Valuing credit default swaps with counterparty risk – A combined copula-LMM approach

Michael Hamp, Janne Kettunen, Gunter Meissner[1]

Abstract

The paper derives a model with a closed form solution for valuing credit default swaps including reference asset – counterparty default correlation. The default correlation between the reference asset and the counterparty is incorporated in two quadruple trees. One tree represents the default swap payoff of the default swap seller; the other tree represents the default swap premium payments of the default swap buyer. Swap valuation techniques are then applied to derive the fair default swap price.

The model incorporates two correlation approaches used in today's credit practice, the Gaussian copula approach and the discrete correlation approach. The Gaussian copula results in a higher credit default swap premium than the discrete approach, since it produces lower joint default probabilities.

The model is represented with three LMM (Libor Market Model) processes. One LMM process simulates risk-free short-term interest rates. Two more LMM processes generate the reference asset default probabilities and the counterparty default probabilities. A Visual Basic open source code version of the model is provided.

Keywords: Default swap pricing, copula, reference asset – counterparty default correlation, Libor Market Model (LMM)

JEL Classification: G12, G13

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[1] Michael Hamp is a Risk Manager with Scotia Bank in Toronto Canada. Janne Kettunen is a graduate student at Hawaii Pacific University (www.hpu.edu). Gunter Meissner is Associate Professor of Finance at Hawaii Pacific University and President of Derivatives Software (www.dersoft.com). Please send questions and comments to gmeissne@aol.com, Tel: USA (808) 524 9311. The authors would like to thank Ranjan Bhaduri for helpful comments and suggestions.

Introduction

The credit derivatives market has grown exponentially since its inception in the mid 1990s. Default swaps currently (year 2005) comprise of roughly 2/3 of the credit derivatives market.

So far no pricing model has achieved dominance in the credit derivative market. Two main approaches exist: Structural models, closely related to the original Merton (1974) model, derive the default swap price by relating a firm's asset to the firm's debt. Merton's restriction that default can principally only occurs at maturity of the debt was relaxed by first-time passage models of Black and Cox (1976), Kim, Ramaswamy, and Sundaresan (1993), Longstaff and Schwarz (1995), and Briys and de Varenne (1997).

Reduced form models or intensity models, pioneered by Jarrow and Turnbull (1995), as structural models, are based on the risk-neutral Merton (1974) framework. In reduced form models, the default is modeled by a stochastic process with an exogenous default intensity or hazard rate h, which multiplied by a time frame, results in the risk-neutral default probability also called pseudo- or martingale default probability. The value of h is derived by calibration of the variables of the stochastic process. Since reduced form models only model the timing of the default not the severity, the recovery rate is usually exogenous. Duffee (1996), Jarrow, Lando and Turnbull (1997), Lando (1998), Schoenbucher (1998), Duffie and Singleton (1999), Das and Sundaram (2000), Hull and White (2000(a) and 2001), and Jarrow and Yildirim (2002) are main contributors to enhancing reduced form modeling. For a survey article comparing the default swap evaluation models of the Jarrow and Turnbull (1995), Brooks and Yan (1998), Duffie and Singleton (1999), Das and Sundaram (2000), and Hull and White (2000(a)), see Cheng (2001).

The model presented here is a reduced form model based on two quadruple trees. The default swap price is derived via discrete swap valuation techniques. The model allows for different netting scenarios in case of default. The model is closest related to Hull-White (2001), who employ continuous swap valuation techniques.

A default swap valuation model should include a variety of input variables. Table 1 gives an overview:

Input variables for deriving the premium s of a default swap
1) Default probability and credit deterioration probability of the reference asset
2) Default probability and credit deterioration probability of the default swap seller
3) Correlation between 1) and 2)
4) Default volatility of the underlying reference asset
5) Default volatility of the default swap seller
6) Correlation between 4) and 5)
7) Maturity of the default swap
8) Expected recovery rate of the reference asset
9) Expected recovery rate of the default swap seller
10) Return of the reference asset (e.g. coupon of the reference bond)
11) Risk-free interest rate term structure used to discount future cash flows
12) Default probability of the default swap buyer in case of periodic credit derivative premium[2]
13) Expected recovery rate of the default swap buyer in case of periodic credit derivative premium
14) Correlation between the default probability of the default swap buyer and the reference asset in case of periodic default swap premium
15) Market risks (as interest rate risk, currency risk, commodity risk and stock price risk) and the correlation between market risk and credit risk
16) Operational risks (e.g. legal risks, documentation risks or settlement risks), which might endanger the enforceability of the payoff and the correlation between operational risk and credit risk
17) Liquidity of the default swap
18) Liquidity of the underlying reference asset
19) BIS risk weight of the default swap seller
20) Urgency of protection (e.g. is an immediate credit deterioration expected or does the protection free up credit lines to enable further business with a client)
21) Transaction costs

Table 1: Input variables for deriving a default swap price

Arguably, the input variables in table 1 are in order of significance with respect to deriving a rigorous default swap price. The model in this paper includes the input variables 1 to 11 of table 1.

