take an N-dimensional (non-Markovian) process $x(t) = \{x_1(t), \cdots, x_N(t)\}$ with an SDE

$$dx_n(t) = \mu_n(t)\,dt + \sigma_n(t) \cdot dW(t)$$

The process $x(t)$ can be mimicked with a Markovian N-dimensional process $x^*(t)$ with the same joint distributions for all components at fixed t.

According to Gyöngy, the process $x^*(t)$ satisfies the SDE

$$dx_n^*(t) = \mu_n^*(t, x^*(t))\,dt + \sigma_n^*(t, x^*(t)) \cdot dW(t)$$

with

$$\begin{aligned} \mu_n^*(t, y) \quad &= \quad E[\mu_n(t)\,|\,x(t) = y] \\ \sigma_n^*(t, y) \cdot \sigma_m^*(t, y) \quad &= \quad E[\sigma_n(t) \cdot \sigma_m(t)\,|\,x(t) = y] \end{aligned}$$

Choice of process components

The first component is the initial rate, $dS = \Sigma(t)\cdot dW(t)$.
The second component should be related to $|\Sigma(t)|^2$.

We fix a shift function $\beta(t)$ (to be determined later) and write the equation for the rate in the form

$$dS = (1 + \beta(t)\,\Delta S(t))\Lambda(t)\cdot dW(t)$$

where

$$\Lambda(t) = \frac{\Sigma(t)}{1 + \Delta S(t)\beta(t)}$$

The second equation is for the variance $V(t) = |\Lambda(t)|^2$,

$$dV(t) = \mu_V(t)\,dt + \sigma_V(t)\cdot dW(t)$$

This completes the SDE's for the non-Markovian pair $\{S(t), V(t)\}$.

Below, for better visualization, we will omit explicit time argument for processes in hand; we will write SDE's using shorthand notations $S(t) \to S$, $V(t) \to V$ etc.
On the other hand, to stress time-dependence for the model parameters we will keep explicit time arguments there.

The initial process pair $\{S(t), V(t)\}$

$$\begin{aligned} dS &= (1 + \beta(t)\, \Delta S)\ \Lambda(t) \cdot dW \\ dV &= \mu_V(t)\, dt + \sigma_V(t) \cdot dW \end{aligned}$$

can be mimicked by a Markovian pair $\{S^*(t), V^*(t)\}$ s.t.

$$\begin{aligned} dS^* &= (1 + \beta(t)\, \Delta S^*)\ \sigma_S^*(t; S^*, V^*)\ \cdot dW \\ dV^* &= \mu_V^*(t; S^*, V^*)\, dt + \sigma_V^*(t; S^*, V^*)\ \cdot dW \end{aligned}$$

where

$$\begin{aligned} |\sigma_S^*(t; s, u)|^2 &= E[|\Lambda(t)|^2 \,|\, S(t) = s, V(t) = u] = u \\ |\sigma_V^*(t; s, u)|^2 &= E[|\sigma_V(t)|^2 \,|\, S(t) = s, V(t) = u] \\ \sigma_S^*(t; s, u) \cdot \sigma_V^*(t; s, u) &= E[\Lambda(t) \cdot \sigma_V(t) \,|\, S(t) = s, V(t) = u] \\ \mu_V^*(t; s, u) &= E[\mu_V(t) \,|\, S(t) = s, V(t) = u] \end{aligned}$$

Our strategy

- For any fixed skew parameter $\beta(t)$ make the projection (in a proper sense) to a shifted Heston model

$$\begin{aligned} dS^* &= (1+\beta(t)\,\Delta S^*)\sqrt{z}\,\sigma_H(t)\cdot dW, & S^*(0) = S(0) \\ dz &= \theta(t)\,(1-z)\,dt + \sqrt{z}\,\sigma_z(t)\cdot dW, & z(0) = 1 \end{aligned}$$

- Find the optimal $\beta(t)$ which minimizes the (properly defined) distance to the projected model

