Valuing Default Swaps on Correlated LMM Processes

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Sources:
www.dersoft.com/presMIT.ppt
www.dersoft.com/Valuing.doc
www.dersoft.com/dslmmkkm.xls
Credit Derivatives: Application, Pricing, and Risk Management

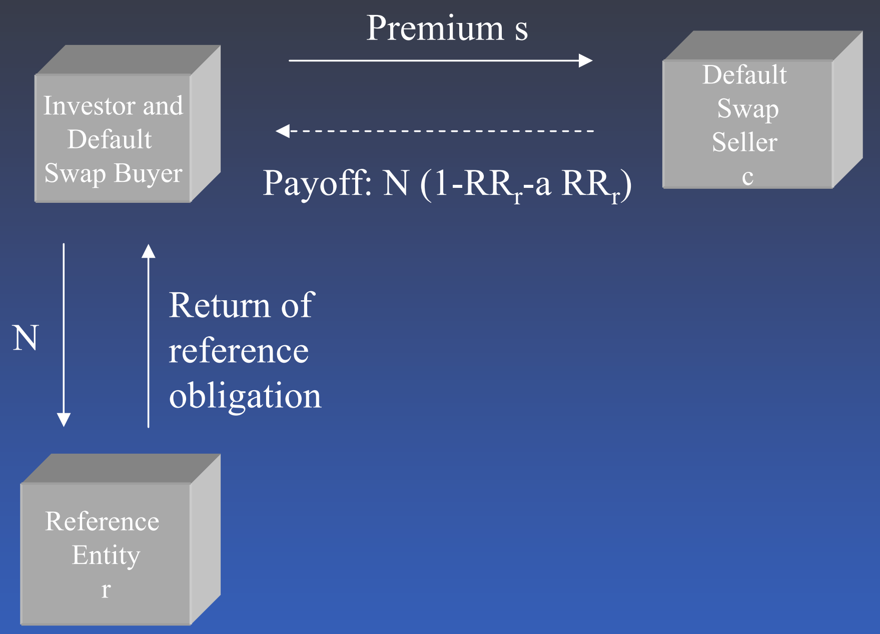
Valuing Default Swaps on Correlated LMM Processes

Agenda

- The Basic Structure of the Model
  - 1a. The Default Swap Payoff Tree
  - 1b. The Default Swap Premium Payment Tree
    - 1b1. Upfront premium
    - 1b2. In arrears premium
  - 1c. Combining the Trees
- 2. The Correlation Concept
- 3. The 3 LMM Processes
- 4. Properties of the Model
- 5. Limitations of the Model
- 6. Conclusion/Summary

Valuing Default Swaps on Correlated LMM Processes

Recall: A plain-vanilla Default swap looks as follows:



Premium s

Investor and Default Swap Buyer

Default Swap Seller c

Payoff: $N\ (1-RR_r\text{-}a\ RR_r)$

N

Return of reference obligation

Reference Entity r

For the investor, which default is worse???

What is ‘worst case scenario’ for the Investor???

Should we include the default probability of the investor???

(see www.dersoft.com/Pricing.doc)

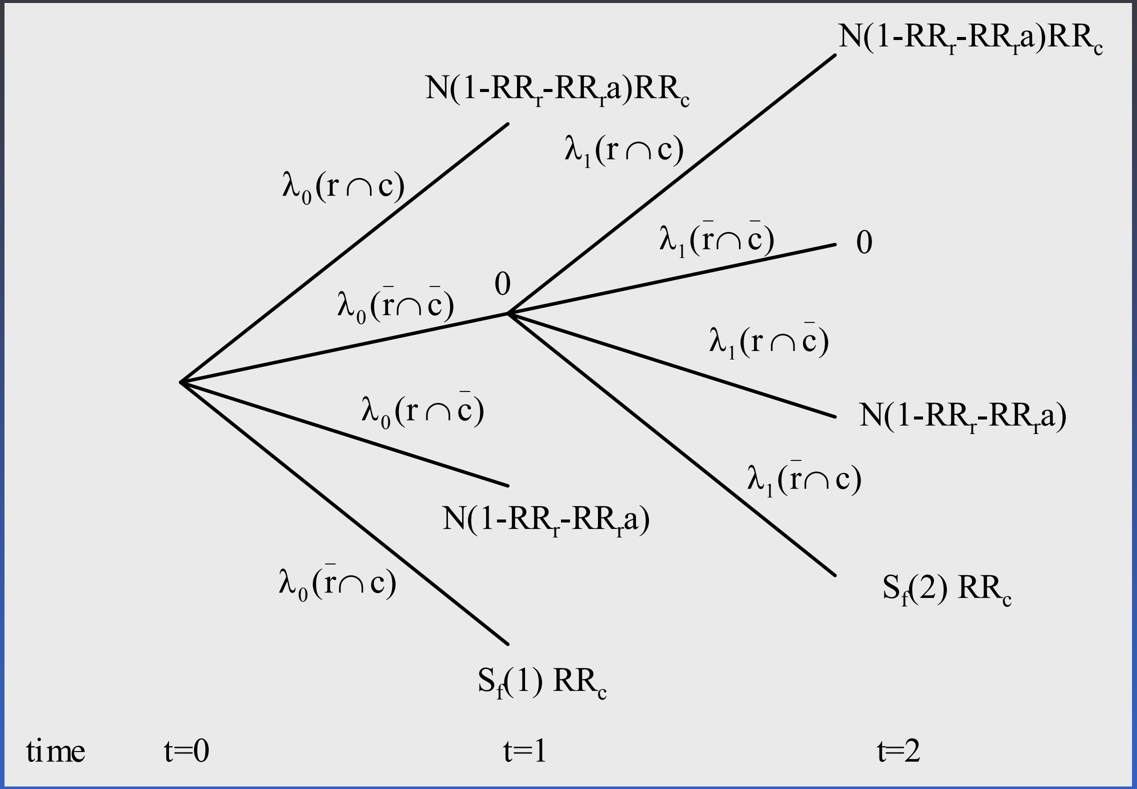
Pricing Default Swaps

What determines the Price of a Default Swap?