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[2] See Meissner, G., "Pricing Default Swaps – Which Default Probabilities, Which Default Correlations Should Be Included?" *Hawaii University Working Paper*

The article is organized as follows. Section 1 provides the definitions of the variables. Section 2 outlines the correlation concept. Section 3 presents the model, where 3a presents default swap payoff tree, 3b presents the default swap premium payment tree, and 3c combines the two trees. Section 4 outlines the model in combination with the Libor Market Model. Section 5 presents the properties and limitations of the model. Section 6 concludes.

1. Definitions

Most reduced form models as Jarrow and Turnbull (1995), Lando (1998), and Duffie and Singleton (1999), model the bankruptcy process with a risk-neutral hazard rate h_t, also called default intensity. The hazard rate, multiplied by a certain time period, $\Delta\tau_t$, $h_t\Delta\tau_t$, gives the default probability for period $\Delta\tau_t$, which starts at t and ends at t+1; h_t is viewed at time t. We will slightly simplify this notation and define $\lambda_t^r(\Delta\tau_t)$ as the default probability of the reference asset for time period $\Delta\tau_t$, which starts at t-1 and ends at t. For ease of notation we will drop $\Delta\tau_t$, hence $\lambda_t^r(\Delta\tau_t) \equiv \lambda_t^r$. As in Hull and White (2000(a) and 2001), our λ_t^r is viewed at time 0, not time t.

Hence we define:

λ_t^r : exogenous, risk-neutral probability of default of reference entity r, under the equivalent martingale measure Q, during time t to t+1, which is expressed in years as $\Delta\tau_t$, viewed at time 0, given no earlier default of the reference entity r

$\Delta\tau_t$: time between nodes t-1 and t, expressed in years

also,

s_t : annual default swap premium to be paid at time t

τ_t : time between time 0 and time t, expressed in years

Td_t : Time of default, measured between time t-1 and default time, expressed in years

N : notional amount of the swap

r_t : risk-free interest rate from time 0 to time t+1

RR_r : exogenous recovery rate of the reference entity

a : accrued interest on the reference obligation from the last coupon date until the default date.

Harrison and Kreps (1979), Harrison and Pliska (1981) and Jarrow and Turnbull (1995) have shown that the existence of risk-neutral probabilities (also termed martingale probabilities or pseudo-probabilities) implies that the market is arbitrage free.

Counterparty default risk is the risk that a counterparty does not honor its obligation. The default swap buyer has counterparty default risk, since the default swap seller has an obligation to the default swap buyer in case of default. Following Hull White (2001), we assume that this obligation is $N(1\text{-}RR_r\text{-}RR_r a)$, where N : notional amount of the swap, RR_r : recovery rate of the reference asset issuer for the reference bond; a : accrued interest of the reference bond from the last coupon date until default. We will refer to the risk of default of the default swap seller as counterparty default risk.[3]

In analogy of the default probability of the reference asset r, λ_t^r, we define the default probability of the counterparty λ_t^c as

λ_t^c : exogenous, risk-neutral probability of default of counterparty c, under the equivalent martingale measure Q, during time t to t+1, which is expressed in years as $\Delta\tau_t$, viewed at time 0, given no earlier default of the counterparty c

RR_c : exogenous recovery rate of the counterparty

$S_f(t)$: Fair value of the default swap at time t excluding counterparty risk (i.e. the swap value (including the notional amount) that gives the default swap a present value of zero, ignoring counterparty default risk). The risk-neutral default probabilities λ_t^r and λ_t^c are derived by calibrating the model.

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[3] In case of a periodic swap premium, the default swap seller also has counterparty default risk, since in case of an increase in the credit quality of the reference asset, the default swap premium decreases (ceteris paribus) and consequently the default swap has a positive present value for the default swap seller. In this article we are ignoring the counterparty default risk of the default swap seller.

