Trading Maximum Drawdown and Options on Maximum Drawdown

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Abstract.

Maximum Drawdown is becoming increasingly important in the risk management and in the portfolio optimization. In this talk, we note that the Maximum Drawdown can be traded as a derivative asset. Several related contracts, such as Call or Put options on the Maximum Drawdown, or barrier option on the Maximum Drawdown (Crash option) are also discussed. These contracts can facilitate risk management for financial institutions concerned with control of the drawdown of their portfolio.

Drawdowns and Drawups

Suppose that we have an underlying asset whose price process at time t is given by S_t. For example, it could be a stock price, index, interest rate or exchange rate. Denote by M_t its running maximum up to time t:

$$M_t = \max_{u \in [0,t]} S_u.$$

Drawdown D_t is defined as the drop of the asset price from its running maximum:

$$D_t = M_t - S_t. \tag{1}$$

Maximum drawdown MDD_t is defined as the maximal drop of the asset price from its running maximum over a given period of time:

$$MDD_t = \max_{u \in [0,t]} D_u. \tag{2}$$

Average drawdown ADD_t is given by

$$ADD_t = \frac{1}{t} \int_0^t D_u du. \tag{3}$$

Similarly, we can define the concept of drawup, maximum drawup and average drawup. *Drawup* U_t is defined as the increase of the asset price from its running minimum:

$$U_t = S_t - m_t, \tag{4}$$

where

$$m_t = \min_{u \in [0,t]} S_u.$$

Maximum drawup MDU_t is given by

$$MDU_t = \max_{u \in [0,t]} U_u, \tag{5}$$

and *Average drawup* ADU_t is defined as

$$ADU_t = \frac{1}{t} \int_0^t U_u du. \tag{6}$$

Figures 1–3 illustrate the concepts of the drawdown and the drawup on data taken from S&P500 for the year 2005. Figures 4–5 show the historical evolution of S&P500 and its maximum drawdown for the period starting in 1970.

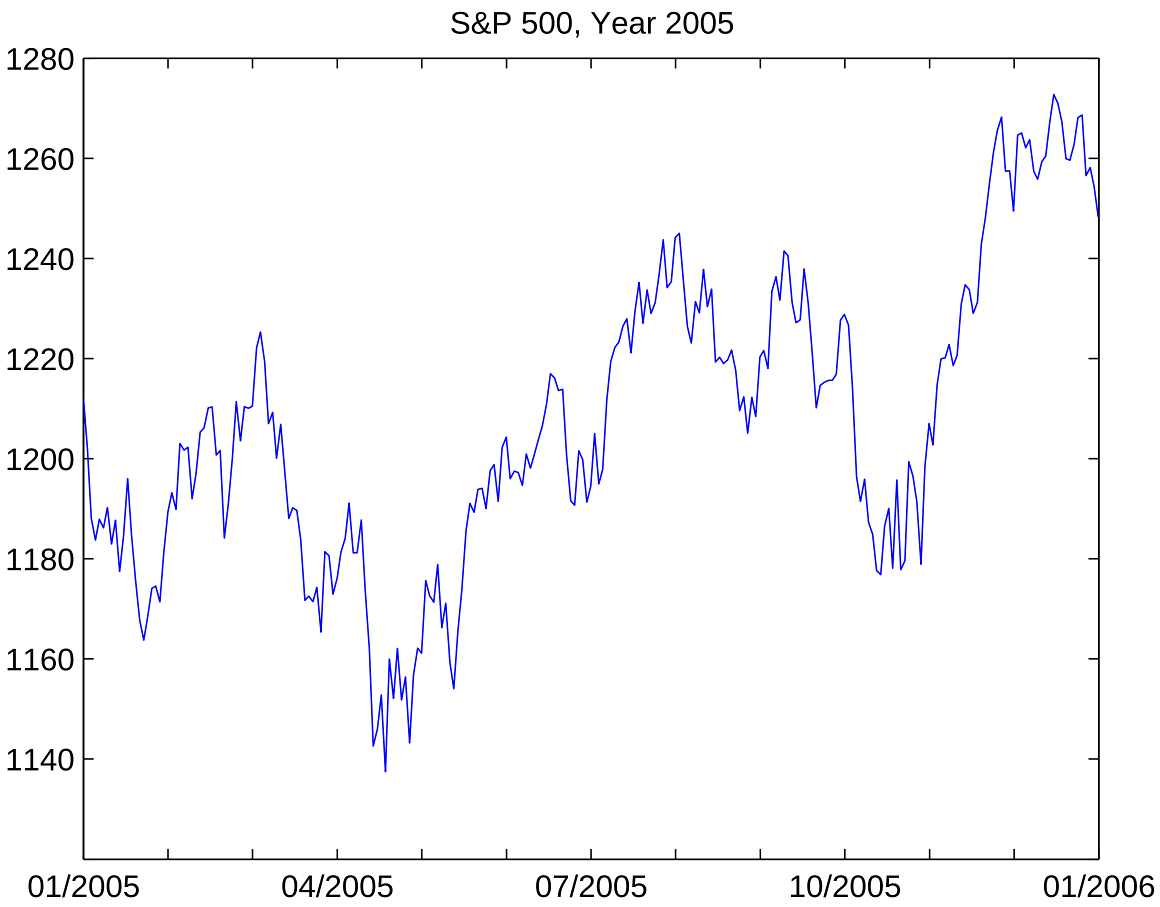

Figure 1: S&P500 daily closing values in 2005.