Input variables for deriving the premium s of a default swap
1) Default probability and credit deterioration probability of the reference asset
2) Default probability and credit deterioration probability of the default swap seller
3) Correlation between 1) and 2)
4) Default volatility of the underlying reference asset
5) Default volatility of the default swap seller
6) Correlation between 4) and 5)
7) Maturity of the default swap
8) Expected recovery rate of the reference asset
9) Expected recovery rate of the default swap seller
10) Return of the reference asset (e.g. coupon of the reference bond)
11) Risk-free interest rate term structure used to discount future cash flows
12) Default probability of the default swap buyer in case of periodic credit derivative premium
13) Expected recovery rate of the default swap buyer in case of periodic credit derivative premium
14) Correlation between the default probability of the default swap buyer and the reference asset in case of periodic default swap premium
15) Market risks (as interest rate risk, currency risk, commodity risk and stock price risk) and the correlation between market risk and credit risk
16) Operational risks (e.g. legal risks, documentation risks or settlement risks), which might endanger the enforceability of the payoff and the correlation between operational risk and credit risk
17) Liquidity of the default swap
18) Liquidity of the underlying reference asset
19) BIS risk weight of the default swap seller
20) Urgency of protection (e.g. is an immediate credit deterioration expected or does the protection free up credit lines to enable further business with a client)
21) Transaction costs

Valuing Default Swaps on Correlated LMM Processes

1a. Payoff Tree



$N(1-RR_r-RR_r a)RR_c$

$N(1-RR_r-RR_r a)RR_c$

$\lambda_1(r \cap c)$

$\lambda_0(r \cap c)$

$\lambda_1(\bar{r} \cap \bar{c})$

0

0

$\lambda_0(\bar{r} \cap \bar{c})$

$\lambda_1(r \cap \bar{c})$

$N(1-RR_r-RR_r a)$

$\lambda_0(r \cap \bar{c})$

$\lambda_1(\bar{r} \cap c)$

$N(1-RR_r-RR_r a)$

$\lambda_0(\bar{r} \cap c)$

$S_f(2)\ RR_c$

$S_f(1)\ RR_c$

time t=0 t=1 t=2

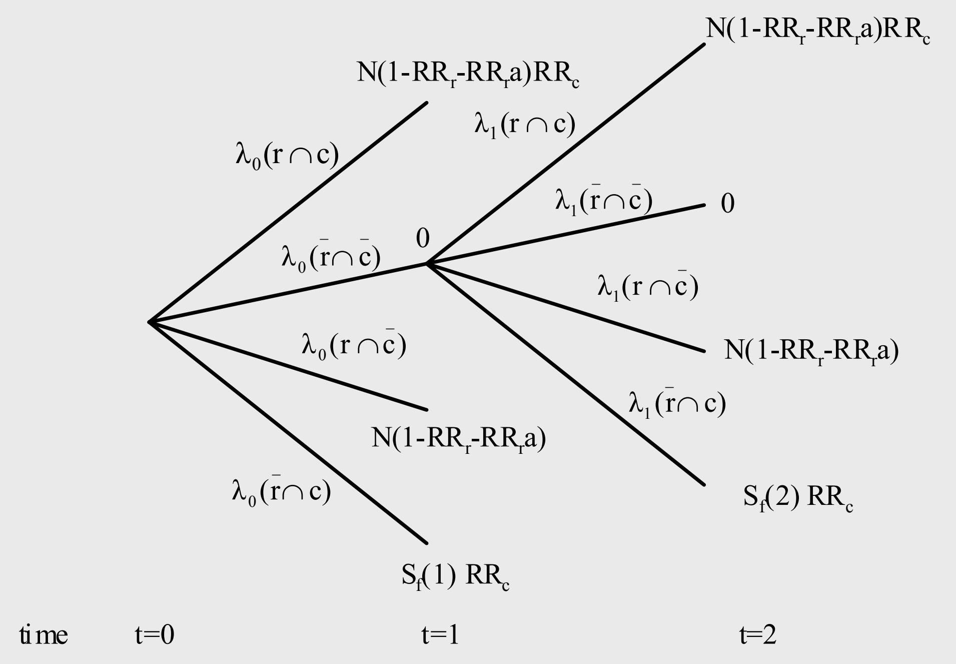
λ(r) : exogenous risk-neutral probability of default of reference entity

λ(c) : exogenous risk-neutral probability of default of the counterparty

N : Notional Amount; RR_r : Recovery Rate of Reference Entity; RR_c : Recovery Rate of Default Swap Seller
S_f : Fair Default Swap Price without Default Possibility of Default Swap Seller