Projection to a shifted Heston

For a fixed skew parameter $\beta(t)$, we look for the unknown volatilities and drift functions $\sigma_S^*(t; S^*, V^*)$, $\sigma_V^*(t; S^*, V^*)$, and $\mu_V^*(t; S^*, V^*)$ as in the Heston model

$$\begin{aligned} dS &= (1 + \beta(t)\,\Delta S)\sqrt{V}\,\frac{\sigma_H(t)}{|\sigma_H(t)|} \cdot dW \\ dV &= \left(V\,\left(\left(\log|\sigma_H(t)|^2\right)' - \theta(t)\right) + \theta(t)\,|\sigma_H(t)|^2\right) dt \\ &+ |\sigma_H(t)|\,\sqrt{V}\,\sigma_z(t) \cdot dW \end{aligned}$$

where $V(t) = z(t)\,|\sigma_H(t)|^2$.

This leads to

$$
\begin{aligned}
E[\mu_V(t) \mid S(t)=s, V(t)=v] &= \mu_\Lambda^*(t;s,v) \\
&\simeq v\,\Big(\big(\log|\sigma_H(t)|^2\big)' - \theta(t)\Big) \\
&+ \theta(t)\,|\sigma_H(t)|^2 \\
E[|\sigma_V(t)|^2 \mid S(t)=s, V(t)=v] &= |\sigma_V^*(t;s,v)|^2 \\
&\simeq |\sigma_H(t)|^2\,|\sigma_z(t)|^2\,v \\
E[\Lambda(t)\cdot\sigma_V(t) \mid S(t)=s, V(t)=v] &= \sigma_S^*(t;s,v)\cdot\sigma_V^*(t;s,v) \\
&\simeq v\,\sigma_z(t)\cdot\sigma_H(t)
\end{aligned}
$$

The last equation depends on the correlation between σ_H and σ_z, i.e.

$$\rho(t) = \frac{\sigma_z(t)\cdot\sigma_H(t)}{|\sigma_H(t)|\,|\sigma_z(t)|}$$

The unknown parameter functions of the shifted Heston model, $|\sigma_H(t)|$, $|\sigma_z(t)|$, $\theta(t)$, and $\rho(t)$, are expressed via unconditional averages, as a result of minimization of the following regression functionals

$$
\begin{aligned}
\chi_1^2(t) &= E\left[\left(\mu_V(t) - V(t)\left(\left(\log|\sigma_H(t)|^2\right)' - \theta(t)\right) - \theta(t)\,|\sigma_H(t)|^2\right)^2\right] \\
\chi_2^2(t) &= E\left[\left(|\sigma_V(t)|^2 - |\sigma_H(t)|^2\,|\sigma_z(t)|^2\,V(t)\right)^2\right] \\
\chi_3^2(t) &= E\left[\left(\Lambda(t)\cdot\sigma_V(t) - \sigma_z(t)\cdot\sigma_H(t)V(t)\right)^2\right]
\end{aligned}
$$

Optimal coefficients

Minimizing the functionals, we obtain the optimal parameters $|\sigma_H(t)|$, $|\sigma_z(t)|$, $\theta(t)$ and $\rho(t)$ of the shifted Heston model in terms of unconditional averages and shift $\beta(t)$,

$$
\begin{aligned}
|\sigma_H(t)|^2 &= E[V(t)] \\
\theta(t) &= (\log E[V(t)])' - \frac{1}{2}\left(\log E[\delta V^2(t)]\right)' + \frac{E[|\sigma_V(t)|^2}{2\,E[\delta V^2(t)]} \\
|\sigma_z(t)|^2 &= \frac{E[V(t)|\sigma_V(t)|^2]}{E[V^2(t)]E[V(t)]} \\
\rho(t) &= \frac{E[V(t)\Lambda(t)\cdot\sigma_V(t)]}{\sqrt{E[V^2(t)]E[V(t)|\sigma_V(t)|^2]}}
\end{aligned}
$$

where $\delta V(t) = V(t) - E[V(t)]$.

Optimal skew function

Choose $\beta(t)$ s.t. it minimizes projection defects, i.e. the values of the functionals $\chi_1^2(t)$, $\chi_2^2(t)$ and $\chi_3^2(t)$ with substituted optimal parameters.