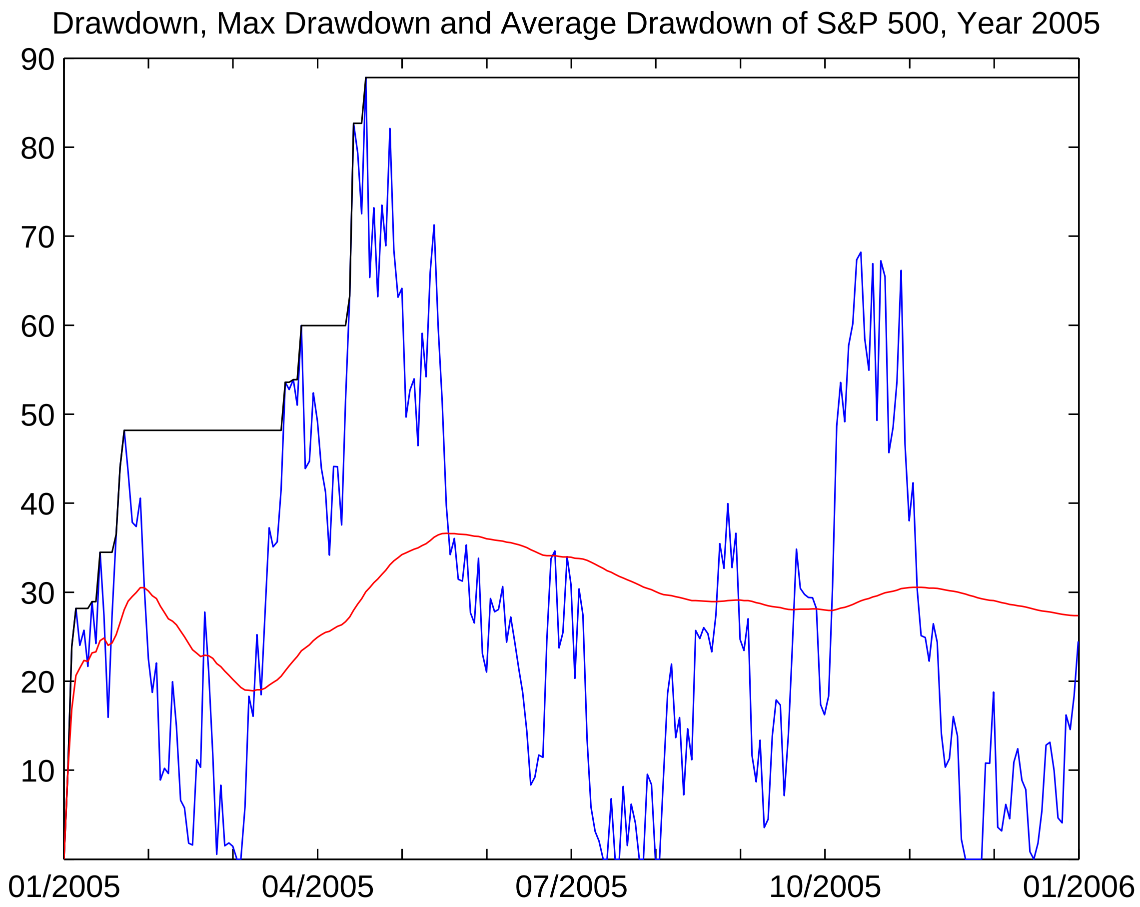

Figure 2: Drawdown (blue), maximum drawdown (black), and average drawdown (red) of S&P500 in 2005.

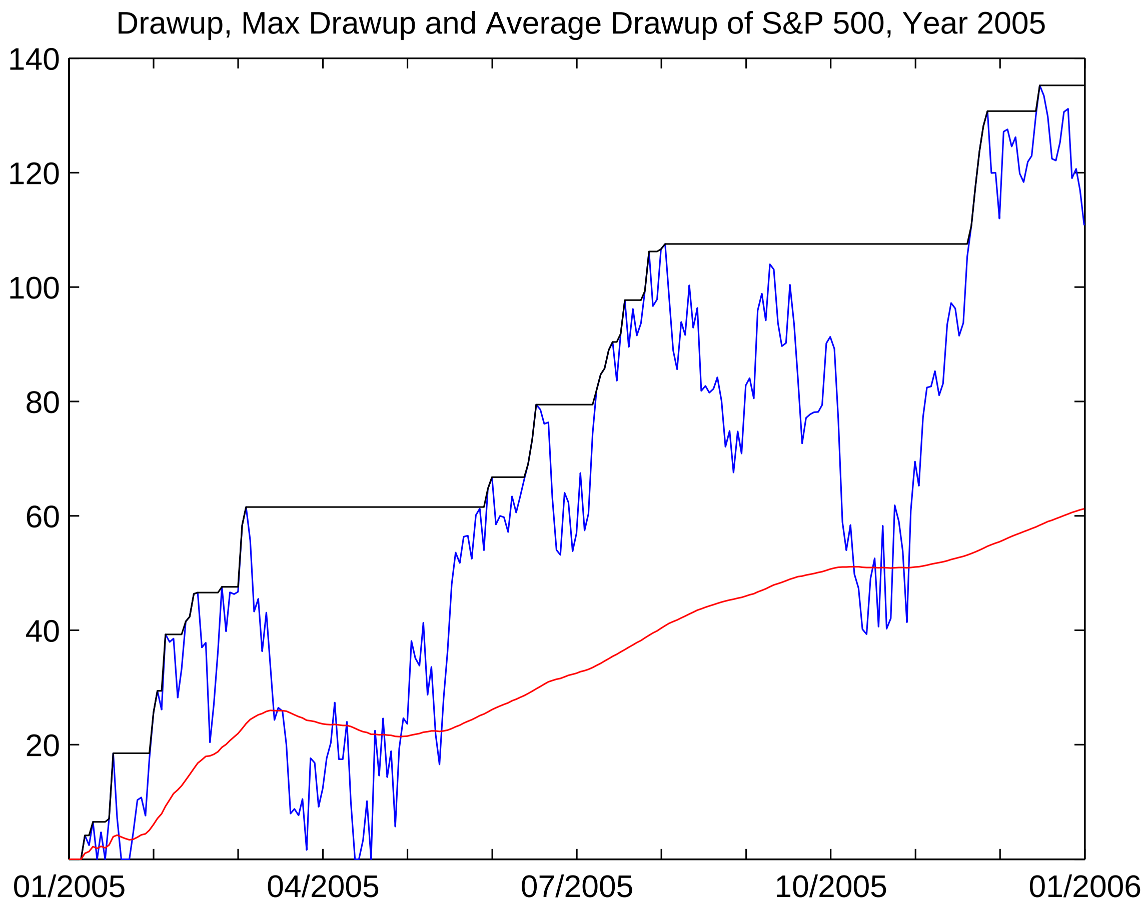

Figure 3: Drawup (blue), maximum drawup (black), and average drawup (red) of S&P500 in 2005.