Valuing Default Swaps on Correlated LMM Processes

1a. Payoff Tree



$N(1\text{-}RR_r\text{-}RR_r a)RR_c$

$\lambda_0(r \cap c)$

$N(1\text{-}RR_r\text{-}RR_r a)RR_c$

$\lambda_1(r \cap c)$

$\lambda_1(\bar{r} \cap \bar{c})$ 0

$\lambda_0(\bar{r} \cap \bar{c})$ 0

$\lambda_1(r \cap \bar{c})$

$N(1\text{-}RR_r\text{-}RR_r a)$

$\lambda_0(r \cap \bar{c})$

$\lambda_1(\bar{r} \cap c)$

$N(1\text{-}RR_r\text{-}RR_r a)$

$\lambda_0(\bar{r} \cap c)$

$S_f(2)\ RR_c$

$S_f(1)\ RR_c$

time t=0 t=1 t=2

Transferring this payoff tree into a mathematical equation, we get for time1

$$[\lambda_0(r \cap c)N(1\text{-}RR_r\text{-}RR_r a)RR_c + \lambda_0(\bar{r} \cap \bar{c})\,N\,0 + \lambda_0(r \cap \bar{c})\,N(1\text{-}RR_r\text{-}RR_r a) + \lambda_0(c \cap \bar{r})\ S_f(1)\ RR_c]\ e^{-r_0 \tau_1} +$$

and for time 2

$$\lambda_0(\bar{r} \cap \bar{c})\ [\lambda_1(r \cap c)\,N\,(1\text{-}RR_r\text{-}RR_r a)RR_c + \lambda_1(\bar{r} \cap \bar{c})\,N\,0 + \lambda_1(r \cap \bar{c})\,N\,(1\text{-}RR_r\text{-}RR_r a) + \lambda_1(c \cap \bar{r})\ (S_f(2)\ RR_c)]\ e^{-r_1 \tau_2}$$

Valuing Default Swaps on Correlated LMM Processes

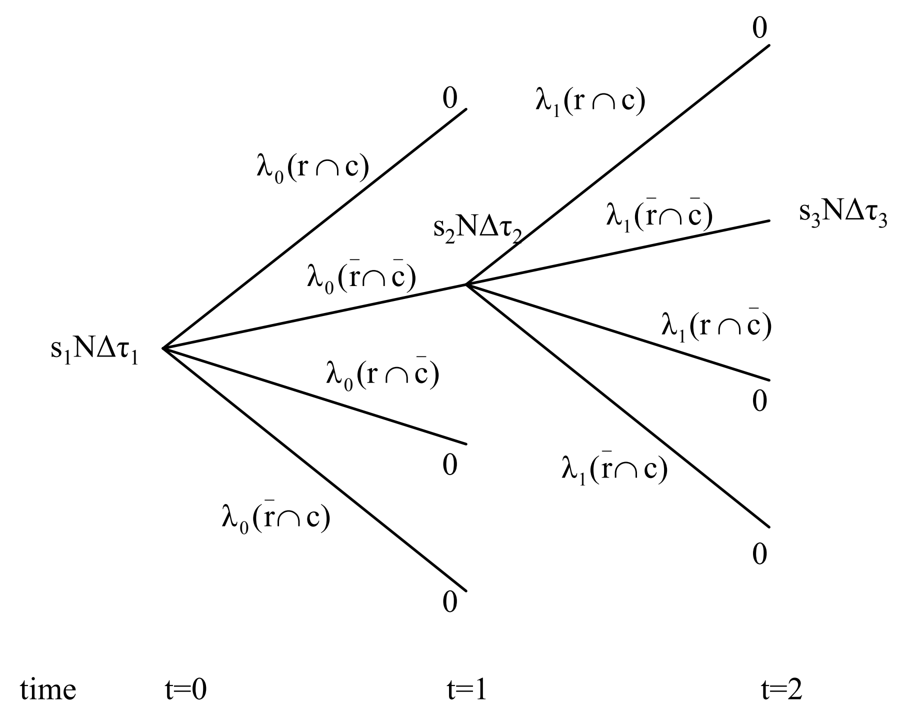
1a. Payoff Tree

Generalizing for T periods, we get the equation for the payoff tree:

$$\sum_{t=1}^{T}\{[\lambda_{t-1}(r\cap c)\,N(1-RR_r-RR_r a)RR_c+\lambda_{t-1}(r\cap\bar{c})\,N(1-RR_r-RR_r a)+ \\ \lambda_{t-1}(c\cap\bar{r})\,S_f(t)RR_c]\,e^{-r_{t-1}\tau_t}\prod_{u=0}^{t-2}\lambda_u(\bar{r}\cap\bar{c})\} \tag{7}$$