Example: projection of a separable process

Recall that a separable process satisfies

$$dS(t) = \sum_n X_n(t) a_n(t) \cdot dW(t),$$

where $a_n(t)$ are deterministic vector functions and $X_n(t)$ obey

$$dX_n(t) = \mu_n(t, \{X_k(t)\})dt + \sigma_n(t, \{X_k(t)\}) \cdot dW(t).$$

where the drift terms μ_n are of the second order in volatilities.

We give closed-form expression of the optimal Heston parameters $|\sigma_H(t)|$, $|\sigma_z(t)|$, $\theta(t)$ and $\rho(t)$ in the leading order in volatilities.

Define two kernels

$$
\begin{aligned}
\Phi(t,\tau) &= \sum_n a_n(t)\cdot\hat{\sigma}(t)\,\hat{\sigma}_n(\tau) \\
\Omega(t,\tau) &= 2\left(\Phi(t,\tau) - \beta(t)\,|\hat{\sigma}(t)|^2\,\hat{\sigma}(\tau)\right)
\end{aligned}
$$

where $\hat{\sigma}(t) = \sum_n X_n(0)\,a_n(t)$ and $\hat{\sigma}_n(t) = \sigma_n(t; X_n(0))$.

<u>Optimal Heston parameters</u>

$$
\begin{aligned}
|\sigma_H(t)| &= |\hat{\sigma}(t)| + O(\sigma^3) \\
\theta(t) &= \left(\log|\sigma_H(t)|^2\right)' - \frac{\int_0^t \frac{\partial}{\partial t}|\Omega(t,\tau)|^2\,d\tau}{2\int_0^t |\Omega(t,\tau)|^2\,d\tau} + O(\sigma^2) \\
|\sigma_z(t)| &= \frac{|\Omega(t,t)|}{|\hat{\sigma}(t)|^2} + O(\sigma^3) \\
\rho(t) &= \frac{\Omega(t,t)\cdot\hat{\sigma}(t)}{|\Omega(t,t)|\,|\hat{\sigma}(t)|} + O(\sigma^2)
\end{aligned}
$$

Optimal skew parameter $\beta(t)$ satisfies a linear ODE,

$$
\begin{aligned}
(\beta(t)\,|\hat\sigma(t)|^2)' \quad & \times \quad \left[\left(\int_0^t \Phi(t,\tau)\cdot\hat\sigma(\tau)\,d\tau \right)^2 \right. \\
& - \quad \left. \int_0^t |\Phi(t,\tau)|^2\,d\tau \int_0^t |\hat\sigma(\tau)|^2\,d\tau \right] \\
+(\beta(t)\,|\hat\sigma(t)|^2) \quad & \times \quad \left[\int_0^t \frac{\partial}{\partial t}\Phi(t,\tau)\cdot\ \Phi(t,\tau)\,d\tau \int_0^t |\hat\sigma(\tau)|^2\,d\tau \right. \\
& - \quad \left. \int_0^t \frac{\partial}{\partial t}\Phi(t,\tau)\cdot\ \hat\sigma(\tau)\,d\tau \int_0^t \Phi(t,\tau)\cdot\hat\sigma(\tau)\,d\tau \right] \\
& + \quad \int_0^t \frac{\partial}{\partial t}\Phi(t,\tau)\cdot\ \hat\sigma(\tau)\,d\tau \int_0^t |\Phi(t,\tau)|^2\,d\tau \\
& - \quad \int_0^t \frac{\partial}{\partial t}\Phi(t,\tau)\cdot\ \Phi(t,\tau)\,d\tau \int_0^t \Phi(t,\tau)\cdot\hat\sigma(\tau)\,d\tau = 0
\end{aligned}
$$

Example: projecting a basket of Heston models

Initial process is a weighted sum

$$S(t) = \sum_i w_i \, S_i(t)$$

of shifted Heston models

$$\begin{aligned} dS_i(t) &= (1 + \Delta S_i(t)\beta_i)\sqrt{z_i(t)}\lambda_i \cdot dW(t) \\ dz_i(t) &= a_i\,(1 - z_i(t))\,dt + \sqrt{z_i(t)}\,\gamma_i \cdot dW(t), \quad z_i(0) = 1 \end{aligned}$$

For simplicity we take time-independent Heston parameters, although the approximation works for time-dependent ones as well.