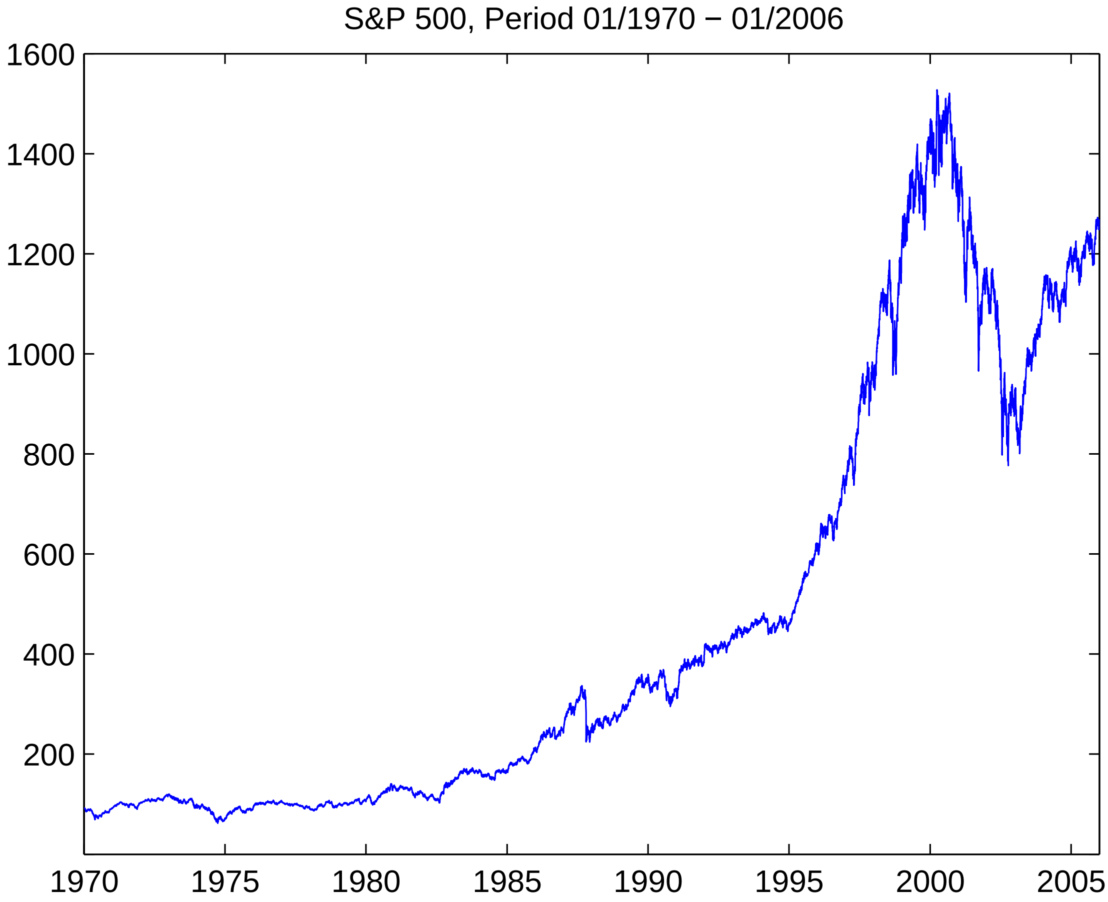

Figure 4: Index S&P 500 from 01/1970 to 12/2005.

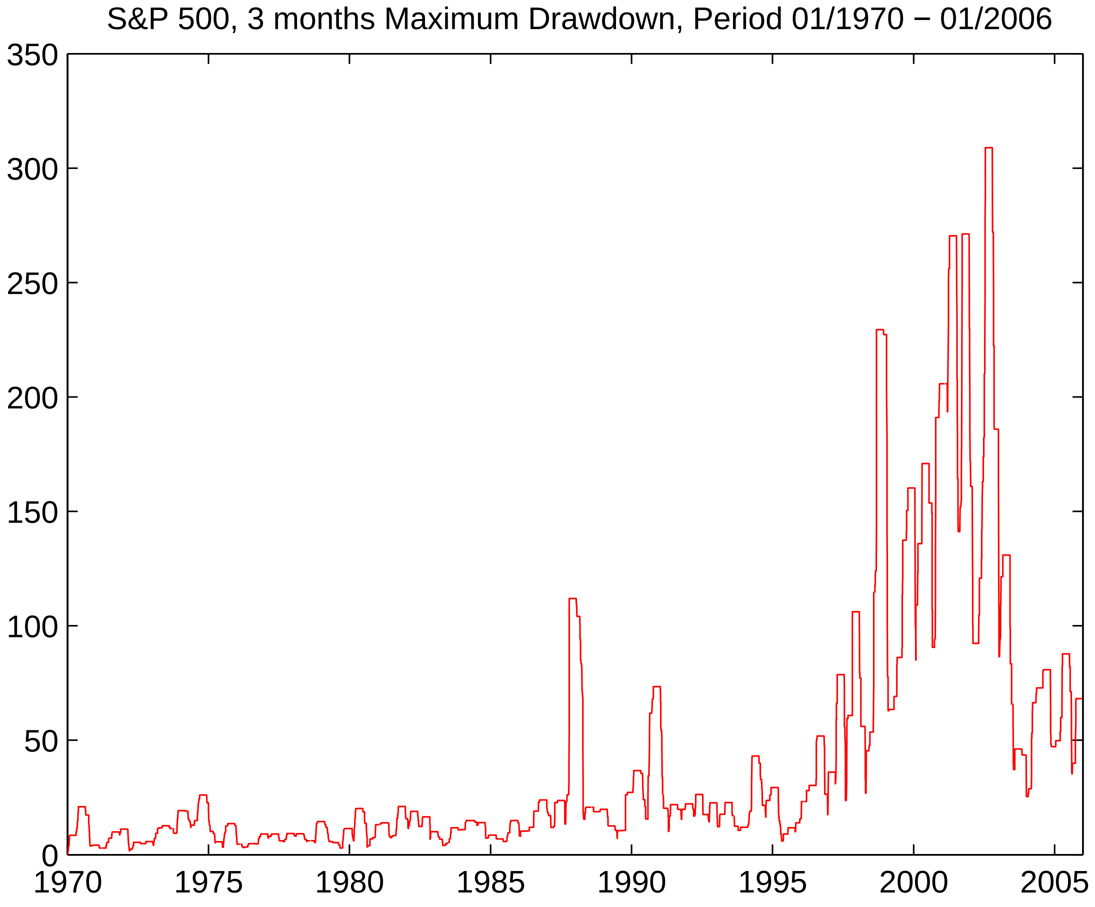

Figure 5: Maximum drawdown taken over 3 months interval, S&P500 index, period from 01/1970 to 12/2005.

The concept of maximum drawdown or average drawdown has been extensively studied in the recent literature. Risk measures based on the maximum or average drawdown can serve as an alternative to the commonly used Value-at-Risk. Portfolio optimization using the drawdown has been considered in Chekhlov, Uryasev and Zabarkin (2005). Analytical results linking the maximum drawdown to the mean return appeared in the paper of Magdon-Ismail and Atiya (2004).