Valuing Default Swaps on Correlated LMM Processes

1b1. Premium s Tree, upfront



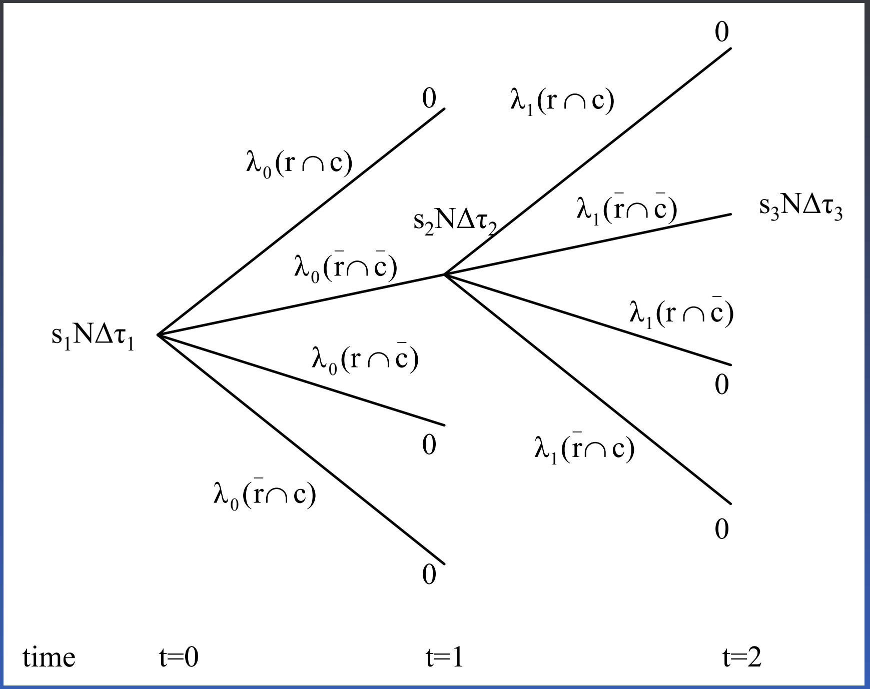
λ(r) : exogenous risk-neutral probability of default of reference entity

λ(c) : exogenous risk-neutral probability of default of the counterparty

s : annual default swap premium; N : Notional Amount; $\Delta\tau_t$: Time between node t-1 and t

Valuing Default Swaps on Correlated LMM Processes

1b1. Premium s Tree, upfront



Transferring this payment tree into a mathematical equation, we get

$$s_1 N \Delta\tau_1 + \lambda_0(\bar{r} \cap \bar{c})\ s_2 N \Delta\tau_2\ e^{-r_0 \tau_1} + \lambda_0(\bar{r} \cap \bar{c})\ \lambda_1(\bar{r} \cap \bar{c})\ s_3 N \Delta\tau_3\ e^{-r_1 \tau_2} + \ldots$$

Valuing Default Swaps on Correlated LMM Processes

1b1. Premium s Tree, upfront

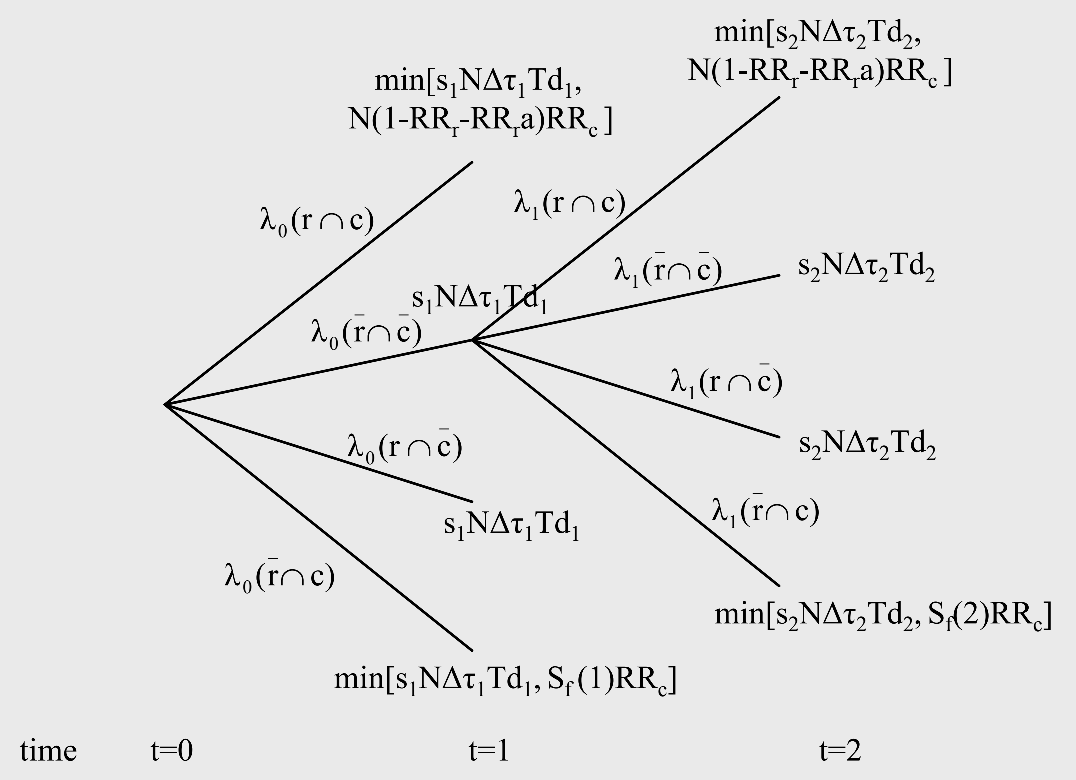
$$s_1 N \Delta\tau_1 + \lambda_0(\bar{r} \cap \bar{c})\; s_2 N \Delta\tau_2\; e^{-r_0 \tau_1} + \lambda_0(\bar{r} \cap \bar{c})\; \lambda_1(\bar{r} \cap \bar{c})\; s_3 N \Delta\tau_3\; e^{-r_1 \tau_2} + \dots$$