2. The Correlation

The default correlation between the reference asset and the counterparty (i.e. the default swap seller) is crucial for deriving a default swap price. Only if *both* the reference asset and counterparty default, will the default swap buyer incur a huge loss.[4] If both default, the default swap buyer will lose his entire investment in the reference asset and the present value of the default swap (assuming that the recovery rates of the reference asset and counterparty are 0).

We will use two approaches to model the default correlation of the reference asset and the counterparty and then compare the results.

1) An approach referred to as discrete or linear default correlation, which is often applied by rating agencies

2) A Gaussian copula approach

1) The discrete default correlation approach

The joint probability of two entities defaulting, $\lambda(r \cap c)$, is simply the multiplication of the individual default probabilities, if the companies' default is not correlated. Hence $\lambda(r \cap c) = \lambda^r \times \lambda^c$. If the default probabilities of the reference entity r and the counterparty c are correlated, in rating practice often equation (1) is applied

$$\lambda(r \cap c) = \rho(\lambda^r, \lambda^c)\sqrt{[\lambda^r - (\lambda^r)^2][\lambda^c - (\lambda^c)^2]} + [\lambda^r \lambda^c] \qquad (1)$$

where $-1 \le \rho(\lambda^r, \lambda^c) \le 1$ is the correlation coefficient, which may be derived from equity correlation. As mentioned above, the individual default probabilities λ^r and λ^c are derived by calibrating the model. From equation (1) we can see that for a default correlation $\rho(\lambda^r, \lambda^c)$ of zero, the joint default probability $\lambda(r \cap c)$ is indeed the

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[4] The default of the counterparty (i.e. the default swap seller) is actually more harmful for the default swap buyer, since in this case he looses his default swap. In case the reference asset defaults, he will be compensated by the counterparty.

product of the individual default probabilities of λ^r and λ^c as stated above, $\lambda(r \cap c) = \lambda^r \text{ x } \lambda^c$.

In equation (1) a rather simple correlation measure is applied. The correlation between

a) a variable that takes the value 1 if the reference asset r defaults and 0 otherwise and
b) a variable that takes the value 1 if the counterparty c defaults and 0 otherwise,

is modeled. In equation (1) it is assumed that the correlation $\rho(\lambda^r, \lambda^c)$ is constant until default swap maturity.

The probability of neither the reference asset nor the counterparty defaulting, i.e. $\lambda(\bar{r} \cap \bar{c})$, can be easily derived from the addition law of basic probability theory, equation (2).

$$\lambda\,(r \cup c) = \lambda^r + \lambda^c - \lambda\,(r \cap c) \tag{2}$$

Equation (2) can be easily verified from figure 1.

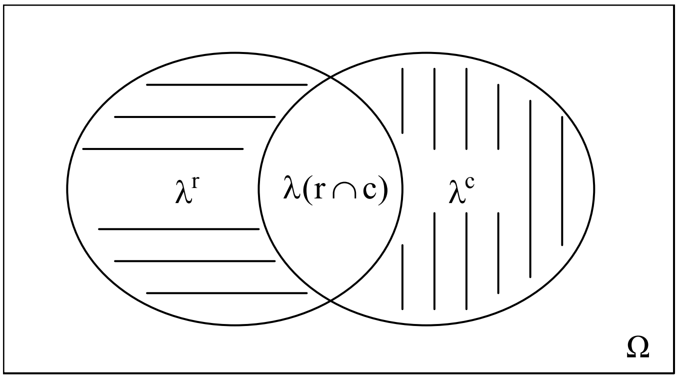

Figure 1: Default probability of reference entity r, λ^r, default probability of counterparty c, λ^c, and the joint default probability $\lambda\,(r \cap c)$.

From equation (2) it follows

$$\lambda\,(\bar{r} \cap \bar{c}) = 1 - \lambda\,(r \cup c) = 1 - [\lambda^r + \lambda^c - \lambda\,(r \cap c)] \tag{3}$$

From equation (2) we can also derive the probability of the reference entity r defaulting and not the counterparty c, $\lambda(r \cap \bar{c})$, (see figure 1 horizontally shaped area) as

$$\lambda(r \cap \bar{c}) = \lambda^{r} - \lambda(r \cap c) \tag{4}$$

Naturally, the probability of the counterparty c defaulting and not the reference entity r, $\lambda(c \cap \bar{r})$, (see vertically shaped area in figure 1), is