A portfolio manager concerned with control of the drawdown might be interested in entering a contract with payoffs depending on the level of the maximum or the average drawdown. The obvious choices are listed in Table 1:

Contract type	**Payoff**
Forward (futures) on the maximum drawdown	$MDD_T - K$
Call on the maximum drawdown	$(MDD_T - K)^+$
Put on the maximum drawdown	$(K - MDD_T)^+$
Forward (futures) on the average drawdown	$ADD_T - K$
Call on the average drawdown	$(ADD_T - K)^+$
Put on the average drawdown	$(K - ADD_T)^+$

Table 1: Contracts depending on the maximum drawdown or the average drawdown.

The price of a futures contract on maximum drawdown or average drawdown can serve as an important risk measure indicator which would be quoted by the market (rather than determined internally). When the market is in a bubble, it is reasonable to expect that the prices of drawdown contracts would be significantly higher. On the other hand, when the market is stable, or when it exhibits mean reversion behavior, the prices of drawdown contracts would become cheaper.

The volume of drawdown trading can be relatively small, but it would carry important information for the entire market in terms of the perception of drawdown risk associated with the given asset or index. If the market enters a bubble, the price of the drawdown contract would increase, indicating the increased risk associated with the given asset. Higher risk leads to a limited exposure to such asset, and possibly to a fast price correction. Transparent quoting of maximum drawdown prices could potentially lead to more stable markets, without long periods of bubbles followed by significant crashes. The market would rather adapt through small and short corrections instead of large and extended drops.

Pricing

As for the pricing, the value $v(t, S_t, M_t, MDD_t)$ of any type of contract depending on the maximum drawdown is given by taking the conditional expectation of the discounted payoff under the risk neutral measure:

$$v(t, S_t, M_t, MDD_t) = \mathbb{E}[e^{-(T-t)} f(\{MDD_u\}_{u=0}^{T}) | S_t, M_t, MDD_t].$$

A similar formula applies for the pricing of contracts that depend on the average drawdown, ADD_t. Here, the function f determines the type of payoff defined by the contract (for instance, $f(\{MDD_t\}_{t=0}^{T}) = MDD_T - K$ for the forward contract, etc.). For the evolution of the underlying asset under the risk neutral measure, we may assume that

$$dS_t = rS_t dt + g(t, S_t) dN_t,$$

for a general martingale N_t (diffusion or jump type process). Other possible evolutions, such as a mean reversion type process, could be considered for the asset dynamics of S_t.

The price can be computed by using standard

Monte Carlo simulations. The pricing via conditional expectations can be linked to partial differential equations through the Feynman-Kac theorem, see for instance Shreve (2004). However, the resulting PDE has 3 spatial dimensions which is difficult to implement and slow to compute. Therefore Monte Carlo methods would be more efficient and easy to implement in this situation. Monte Carlo also allows for more complicated dynamics of the asset price, such as stochastic volatility or jumps.

Figures 6–7 show the price of the forward contract on the maximum drawdown MDD_T as a function of maturity written on the S&P500 index, assuming the geometric Brownian motion model. It also shows the price of the call option on the maximum drawdown with maturity 1 year as a function of strike. Figures 8–9 illustrate the same contracts written on the average drawdown ADD_T.

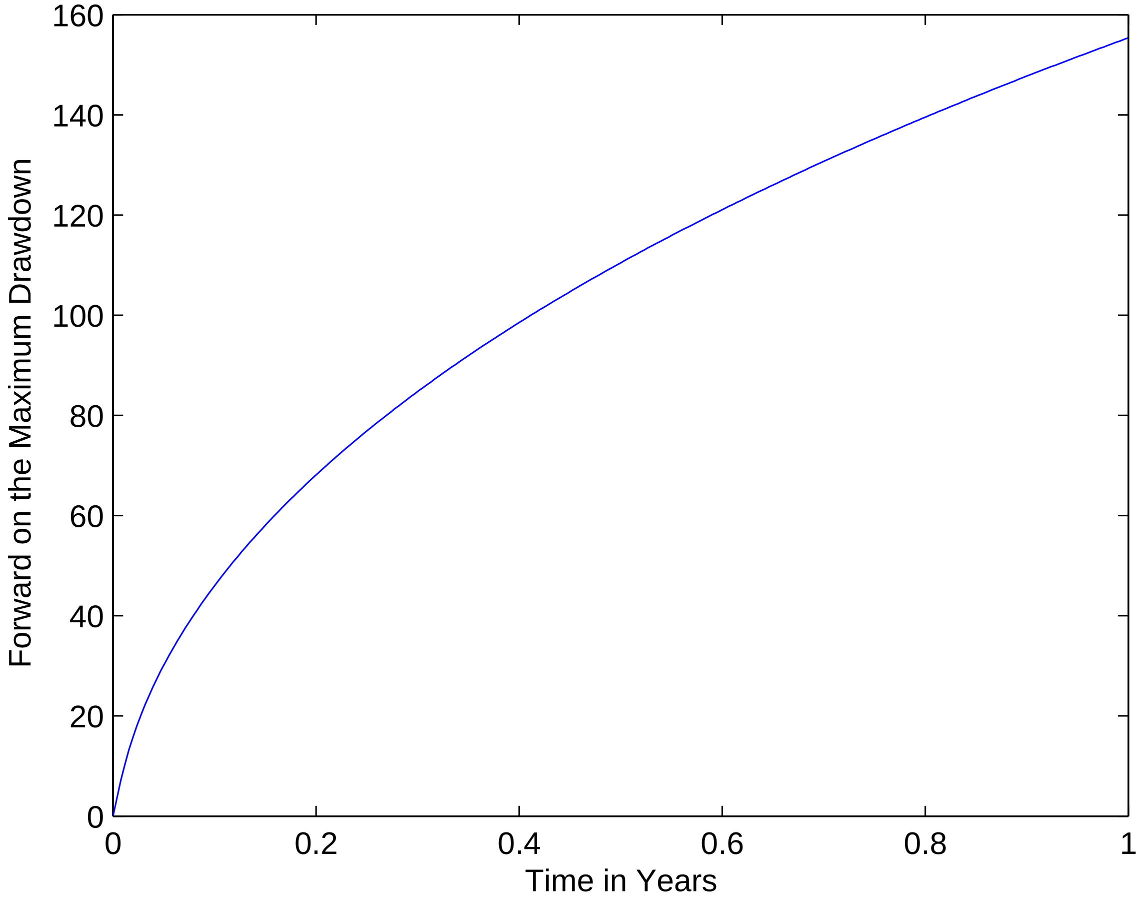

Figure 6: The price of the contract with the payoff MDD_T (maximum drawdown) as a function of maturity T. The underlying asset is the index S&P500 on 01/03/2005: opening value 1,211.92, interest rate $r = 0.03$, volatility $\sigma = 12\%$.