Generalizing for T periods, we get the equation for the upfront payment tree:

$$s_1 N \Delta \tau_1 + \sum_{t=1}^{T} s_{t+1}\, N\, \Delta\tau_{t+1}\, e^{-r_{t-1}\tau_t} \prod_{u=0}^{t-1} \lambda_u(\bar{r} \cap \bar{c}) \tag{8}$$

Valuing Default Swaps on Correlated LMM Processes

1b2. Premium s Tree, in arrears



$\min[s_2 N\Delta\tau_2 Td_2,$
$N(1\text{-}RR_r\text{-}RR_r a)RR_c\,]$

$\min[s_1 N\Delta\tau_1 Td_1,$
$N(1\text{-}RR_r\text{-}RR_r a)RR_c\,]$

$\lambda_1(r \cap c)$

$\lambda_0(r \cap c)$

$\lambda_1(\bar{r} \cap \bar{c})$

$s_2 N\Delta\tau_2 Td_2$

$s_1 N\Delta\tau_1 Td_1$

$\lambda_0(\bar{r} \cap \bar{c})$

$\lambda_1(r \cap \bar{c})$

$s_2 N\Delta\tau_2 Td_2$

$\lambda_0(r \cap \bar{c})$

$s_1 N\Delta\tau_1 Td_1$

$\lambda_1(\bar{r} \cap c)$

$\lambda_0(\bar{r} \cap c)$

$\min[s_2 N\Delta\tau_2 Td_2, S_f(2)RR_c]$

$\min[s_1 N\Delta\tau_1 Td_1, S_f(1)RR_c]$

time t=0 t=1 t=2

λ(r) : exogenous risk-neutral probability of default of reference entity

λ(c) : exogenous risk-neutral probability of default of the counterparty

N : Notional Amount; RR_r : Recovery Rate of Reference Entity; RR_c : Recovery Rate of Default Swap Seller
S_f : Fair Default Swap Price without Default Possibility of Default Swap Seller; Td: Time of default

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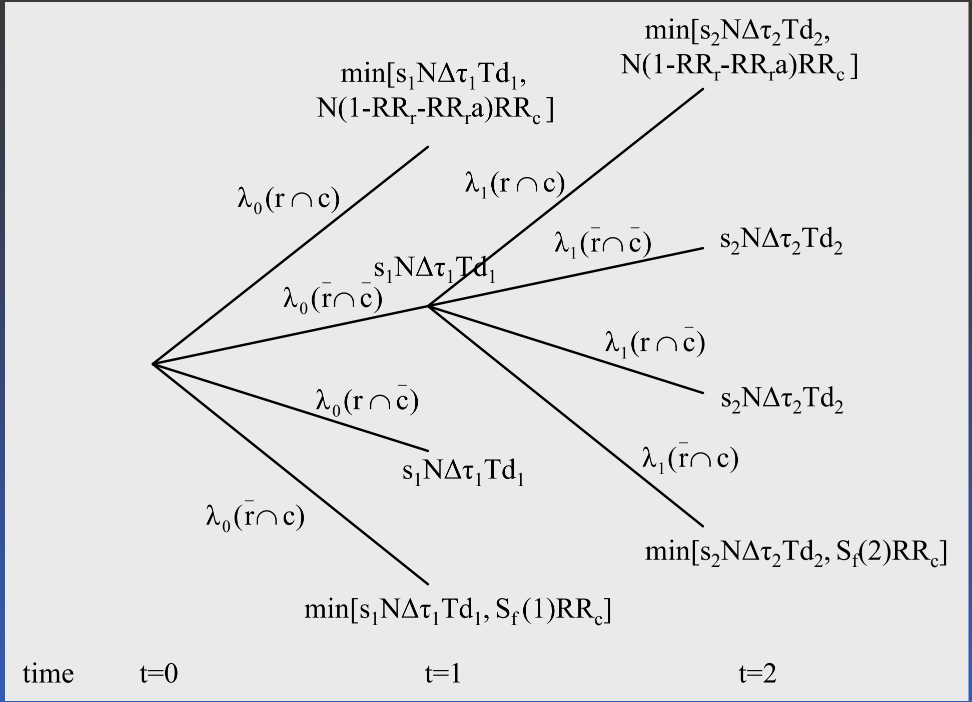
1b2. Premium s Tree, in arrears



Transferring this payoff tree into a mathematical equation, we get for time1

$$\{\lambda_0(r \cap c)\ \min[sN\Delta\tau_1 Td_1, N(1\text{-}RR_r\text{-}RR_r a)RR_c] + \lambda_0(\bar{r} \cap \bar{c})\ s_1 N\,\Delta\tau_1\, Td_1 + \lambda_0(r \cap \bar{c})\ s_1 N\,\Delta\tau_1\, Td_1 + \lambda_0(c \cap \bar{r})\ \min[sN\Delta\tau_1 Td_1,\ S_f(1)\,RR_c]\}\ e^{-r_0\tau_1} +$$

and for time 2

$$\lambda_0(\bar{r} \cap \bar{c})\{\lambda_1(r \cap c)\min[s_2 N\,\Delta\tau_2\, Td_2, N(1\text{-}RR_r\text{-}RR_r a)RR_c] + \lambda_1(\bar{r} \cap \bar{c})\ s_2 N\,\Delta\tau_2\, Td_2 + \lambda_1(r \cap \bar{c})\ s_2 N\,\Delta\tau_2\, Td_2 + \lambda_0(c \cap \bar{r})\min[s_2 N\,\Delta\tau_2\, Td_2,\ S_f(2)\,RR_c]\}\ e^{-r_1\tau_2}$$