$$\lambda(c \cap \bar{r}) = \lambda^{c} - \lambda(r \cap c) \tag{5}$$

2) The Gaussian copula approach

A more complex correlation measure that has gained market popularity in the recent past is the copula model. Here a joint distribution function is formed from marginal functions, incorporating the dependence structure. For a good introduction to copulas see Romano (2000) and Li (2000). We will use the popular Gaussian copula to model the default correlation between the reference asset and the counterparty (i.e. default swap seller). The bivariate Gaussian copula is defined as

$$M\,[N^{-1}(\lambda^{r}), N^{-1}(\lambda^{c});\ \bar{\rho}(\lambda^{r},\lambda^{c})] \qquad 0 \le \bar{\rho}(\lambda^{r},\lambda^{c}) \le 1 \tag{6}$$

where M is a bivariate normal distribution function, representing the joint probability of default, N^{-1} is the inverse of a univariate normal distribution, representing the marginal distributions and $\bar{\rho}$ is the correlation parameter, which is typically bigger than ρ in equation (1).

Combining equations (1) and (6), we derive for the joint default probability M using a Gaussian bivariate copula approach

$$M\,[N^{-1}(\lambda^r), N^{-1}(\lambda^c);\ \bar{\rho}(\lambda^r,\lambda^c)] = \bar{\rho}(\lambda^r,\lambda^c)\,\sqrt{[\lambda^r-(\lambda^r)^2][\lambda^c-(\lambda^c)^2]} + [\lambda^r\,\lambda^c] \tag{7}$$

all variables defined as in equation (1) and (6).

3. The Model

3a. The default swap payoff tree

We assume that if both reference entity and the counterparty default, $\lambda(r \cap c)$, the standard payoff in case of default of the reference asset will be reduced by the recovery rate of the counterparty. Hence the payoff will be $N(1\text{-}RR_r\text{-}RR_r a)RR_c$. There will be no payoff if neither the reference entity nor the counterparty default, $\lambda(\bar{r} \cap \bar{c})$. There will be the standard payoff $N(1\text{-}RR_r\text{-}RR_r a)$ if only the reference entity defaults, $\lambda(r \cap \bar{c})$. We assume that if only the counterparty defaults, $\lambda(c \cap \bar{r})$, the counterparty will pay the time t value of the default swap, $S_f(t)$, multiplied by the recovery rate of the counterparty, hence $S_f(t)\ RR_c$. Graphically we derive figure 2:

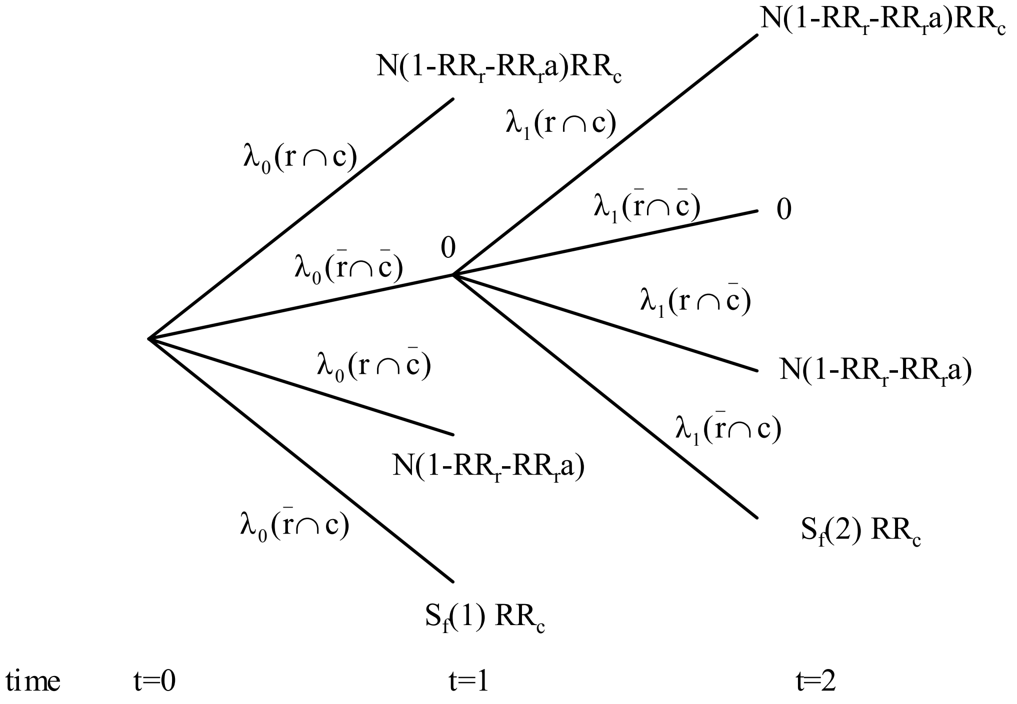

Figure 2: Two-period payoff tree of a default swap including counterparty default risk