Basket of Heston models as a separable process

We deal with the non-trivial drift in $z_i(t)$ by writing it in the form

$$z_i(t) = 1 + y_i(t)\, e^{-\int_0^t a_i(s)\, ds}$$

where the process $y_i(t)$ is a martingale,

$$dy_i(t) = e^{\int_0^t a_i(s)\, ds} \sqrt{1 + y_i(t)\, e^{-\int_0^t a(s)\, ds}}\, \gamma_i(t) \cdot dW(t), \quad y_i(0) = 0$$

Keeping only linear terms in ΔS_i and y_i,

$$(1 + \Delta S_i(t)\beta_i)\sqrt{z_i(t)}\lambda_i \simeq \lambda_i \left(1 + \Delta S_i(t)\, \beta_i + \frac{1}{2}\, y_i(t)\, e^{-\int_0^t a_i(s)\, ds}\right)$$

Note that both ΔS_i and y_i have no drift.

In this way, we go from

$$dS(t) = \sum_i w_i \, dS_i(t) = \sum_i w_i \, (1 + \Delta S_i(t)\beta_i) \sqrt{z_i(t)} \lambda_i \cdot dW(t)$$

to an approximate representation

$$dS(t) \simeq \sum_i w_i \, \lambda_i \, \left(1 + \Delta S_i(t) \, \beta_i + \frac{1}{2} \, y_i(t) \, e^{- \int_0^t a_i(s) \, ds} \right) \cdot dW(t)$$

Now we can use the generic result for the effective Heston parameters of a separable process.

Explicit formulas for the basket of Heston models

$$
\begin{aligned}
\sigma_H &= \sum_i w_i \lambda_i \\
\sigma_z &= 2\frac{\sum_i w_i\, d_i \left(\beta_i\ \lambda_i + \frac{1}{2}\,\gamma_i\right)}{|\hat{\sigma}(t)|^2} - 2\beta\,\sigma_H \\
\theta(t) &= -\frac{\int_0^t \frac{\partial}{\partial t}|\Omega(t,\tau)|^2\, d\tau}{2\ \int_0^t |\Omega(t,\tau)|^2\, d\tau}
\end{aligned}
$$

where $d_i = \lambda_i \cdot \sigma_H$ and

$$
\begin{aligned}
\Phi(t,\tau) &= \sum_i w_i\, d_i \left(\beta_i\ \lambda_i + \frac{1}{2}\, e^{-t a_i}\, \gamma_i\, e^{\tau a_i}\right) \\
\Omega(t,\tau) &= 2\left(\Phi(t,\tau) - \beta(t)\,|\sigma_H|^2\,\sigma_H\right)
\end{aligned}
$$

Optimal shift $\beta(t)$ was calculated by the above linear ODE with the initial value taken from the MP to DD

$$\beta(0) = \frac{\sum_i w_i \beta_i d_i^2}{|\sigma|^4}$$

Numerical results

Basket of 5 assets with the following Heston model parameters,

	Asset 1	Asset 2	Asset 3	Asset 4	Asset 5
vol of rate (%)	14	15	16	17	18
beta skew (%)	30	40	50	60	70
reversion (%)	10	10	10	10	10
vol-of-vol (%)	70	75	80	85	90

- Initial values $S_i(0) = 1$ and weights $w_i = 1/5$ for all i
- Correlations between rates: $\frac{\lambda_i \cdot \lambda_j}{|\lambda_i|\,|\lambda_j|} = 70\%$ for all i, j
- Correlations between stochastic vols: $\frac{\gamma_i \cdot \gamma_j}{|\gamma_i|\,|\gamma_j|} = 90\%$ for all i, j
- Cross-correlations: $\frac{\lambda_i \cdot \gamma_j}{|\lambda_i|\,|\gamma_j|} = -20\%$ for all i, j

Optimal parameters of the effective Heston model

All optimal parameters are *almost* flat

- volatility of rate $|\sigma_H| = 13.95\%$
- skew $\beta(t) \simeq 63.24\%$
- vol-of-vol $|\sigma_z| \simeq 77.43\%$
- reversion $\theta(t) \simeq 9.99\%$
- correlation $\rho(t) \simeq -23.9\%$

We compare:

exact implied volatilities, calculated by a direct simulation with a large number of paths

against

approximate implied volatilities, calculated by the projection to the Heston model.