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1b2. Premium s Tree, in arrears

Generalizing for T periods, we get the equation for the premium tree:

$$\sum_{t=1}^{T} \{[\,\lambda_{t-1}(r \cap c)\ \min[x_t] + \lambda_{t-1}(\bar{r} \cap \bar{c})\, s\, N\, \Delta\tau_t\, Td_t + \lambda_{t-1}(r \cap \bar{c})\, s\, N\, \Delta\tau_t Td_t + \lambda_{t-1}(c \cap \bar{r})\, \min[y_t]\,]e^{-r_{t-1}\tau_t} \prod_{u=0}^{t-2} \lambda_u(\bar{r} \cap \bar{c})\} \tag{9}$$

where $\min[sN\Delta\tau_{t-1}, N(1\text{-}RR_r\text{-}RR_r a)RR_c] \equiv \min[x_t]$ and
$\min[sN\Delta\tau_{t-1}, S_f(t)RR_c] \equiv \min[y_t]$,

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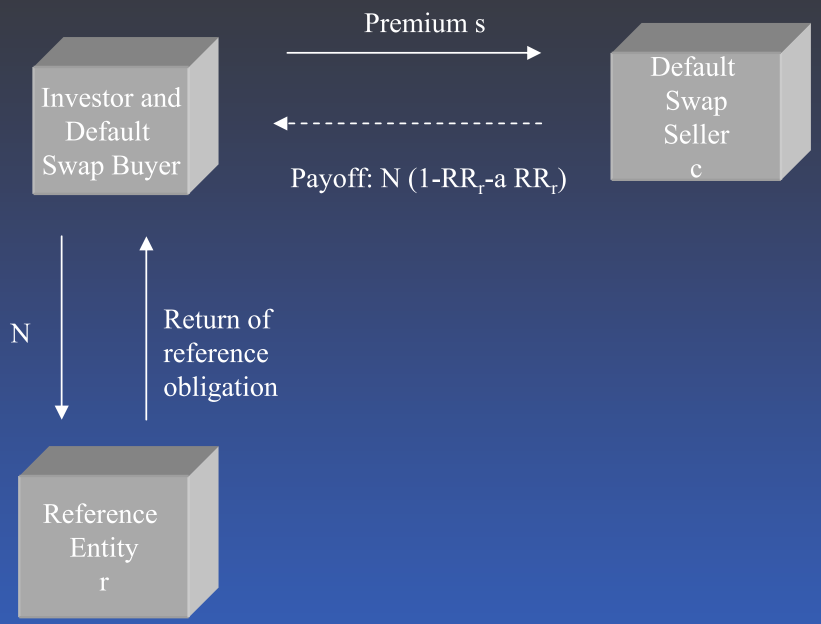
1c. We can now use swap evaluation techniques to find s



Premium s

Investor and Default Swap Buyer

Default Swap Seller c

Payoff: $N\,(1-RR_r-a\,RR_r)$

N

Return of reference obligation

Reference Entity r

From the investor’s viewpoint the swap is worth

PV (Sum of all payoffs) – PV (Sum of all premiums), hence

Eq. (7) – Eq. (8) for upfront premium and

Eq. (7) – Eq. (9) for in arrears premium

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Eq. (7) – Eq. (8) for up front premium is

$$\sum_{t=1}^{T}\{[\lambda_{t-1}(r\cap c)\,N(1-RR_r-RR_r a)RR_c+\lambda_{t-1}(r\cap\bar{c})\,N(1-RR_r-RR_r a)+\lambda_{t-1}(c\cap\bar{r})\,S_f(t)RR_c]\,e^{-r_{t-1}\tau_t}\prod_{u=0}^{t-2}\lambda_u(\bar{r}\cap\bar{c})\}$$

$$-$$

$$s_1 N\Delta\tau_1+\sum_{t=1}^{T}s_{t+1}\,N\,\Delta\tau_{t+1}\,e^{-r_{t-1}\tau_t}\prod_{u=0}^{t-1}\lambda_u(\bar{r}\cap\bar{c})$$

Setting this to 0, and solving for s, we get

$$s=\frac{\sum_{t=1}^{T}\{[\lambda_{t-1}(r\cap c)\,N(1-RR_r-RR_r a)RR_c+\lambda_{t-1}(r\cap\bar{c})\,N(1-RR_r-RR_r a)+\lambda_{t-1}(c\cap\bar{r})\,S_f(t)RR_c]\,e^{-r_{t-1}\tau_t}\prod_{u=0}^{t-2}\lambda_u(\bar{r}\cap\bar{c})\}}{N\Delta\tau_t+\sum_{t=1}^{T}N\,\Delta\tau_{t+1}\,e^{-r_{t-1}\tau_t}\prod_{u=0}^{t-1}\lambda_u(\bar{r}\cap\bar{c})}\tag{11}$$