Including discount factors, we derive from figure 2 the present value of the payoff of a two-period default swap as

$$[\lambda_0(r \cap c)N(1\text{-}RR_r\text{-}RR_r a)RR_c \;+\; \lambda_0(\bar{r} \cap \bar{c})N\;\;0 \;+\; \lambda_0(r \cap \bar{c})N(1\text{-}RR_r\text{-}RR_r a) \;+\; \lambda_0(c \cap \bar{r})\; S_f(1)\, RR_c]\; e^{-r_0 \tau_1} +$$

$$\lambda_0(\bar{r} \cap \bar{c})\;[\lambda_1(r \cap c)N\;(1\text{-}RR_r\text{-}RR_r a)RR_c + \lambda_1(\bar{r} \cap \bar{c})N\;0 + \lambda_1(r \cap \bar{c})N\;(1\text{-}RR_r\text{-}RR_r a) + \lambda_1(c \cap \bar{r})\;(S_f(2)\,RR_c)]\; e^{-r_1 \tau_2} \qquad (8)$$

Generalizing equation (8) for T periods, we derive

$$\sum_{t=1}^{T}\{[\lambda_{t-1}(r \cap c)\,N(1 - RR_r - RR_r a)RR_c + \lambda_{t-1}(r \cap \bar{c})\,N(1 - RR_r - RR_r a) + \lambda_{t-1}(c \cap \bar{r})\,S_f(t)RR_c]\,e^{-r_{t-1}\tau_t}\prod_{u=0}^{t-2}\lambda_u(\bar{r} \cap \bar{c})\} \qquad (9)$$

3b1. The default swap premium payment tree; Premium payment upfront

Let's now look at the default swap premium payment tree. We will firstly consider the case of up front premium payment, i.e., the default swap premium is paid at the beginning of each period.

As seen if figure 3, only in case of no default of the reference asset and the counterparty, $\lambda(\bar{r} \cap \bar{c})$ will the premium payment process continue.

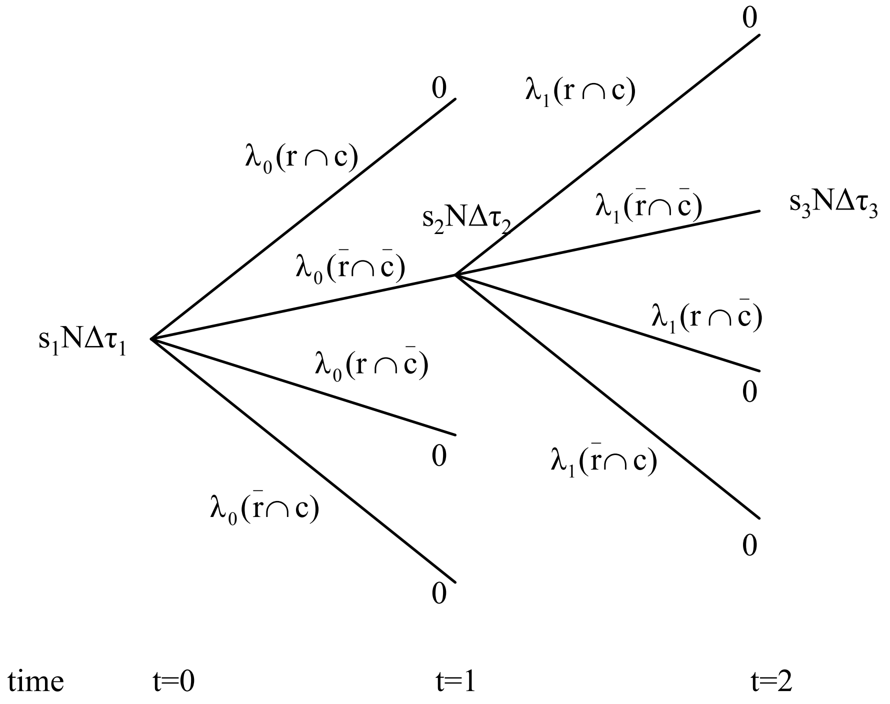

Figure 3: Default swap premium payment tree; Premium paid up front

From figure 3 we get for the premium payment

$$s_1 N \Delta\tau_1 + \lambda_0(\bar{r} \cap \bar{c})\ s_2 N \Delta\tau_2\ e^{-r_0\tau_1} + \lambda_0(\bar{r} \cap \bar{c})\ \lambda_1(\bar{r} \cap \bar{c})\ s_3 N \Delta\tau_3\ e^{-r_1\tau_2} + \ldots$$