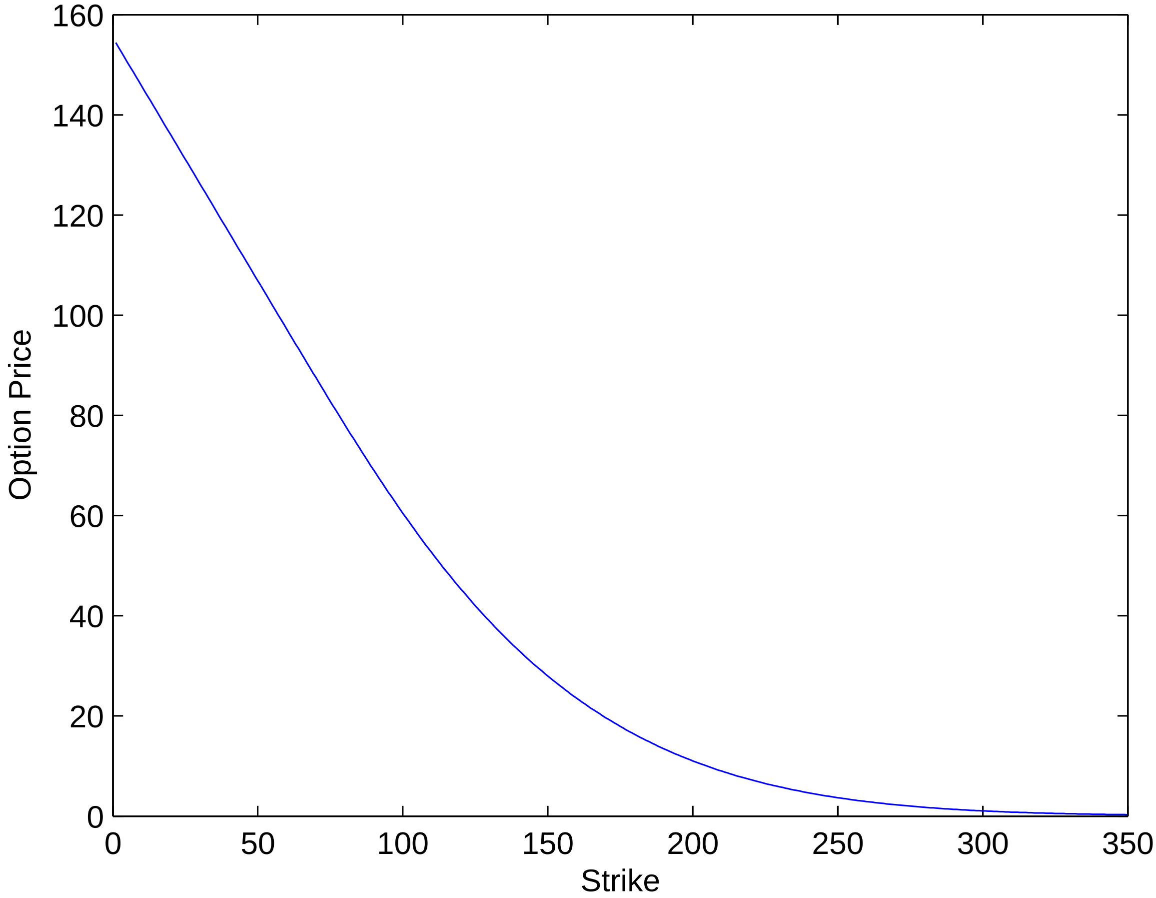

Figure 7: The price of the call option $(MDD_1 - K)^+$ as a function of the strike K, same underlying index S&P500, the value of the option on 01/03/2005 with maturity 12/30/2005.

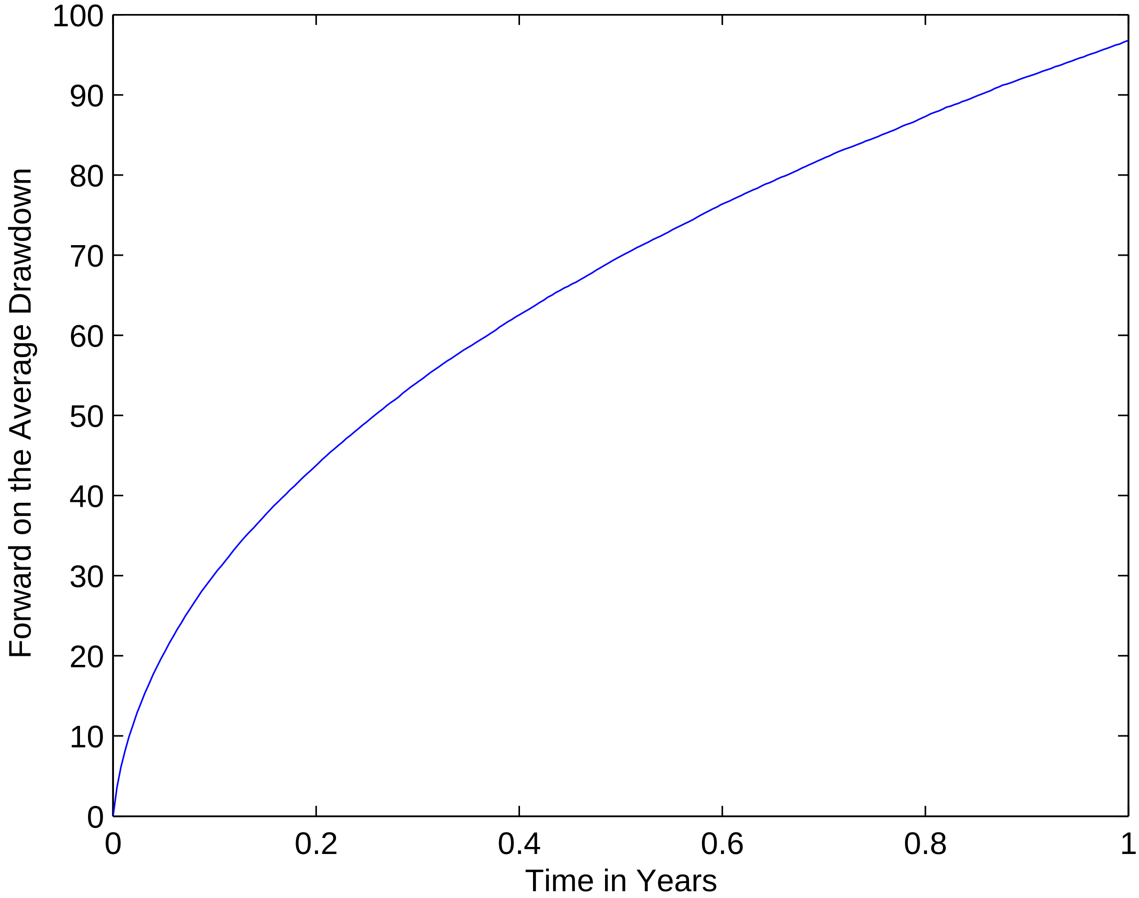

Figure 8: The price of the contract with the payoff ADD_T (average drawdown) as a function of maturity T. The underlying asset is the index S&P500 on 01/03/2005: opening value 1,211.92, interest rate $r = 0.03$, volatility $\sigma = 12\%$.