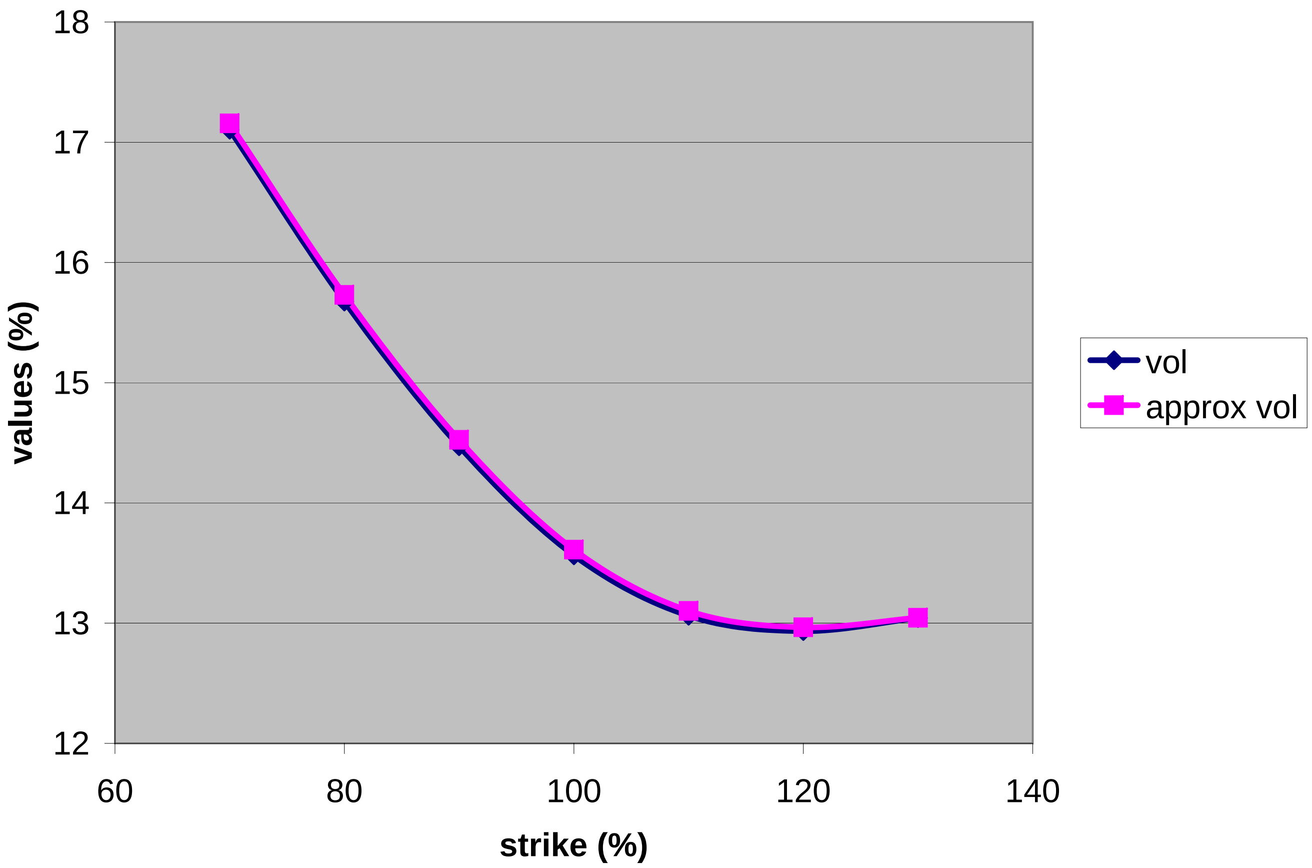

Figure 4: Heston basket option implied volatilities for 1Y maturity

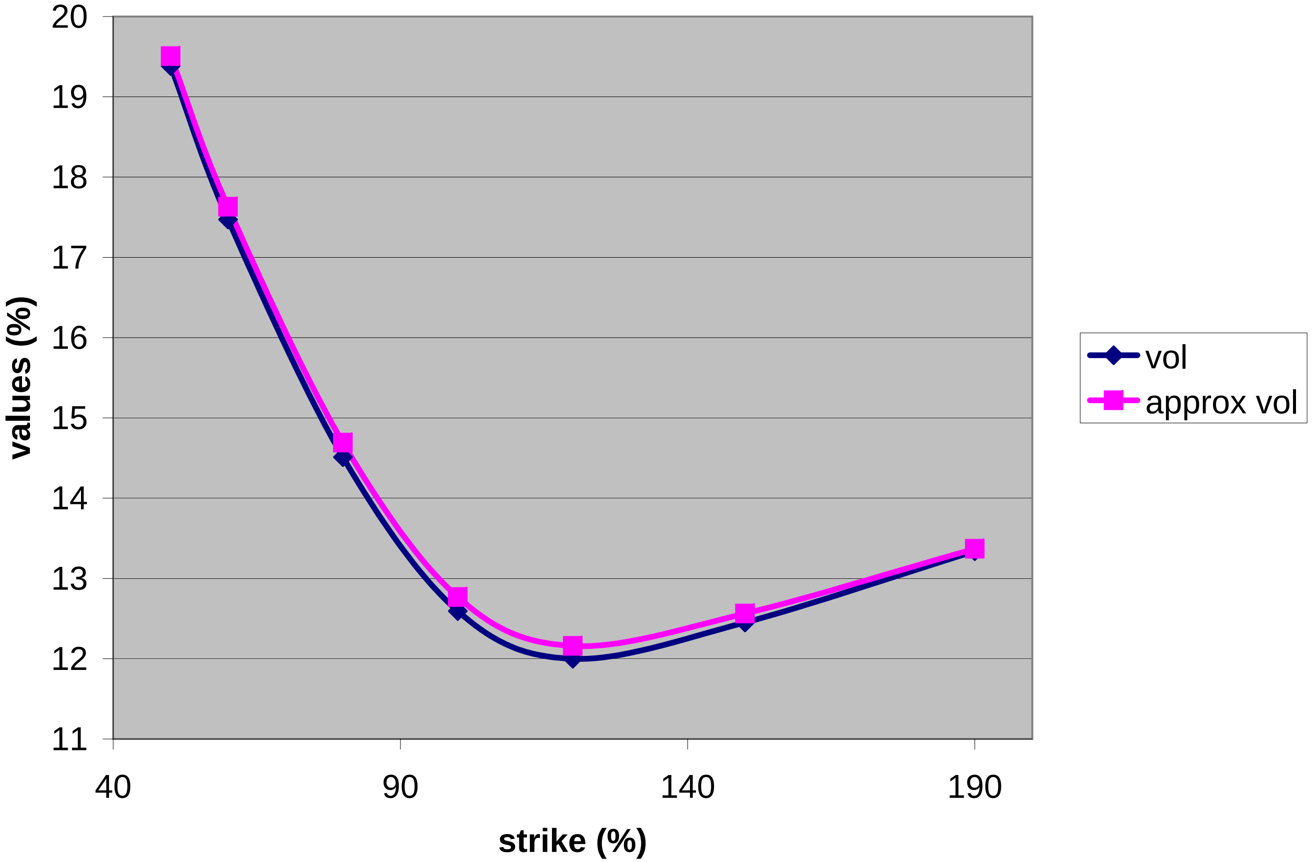

Figure 5: Heston basket option implied volatilities for 5Y maturity

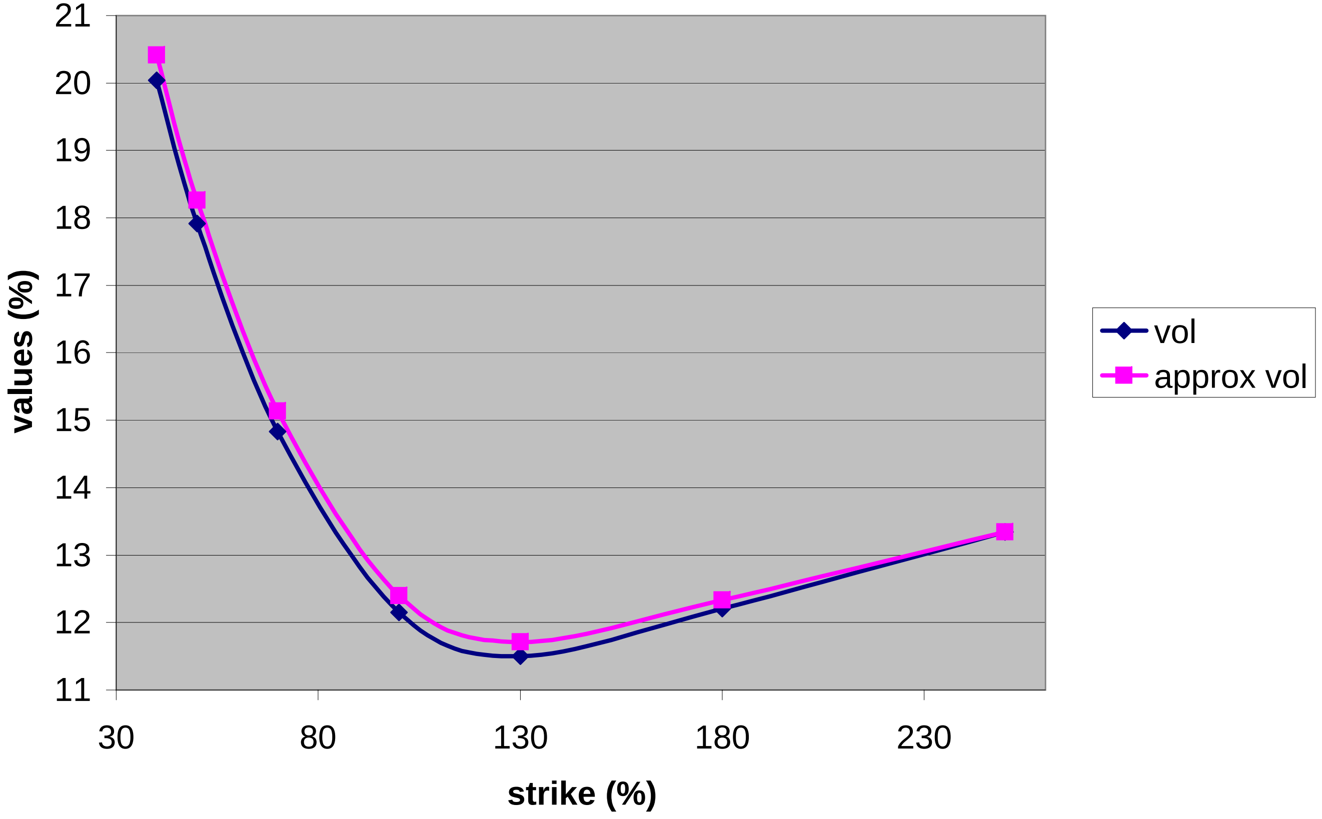

Figure 6: Heston basket option implied volatilities for 10Y maturity

Observations:

- Analytics captures both skew and smile.
- Good approximation quality for the optimal analytics: the biggest error for 10Y time horizon does not exceed 20 bps (for ATM) and 25 bps (for out-of-money).
- In general, the errors are of the same order as MC standard deviation for 10 sec simulation (Pentium M 1.8 GHz).

Possible improvements:

- More careful choice of the initial skew $\beta(0)$ (we could reduce the error significantly by varying it).
- Calculation of the averages to higher orders in volatility.

Summary

We presented

- a review of the Markovian projection to the displaced diffusion
- new results for the Markovian projection to the shifted Heston model, taking into account skew and smile effects
- numerical results for a basket of Heston processes

Details of the calculations can be found in the upcoming

A. Antonov and T. Misirpashaev (2007),
"*Reducing dimensionality of smile models by projection on an effective Heston model*",
working paper

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Contact information

www.numerix.com

UK NumeriX Software Ltd, 2nd floor 41 Eastcheap London EC3
Tel +44 (0) 207 648 6100, Fax +44 (0) 207 648 6139

antonov@numerix.com