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Eq. (7) – Eq. (9) for in arrears premium is

$$\begin{aligned}&\sum_{t=1}^{T}\{[\lambda_{t-1}(r\cap c)\,N(1-RR_r-RR_r a)RR_c+\lambda_{t-1}(r\cap\bar{c})\,N(1-RR_r-RR_r a)+\\&\lambda_{t-1}(c\cap\bar{r})\,S_f(t)RR_c]e^{-r_{t-1}\tau_t}\prod_{u=0}^{t-2}\lambda_u(\bar{r}\cap\bar{c})\}\end{aligned}\quad-$$

$$\begin{aligned}&\sum_{t=1}^{T}\{[\lambda_{t-1}(r\cap c)\ \min[x_t]+\lambda_{t-1}(\bar{r}\cap\bar{c})\,s\,N\,\Delta\tau_t\ Td_t+\lambda_{t-1}(r\cap\bar{c})\,s\,N\,\Delta\tau_t Td_t+\\&\lambda_{t-1}(c\cap\bar{r})\min[y_t]]e^{-r_{t-1}\tau_t}\prod_{u=0}^{t-2}\lambda_u(\bar{r}\cap\bar{c})\}\end{aligned}$$

Setting this to 0, and solving for s, we get

$$s=\frac{\begin{aligned}&\sum_{t=1}^{T}\{[\lambda_{t-1}(r\cap c)\,N(1-RR_r-RR_r a)RR_c+\lambda_{t-1}(r\cap\bar{c})\,N(1-RR_r-RR_r a)+\\&\lambda_{t-1}(c\cap\bar{r})\,S_f(t)RR_c\ ]e^{-r_{t-1}\tau_t}\prod_{u=0}^{t-2}\lambda_u(\bar{r}\cap\bar{c})\}\end{aligned}}{\begin{aligned}&\sum_{t=1}^{T}\{[\lambda_{t-1}(r\cap c)\ \min[x_t]/s+\lambda_{t-1}(\bar{r}\cap\bar{c})\,N\,\Delta\tau_t\ Td_t+\lambda_{t-1}(r\cap\bar{c})\,N\,\Delta\tau_t Td_t+\\&\lambda_{t-1}(c\cap\bar{r})\min[y_t]/s\,]e^{-r_{t-1}\tau_t}\prod_{u=0}^{t-2}\lambda_u(\bar{r}\cap\bar{c})\}\end{aligned}}\qquad(13)$$

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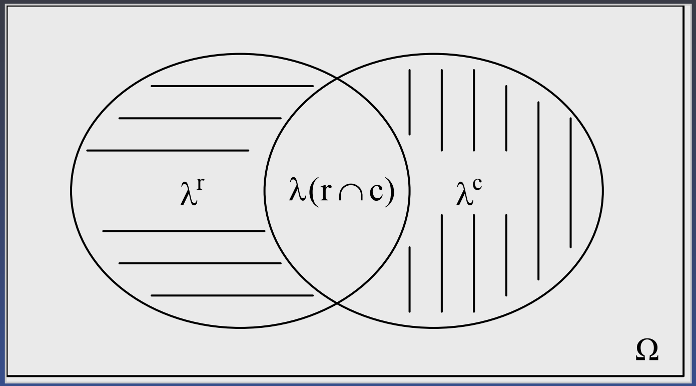
2. Including the Correlation



From this figure, we get

$$\lambda(c \cup r) = \lambda^{r} + \lambda^{c} - \lambda(r \cap c) \quad \text{where}$$

$$\lambda(c \cap r) = \rho(\lambda^{r}, \lambda^{c})\sqrt{[\lambda^{r} - (\lambda^{r})^{2}][\lambda^{c} - (\lambda^{c})^{2}]} + \lambda^{r}\lambda^{c}$$

$$\lambda(c \cap \bar{r}) = \lambda^{c} - \lambda(r \cap c)$$

$$\lambda(r \cap \bar{c}) = \lambda^{r} - \lambda(r \cap c) \quad \text{only this term remains for } \lambda^{c} = 0$$

$$\lambda(\bar{r} \cap \bar{c}) = 1 - [\lambda^{r} + \lambda^{c} - \lambda(r \cap c)]$$

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$$s = \frac{\sum_{t=1}^{T}\{[\lambda_{t-1}(r\cap c)\,N(1-RR_r-RR_r a)RR_c+\lambda_{t-1}(r\cap\bar{c})\,N(1-RR_r-RR_r a)+ \lambda_{t-1}(c\cap\bar{r})\,S_f(t)RR_c\,]e^{-r_{t-1}\tau_t}\prod_{u=0}^{t-2}\lambda_u(\bar{r}\cap\bar{c})\}}{\sum_{t=1}^{T}\{[\lambda_{t-1}(r\cap c)\;\min[x_t]/s+\lambda_{t-1}(\bar{r}\cap\bar{c})\,N\,\Delta\tau_{t-1}+\lambda_{t-1}(r\cap\bar{c})\,N\,\Delta\tau_{t-1} \lambda_{t-1}(c\cap\bar{r})\min[y_t]/s\,]e^{-r_{t-1}\tau_t}\prod_{u=0}^{t-2}\lambda_u(\bar{r}\cap\bar{c})\}} \qquad (13)$$

Conclusion:

1) To exclude counterparty default, we use equation (13) with $\lambda^c = 0$

(In equation (13) all terms except $\lambda_{t-1}(r\cap\bar{c})\,N(1-RR_r-RR_r a)$ and $\lambda_{t-1}(r\cap\bar{c})\,N\,\Delta\tau_{t-1}$ are 0).