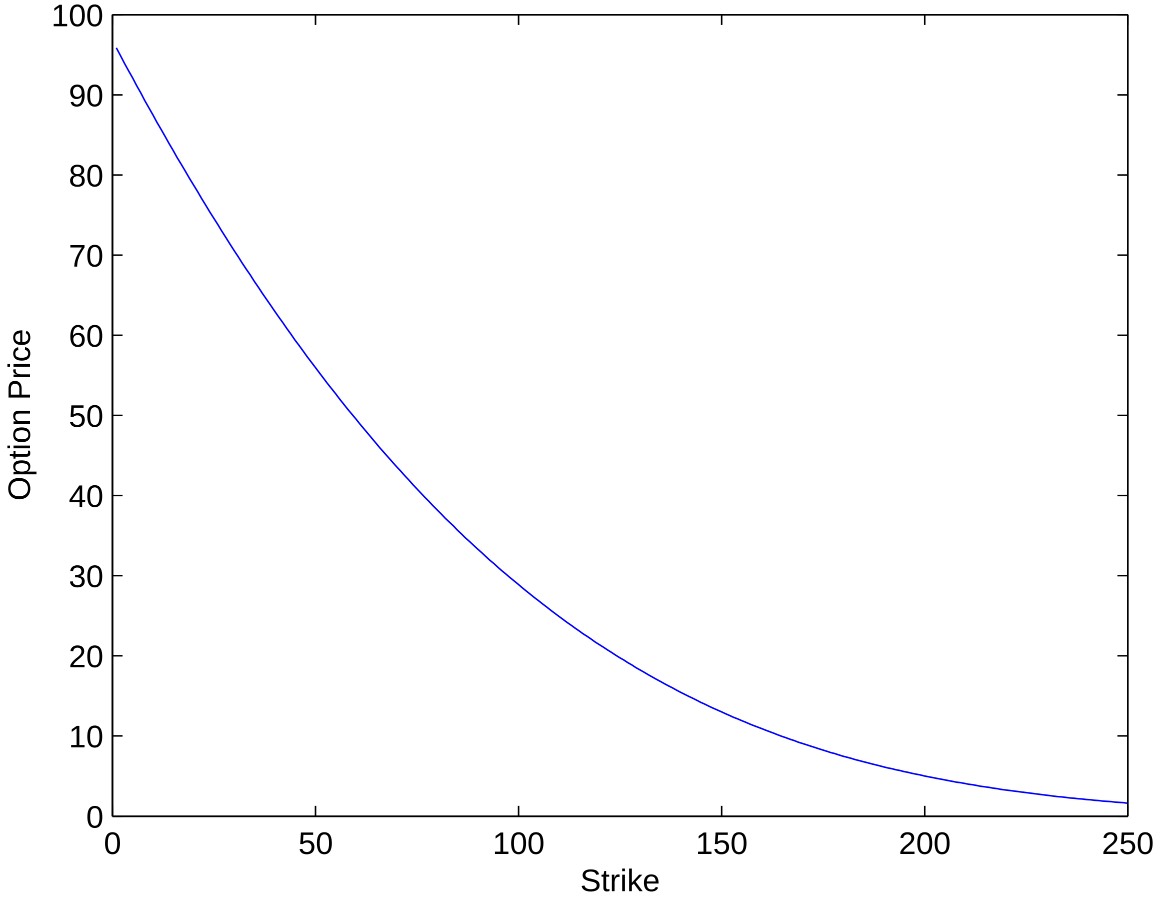

Figure 9: The price of the call option $(ADD_1 - K)^+$ as a function of the strike K, same underlying index S&P500, the value of the option on 01/03/2005 with maturity 12/30/2005.

Table 2 lists realized values and theoretical prices of contracts depending on the maximum and average drawdown or drawup written on S&P500 in the year 2005. Notice that in that period, theoretical prices of 1 year contracts written on both drawdown and drawup exceeded the realized values. This might be due to the effect of mean reversion – stable real markets could stay within a more conservative range in the long run if compared to models based on geometric Brownian motion model. We assumed a rather small estimate of volatility $\sigma = 12\%$.

Contracts Expiring on 12/30/2005, S&P500	Realized Values		Theoretical Prices	
	3 months	1 year	3 months	1 year
Forward on MDD	51.97	87.81	76.84	155.39
Forward on ADD	17.30	27.35	49.26	96.81
Forward on MDU	95.90	135.24	84.44	185.27
Forward on ADU	51.16	61.28	56.40	124.19

Table 2: The realized values and theoretical prices of forward contracts on the maximum drawdown, the average drawdown, the maximum drawup, and the average drawup, where the underlying index is S&P500. We assume that the 3 month contract starts on Oct 3, 2005 and expires on Dec 30, 2005; the 1 year contract starts on Jan 3, 2005 and expires on Dec 30, 2005. We use opening value 1,211.92 of the index for the one year contract, opening value 1228.81 of the index for the 3 months contract, interest rate $r = 0.03$, volatility $\sigma = 12\%$. Theoretical prices are obtained from Monte Carlo simulation, standard deviations do not exceed 0.01.

Crash Option, Rally Option and Other Contracts

Let us define the event of the absolute value market crash as the first time the stock price S_t drops by a constant a from its running maximum, i.e.,

$$T_a = \inf\{t \geq 0 : D_t = M_t - S_t \geq a\} = \inf\{t \geq 0 : MDD_t \geq a\},$$

which is obviously the same event as the first time the maximum drawdown exceeds level a.

A *crash option* is defined as a contract which pays off a at the time of the market crash T_a if $T_a < T$, where T is the maturity. Other payoffs, such as $M_{T_a} - S_{T_a}$ ($M_{T_a} - S_{T_a}$ could be greater than a if the market exhibits price discontinuity at the time of the crash), are possible to consider. The crash option can be regarded as a barrier option on the maximum drawdown.