Assuming a constant swap premium s, i.e, $s_1 = s_2 = s_3 \ldots$, and generalizing for T periods, we get

$$s_1 N \Delta\tau_0 + \sum_{t=1}^{T} s_{t+1}\ N\ \Delta\tau_{t+1}\ e^{-r_{t-1}\tau_t} \prod_{u=0}^{t-1} \lambda_u(\bar{r} \cap \bar{c}) \qquad (10)$$

3b2. The default swap premium payment tree; Premium payment in arrears

In most default swap contract, the default swap premium is paid in arrears, i.e. at the end of each period. In case of default, the default swap buyer typically has to pay the accrued premium amount from the last premium payment date to the default date. Let’s now discuss this case.

In case of both the reference entity and the counterparty defaulting, $\lambda(r \cap c)$, the final swap premium payment of the default swap buyer depends on the national bankruptcy law and the specific terms of the default swap contract. Principally three scenarios exist

Scenario 1: The default swap buyer makes no final accrual payment and receives the payoff $N(1\text{-}RR_r\text{-}RR_r a)RR_c$.

Scenario 2: The default swap buyer makes a final accrual payment of the minimum of his obligation and the payment of the default swap seller: min [$s\ N\ \Delta\tau\ Td$, $N(1\text{-}RR_r\text{-}RR_r a)RR_c$]. This scenario nets the obligations in case of $s\ N\ \Delta\tau\ Td \geq N(1\text{-}RR_r\text{-}RR_r a)RR_c$ and gives a payoff of $N(1\text{-}RR_r\text{-}RR_r a)RR_c - s\ N\ \Delta\tau\ Td$ in case of $N(1\text{-}RR_r\text{-}RR_r a)RR_c \geq s\ N\ \Delta\tau\ Td$.

Scenario 3: The default swap buyer makes a final accrual payment of $s\ N\ \Delta\tau$. However, this payment may be higher than the reduced, recovery rate dependent final payment of the default swap seller $N(1\text{-}RR_r\text{-}RR_r a)RR_c$

In case of the counterparty defaulting but not the reference entity, $\lambda(c \cap \bar{r})$, the final swap premium payment of the default swap buyer depends again on the national bankruptcy law and the specific terms of the default swap contract. Principally the modified three scenarios are now:

Scenario 1: The default swap buyer makes no final accrual payment and receives the payoff $N(1\text{-}RR_r\text{-}RR_r a)RR_c$.

Scenario 2: The default swap buyer makes a final accrual payment of the minimum of his obligation and the payment of the default swap seller: min [$s\ N\ \Delta\tau\ Td$, $S_f(t)\ RR_c$]. This scenario nets the obligations in case of $s\ N\ \Delta\tau\ Td \geq S_f(t)\ RR_c$ and gives a payoff of $S_f(t)\ RR_c - s\ N\ \Delta\tau\ Td$ in case of $S_f(t)\ RR_c \geq s\ N\ \Delta\tau\ Td$.

Scenario 3: The default swap buyer makes a final accrual payment of $s\ N\ \Delta\tau\ Td$. However, this payment may be higher than the reduced, recovery rate dependent final payment of the default swap seller $S_f(t)\ RR_c$.

Of all scenarios, scenario 2 and 3 reflect the international and American bankruptcy law best. Scenario 1 is based on a "walk away clause", that allows the solvent party to cease payments but receive the recovery rate of the defaulting party.

Most derivatives contracts do not include such a clause. Scenario 2 is based on netting agreements, which are a standard provision of the legal documentation of OTC derivatives, making scenario 2 a realistic choice. However, also scenario 3 can also be considered realistic: Often solvent parties honor their obligation to their defaulting counterparty, even though the defaulting counterparty cannot honor its obligations. This is done for public relations reasons.