2) To include counterparty default λ^c which is not correlated to reference asset default λ^r we use equation (13) with

$$\lambda(c\cap r) = \rho(\lambda^r,\lambda^c)\sqrt{[\lambda^r-(\lambda^r)^2][\lambda^c-(\lambda^c)^2]}+\lambda^r\lambda^c \text{ with } \rho(\lambda^r,\lambda^c)=0$$

3) To include the default correlation between r and c, we use equation (13) in combination $\rho(\lambda^r,\lambda^c)\neq 0$

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3. The Model in Combination with 3 LMM Processes

The Model uses 3 LMM (Libor Market Model) Processes. One for the interest rate process, one the reference asset default process and one for the counterparty default process.

Hull and White (2000) show that a one-factor Libor Market Model can be expressed as

$$F_k(t_{j+1}) = F_k(t_j)\exp\left[\left(\sum_{i=j+1}^{k}\frac{\Delta\tau_i F_i(t_j)\Lambda_{i-j-1}\Lambda_{k-j-1}}{1+\Delta\tau_i F_i(t_j)} - \frac{\Lambda_{k-j-1}^2}{2}\right)\Delta\tau_j + \Lambda_{k-j-1}\,\varepsilon\sqrt{\Delta\tau_j}\right]$$

$F_k(t)$: Forward interest rate and forward default probabilities of reference asset and counterparty between time k and k+1, seen at time t, with compounding of $\Delta\tau_i$

$\Delta\tau_i$: time between horizontal nodes i and i+1, expressed in years

ε : random drawing from standard normal distribution

Λ_k : forward rate volatility term for time t_k to t_{k+1}. Assuming $\Delta\tau$ is constant, Λ can be derived iteratively using

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$$\Lambda_{k-1} = \sqrt{k\,\sigma_k^2 - \sum_{v=0}^{k-2} \Lambda_v^2}$$

where σ_k is caplet's volatility for interest rates and the forward default volatility counterparty and reference asset between time t_k and t_{k+1}.

The programmed model has nine steps, which can be summed up in the following table

Default Scenario	Payoff	Premium Payment	Paid at	How to Proceed
$\lambda(\bar{r} \cap \bar{c})$	0	$s\ N\ \Delta\tau_t$	t_{k+1}	Continue to next time step (if not maturity)
$\lambda(r \cap \bar{c})$	$N(1-RR_r-RR_r c\ Td_t)$	$s\ N\ \Delta\tau_t\ Td_t$	t_{k+1}	Stop trial
$\lambda(c \cap \bar{r})$	$S_f(Td_t)\ RR_c$	$\mathrm{Min}(s\ N\ \Delta\tau_t\ Td_t,$ $S_f(T_d)\ RR_c)$	t_{k+1}	Stop trial
$\lambda(c \cap r)$	$N(1-RR_r-RR_r c\ Td_t)$ RR_c	$\mathrm{Min}[s\ N\ \Delta\tau_t\ Td_t,$ $N(1-RR_r-RR_r c T_d)\ RR_c]$	t_{k+1}	Stop trial

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4. Properties of the Model

a)

$$\partial s/\partial\lambda^{r} \geq 0$$

If $\lambda^{r} = 0$ (and $RR_{r} = 0) \rightarrow s =$ 0

If $\lambda^{r} = 1$ (and $RR_{r} = 0) \rightarrow s =$ 100% of notional if premium payment is upfront

For in arrears premium payment, s can be get > 100% of notional !!!

b)

$$\partial s/\partial\lambda^{c} \leq 0$$

If $\lambda^{c} = 1$ (and $RR_{c} = 0) \rightarrow s =$ 0

c)

$$\partial s/\partial\rho(\lambda^{r},\lambda^{c}) \leq 0$$

If $\rho(\lambda^{r},\lambda^{c}) = 1$ and RR_r and RR_c = 0, if follows that s = 0

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4. Properties of the Model

d) $$\partial s/\partial RR_r \leq 0$$

Expected result, because RR_r is deducted from the payoff $N(1-RR_r-aRR_r)$.
If $RR_r = 1$, it follows $s = 0$

e) $$\partial s/\partial RR_c \geq 0$$

Expected result, since the higher the recovery rate
of the counterparty (insurance seller), the more valuable the default swap

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5. Limitations of the Model

- As in most reduced form models, the direct economic reasons for default, i.e. the company's specific asset-liability structure or the company's liquidity are not part of the analysis.

- The recovery rates of the reference asset and the counterparty of the model do not depend on the model variables, but are exogenous and assumed constant over the life of the default swap.

- The model does not incorporate the default risk of the default swap buyer in case of periodic default swap premium (see Meissner, G., "Pricing Default Swaps – Which Default Probabilities, Which Default Correlations should be Included?" Hawaii Pacific University Working Paper)

- Also, the model awaits calibration of the input variables, especially the forward default probabilities and forward default volatilities of the reference asset and the counterparty.

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6. Conclusion/Summary

- The models derives a closed form solution for the default swap premium including reference asset – counterparty default correlation

- Since it is a Reduced Form Model, it has the typical Reduced Form Model limitations (no company specific data as asset-liability structure to explain default)

- The model can be expanded in many ways
  - Incorporate default swap buyer default probability
  - Use a more elaborate correlation concept as the Copula approach
  - Endogenize recovery rate of reference asset and counterparty
  - Integrate Market risks and Operational Risks

- The model awaits calibration