The closest contract to the crash option that has been studied in the literature is the Russian option. It is a perpetual option that pays off at the time of the exercise the running maximum (discounted by a rate $\alpha > r$) of the asset price. It turns out that in the Black-Scholes model, the optimal exercise time is the time of a drawdown of a certain size a. See, for instance, Shepp and Shiryaev (1993).

The crash option resets its holder to the historical maximum of the asset price during the lifetime of the contract at the time of the market crash. If the crash does not happen up to the time of the maturity of the contract, the option expires worthless. Since the market crash is a rather extreme event, most of the time the option will not end up in the money if we are considering a large drop. This feature will make the contract cheap. It is possible to construct a contract which would set the wealth of its holder to a different level than the running maximum at the time of the crash, for instance to the running average, or to some intermediate point between the running maximum and the crash value.

Drop Level	Maturity 3M	1Y
50	32.80	49.72
75	27.56	72.24
100	18.00	83.90
150	4.32	72.27
200	0.51	45.10

Table 3: The price of the crash option (with daily monitoring) for selected drop levels and maturities 3 months and 1 year. We assume that the 3 month contract starts on Oct 3, 2005 and expires on Dec 30, 2005; the 1 year contract starts on Jan 3, 2005 and expires on Dec 30, 2005. We use opening value 1,211.92 of the index for the one year contract, opening value 1228.81 of the index for the 3 months contract, interest rate $r = 0.03$, volatility $\sigma = 12\%$. Theoretical prices are obtained from Monte Carlo simulation, standard deviations do not exceed 0.01.

Table 3 gives theoretical prices of crash option contract written on S&P500 in the year 2005 for selected drop levels and maturities. The prices were computed as discounted payoff under the risk neutral measure, using Monte Carlo simulation. Figure 4 plots the theoretical price of the crash option as a function of the barrier level. Notice that small drawdowns typically happen within the given time framework just from the random nature of the price process, making the price of crash option increase almost linearly around zero. However, drops of larger sizes become increasingly unlikely, making the corresponding option price cheaper.

A *rally option* insures the opposite event of a crash, the case of a market rally. Define the rally event in the absolute value as the first time the stock price raises by a constant b from its running minimum, i.e.,

$$T_b = \inf\{t \geq 0 : S_t - m_t \geq b\} = \inf\{t \geq 0 : MDU_t \geq b\}.$$

The rally option is defined as a contract which pays off b at the time of the rally T_b if $T_b < T$, where T is the maturity. Other payoffs, such as $S_{T_b} - m_{T_b}$, are possible to consider.

A *range barrier option* pays off c at the time U_c if $U_c < T$, where T is the maturity, and U_c is first time the stock exceeds a range of the level c, i.e.,

$$U_c = \inf\{t \geq 0 : M_t - m_t \geq c\}.$$

All contracts described in this paper can be also defined on the percentage (relative value) drawdown or drawup as opposed to the absolute values. For instance, we can define the *maximum relative drawdown* $MRDD_t$ as the largest relative drop of the asset with respect to its running maximum up to time t:

$$MRDD_t = \max_{u \in [0,t]} (1 - \frac{S_u}{M_u}).$$

One can trade various options with maximum relative drawdown as the underlying process. For example, the barrier option on $MRDD_t$ represents the relative value crash option with the payoff $a^* \cdot M_{T_{a^*}}$ at the first time T_{a^*} when the relative drawdown exceeds $100 \cdot a^*$ percent:

$$T_{a^*} = \inf\{t \geq 0 : MRDD_t \geq a^*\}.$$

Similarly, we can define the *maximum relative drawup* as

$$MRDU_t = \max_{u \in [0,t]} \frac{S_u}{m_u} - 1,$$

and the corresponding contracts written on $MRDU_t$ as the underlying process.

Example of PDE approach: Crash Options on MRDD

For the crash option on the maximum relative drawdown, its value is given by

$$v(t,x,y) = a^*\mathbb{E}[e^{-r(T^*_{a^*}-t)}I(T^*_{a^*} < T)M_{T^*_{a^*}}|S_t = x, M_t = y],$$

We have the partial differential equation

$$v_t(t,x,y)+rxv_x(t,x,y)+\frac{1}{2}\sigma^2x^2v_{xx}(t,x,y) = rv(t,x,y),$$

satisfied in the region $\{(t,x,y); 0 \le t < T, x \le y \le \frac{1}{1-a^*}x\}$, and with the boundary conditions

$$\begin{aligned} v(t,(1-a^*)y,y) &= a^*y, \quad 0 \le t \le T,\ y > 0, \\ v_y(t,y,y) &= 0, \quad 0 \le t \le T,\ y > 0, \\ v(T,x,y) &= 0, \quad x \le y < \frac{1}{1-a^*}x. \end{aligned}$$

Since the percentage value crash option satisfies the linear scaling property

$$v(t,\lambda x,\lambda y) = \lambda v(t,x,y),$$

we may reduce the dimensionality of the problem by introducing function u by

$$u(t,z) = v(t,z,1), \quad 0 \le t \le T,\ 1-a^* \le z \le 1.$$

Then

$$v(t,x,y) = yu(t,\tfrac{x}{y}).$$

It is easy to verify that u satisfies

$$u_t(t,z) + rzu_z(t,z) + \tfrac{1}{2}\sigma^2 z^2 u_{zz}(t,z) = ru(t,z),$$

$$0 \le t \le T,\ 1 - a^* \le z \le 1,$$

with boundary conditions

$$\begin{aligned} u(T,z) &= 0, \quad 1 - a^* < z \le 1, \\ u(t,1) &= u_z(t,1), \quad 0 \le t < T, \\ u(t,1-a^*) &= a^*, \quad 0 \le t \le T. \end{aligned}$$

Drop Percentage	**Maturity**						
	1M	**3M**	**6M**	**1Y**	**5Y**	**25Y**	∞
5%	1.34%	3.83%	4.94%	5.25%	5.26%	5.26%	5.26%
10%	0.04%	1.42%	3.99%	7.35%	11.07%	11.11%	11.11%
15%	0.00%	0.16%	1.38%	4.60%	15.65%	17.65%	17.65%
20%	0.00%	0.01%	0.25%	1.95%	15.21%	24.87%	25.00%
25%	0.00%	0.00%	0.02%	0.56%	11.69%	31.02%	33.33%

Table 4: The price of the Percentage Crash Option for selected drop levels and selected maturities. The price of the option is given as a percentage of the initial asset price using the parameters $r = 3\%$, $\sigma = 12\%$. The perpetual option has the price $\frac{a^*}{1-a^*}$. The percentage drop of any level happens in finite time, the payoff of the option being $a^* M_{T^*_{a^*}}$, which is a value exceeding $a^* S_0$.