In the following analysis, we will apply scenario 2: Hence, we derive the swap premium payment tree as seen in figure 4:

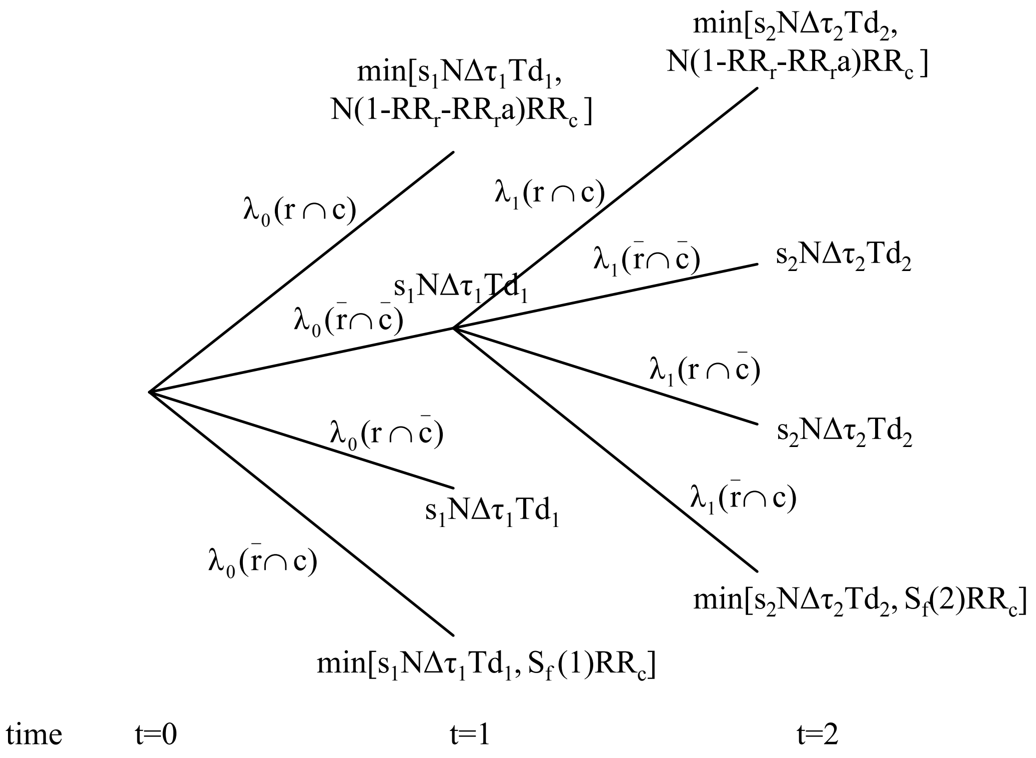

Figure 4: Two-period swap premium tree of a default swap including counterparty default risk

From figure 4 we get for the present value of the swap premium payments

$$\{\lambda_0(r\cap c)\ \min[sN\Delta\tau_1Td_1,\ N(1-RR_r-RR_ra)RR_c] + \lambda_0(\bar{r}\cap\bar{c})\ s_1\ N\ \Delta\tau_1\ Td_1 +$$

$$\lambda_0(r\cap\bar{c})\ s_1 N\ \Delta\tau_1\ Td_1 + \lambda_0(c\cap\bar{r})\ \min[sN\Delta\tau_1 Td_1,\ S_f(1)RR_c]\}\ e^{-r_0\tau_1} +$$

$$\lambda_0(\bar{r}\cap\bar{c})\{\lambda_1(r\cap c)\min[s_2\, N\, \Delta\tau_2\, Td_2,\ N(1\text{-}RR_r\text{-}RR_r a)RR_c] + \lambda_1(\bar{r}\cap\bar{c})\ s_2\, N\, \Delta\tau_2\, Td_2 + \lambda_1(r\cap\bar{c})\ s_2\, N\, \Delta\tau_2\, Td_2 + \lambda_0(c\cap\bar{r})\min[s_2\, N\, \Delta\tau_2\, Td_2,\ S_f(2)\, RR_c]\}\ e^{-r_1\tau_2}$$

Assuming a constant swap premium s, i.e. $s_1 = s_2 = s_3\ldots$, generalizing for T periods and simplifying the notation by using $\min[s\, N\, \Delta\tau_t\, Td_t, N(1\text{-}RR_r\text{-}RR_r a)RR_c] \equiv \min[x_t]$ and $\min[s\, N\, \Delta\tau_t\, Td_t, S_f(t)RR_c] \equiv \min[y_t]$, we derive

$$\sum_{t=1}^{T}\{[\lambda_{t-1}(r\cap c)\ \min[x_t] + \lambda_{t-1}(\bar{r}\cap\bar{c})\, s\, N\, \Delta\tau_t\ Td_t + \lambda_{t-1}(r\cap\bar{c})\, s\, N\, \Delta\tau_t Td_t + \lambda_{t-1}(c\cap\bar{r})\min[y_t]]e^{-r_{t-1}\tau_t}\prod_{u=0}^{t-2}\lambda_u(\bar{r}\cap\bar{c})\} \tag{11}$$