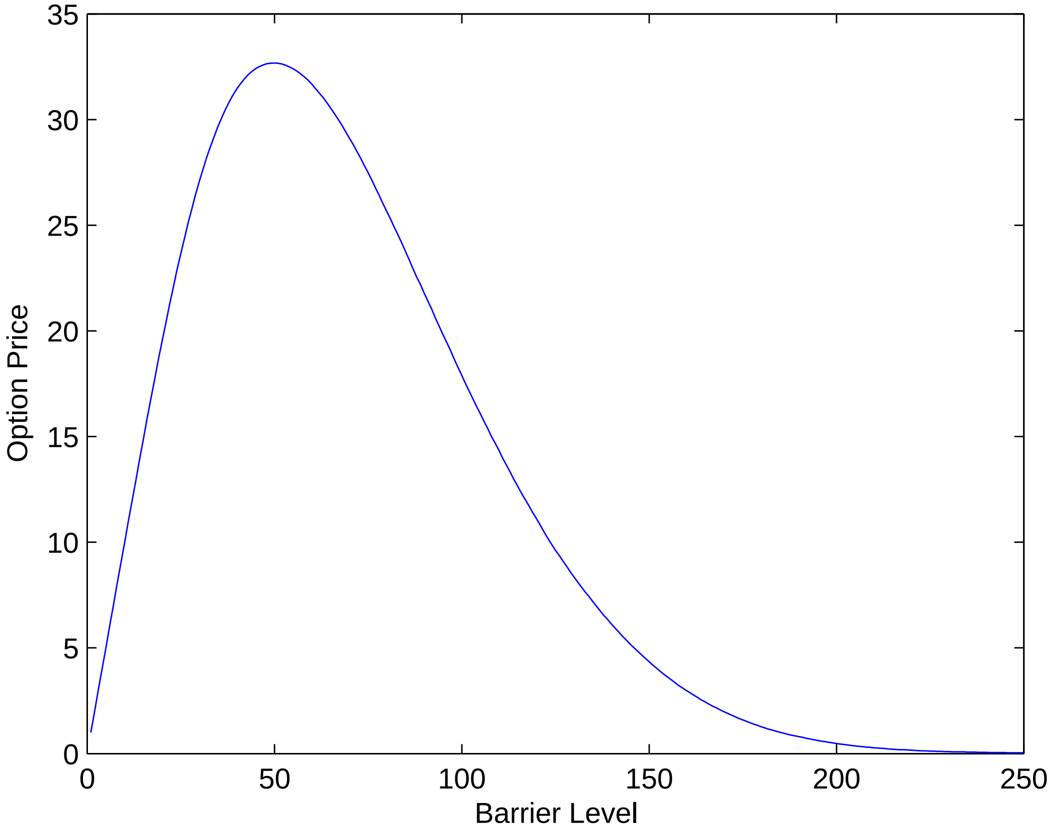

Figure 10: Price of the crash option (with daily monitoring) as a function of barrier level and maturity 3 months. We assume that the 3 month contract starts on Oct 3, 2005 and expires on Dec 30, 2005; the 1 year contract starts on Jan 3, 2005 and expires on Dec 30, 2005. We use opening value 1,211.92 of the index for the one year contract, opening value 1228.81 of the index for the 3 months contract, interest rate $r = 0.03$, volatility $\sigma = 12\%$.

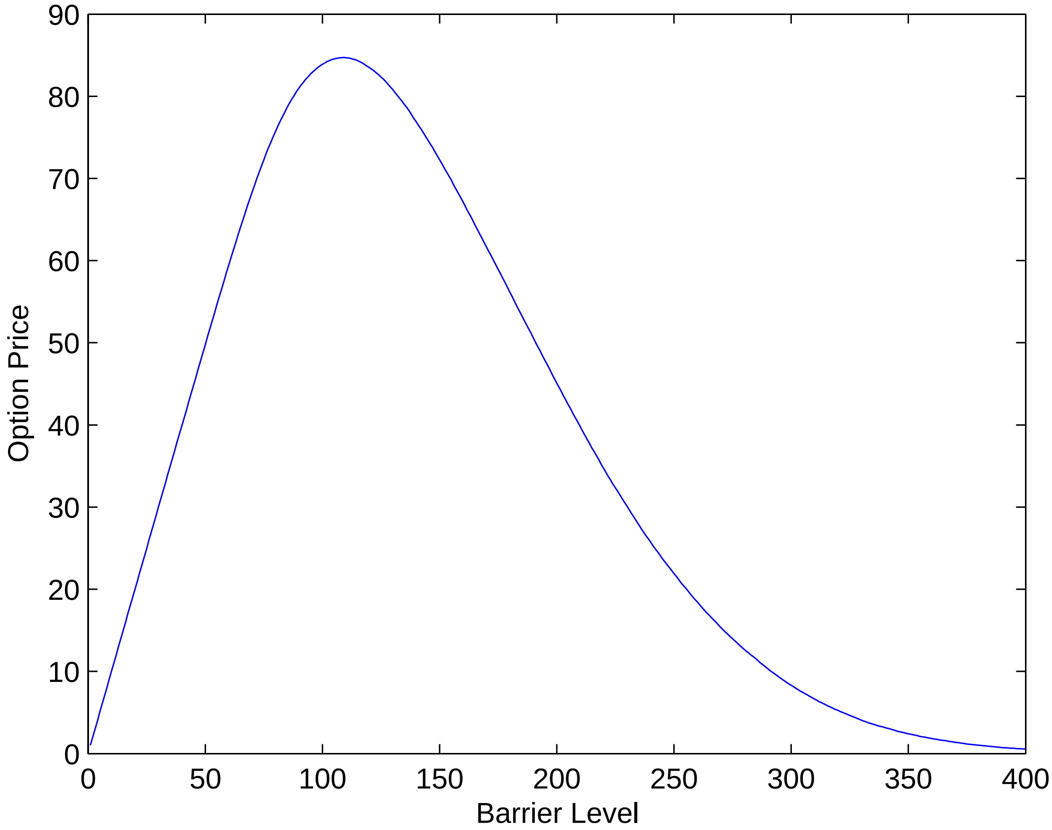

Figure 11: Price of the crash option (with daily monitoring) as a function of barrier level and maturity 1 year. We assume that the 3 month contract starts on Oct 3, 2005 and expires on Dec 30, 2005; the 1 year contract starts on Jan 3, 2005 and expires on Dec 30, 2005. We use opening value 1,211.92 of the index for the one year contract, opening value 1228.81 of the index for the 3 months contract, interest rate $r = 0.03$, volatility $\sigma = 12\%$.