3c. Combining the payoff tree and the premium payment tree

3c1: In case the default swap premium is paid upfront, we derive the value of the default swap from the viewpoint of the default swap buyer by subtracting equation (10) from equation (9):

$$\sum_{t=1}^{T}\{[\lambda_{t-1}(r\cap c)\ N(1-RR_r-RR_r a)RR_c + \lambda_{t-1}(r\cap\bar{c})\ N(1-RR_r-RR_r a) + \lambda_{t-1}(c\cap\bar{r})\,S_f(t)RR_c]\,e^{-r_{t-1}\tau_t}\prod_{u=0}^{t-2}\lambda_u(\bar{r}\cap\bar{c})\}$$
$$-\ s_1\, N\, \Delta\tau_0 + \sum_{t=1}^{T} s_{t+1}\ N\, \Delta\tau_{t+1}\ e^{-r_{t-1}\tau_t}\prod_{u=0}^{t-1}\lambda_u(\bar{r}\cap\bar{c}) \tag{12}$$

Setting equation (10) to zero, assuming identical premium payments, i.e. s1 = s2 = s3… and solving for the fair default swap premium s, which gives the default swap a value of zero, we derive

$$s=\frac{\begin{array}{l}\sum_{t=1}^{T}\{[\lambda_{t-1}(r\cap c)\,N(1-RR_r-RR_r a)RR_c+\lambda_{t-1}(r\cap\bar{c})\,N(1-RR_r-RR_r a)+\\ \lambda_{t-1}(c\cap\bar{r})\,S_f(t)RR_c\,]e^{-r_{t-1}\tau_t}\prod_{u=0}^{t-2}\lambda_u(\bar{r}\cap\bar{c})\}\end{array}}{N\,\Delta\tau_t+\sum_{t=1}^{T}N\,\Delta\tau_{t+1}\,e^{-r_{t-1}\tau_t}\prod_{u=0}^{t-1}\lambda_u(\bar{r}\cap\bar{c})}$$

(13)

3c2: In case the default swap premium is paid in arrears, we derive the value of the default swap from the viewpoint of the default swap buyer by subtracting equation (11) from equation (9):

$$\begin{array}{l}\sum_{t=1}^{T}\{[\lambda_{t-1}(r\cap c)\,N(1-RR_r-RR_r a)RR_c+\lambda_{t-1}(r\cap\bar{c})\,N(1-RR_r-RR_r a)+\\ \lambda_{t-1}(c\cap\bar{r})\,S_f(t)RR_c\,]\,e^{-r_{t-1}\tau_t}\prod_{u=0}^{t-2}\lambda_u(\bar{r}\cap\bar{c})\}\\ -\sum_{t=1}^{T}\{[\,\lambda_{t-1}(r\cap c)\ \min[x_t]+\lambda_{t-1}(\bar{r}\cap\bar{c})\,s\,N\,\Delta\tau_t\ Td_t+\lambda_{t-1}(r\cap\bar{c})\,s\,N\,\Delta\tau_t Td_t+\\ \lambda_{t-1}(c\cap\bar{r})\,\min[y_t]\,]e^{-r_{t-1}\tau_t}\prod_{u=0}^{t-2}\lambda_u(\bar{r}\cap\bar{c})\}\end{array}$$

(14)

Setting equation (14) to zero and solving for the fair default swap premium s, which gives the default swap a value of zero, we derive

$$s=\frac{\begin{array}{l}\sum_{t=1}^{T}\{[\lambda_{t-1}(r\cap c)\,N(1-RR_r-RR_r a)RR_c+\lambda_{t-1}(r\cap\bar{c})\,N(1-RR_r-RR_r a)+\\ \lambda_{t-1}(c\cap\bar{r})\,S_f(t)RR_c\,]e^{-r_{t-1}\tau_t}\prod_{u=0}^{t-2}\lambda_u(\bar{r}\cap\bar{c})\}\end{array}}{\begin{array}{l}\sum_{t=1}^{T}\{[\,\lambda_{t-1}(r\cap c)\ \min[x_t]/s+\lambda_{t-1}(\bar{r}\cap\bar{c})\,N\,\Delta\tau_t\ Td_t+\lambda_{t-1}(r\cap\bar{c})\,N\,\Delta\tau_t Td_t+\\ \lambda_{t-1}(c\cap\bar{r})\,\min[y_t]/s\,]e^{-r_{t-1}\tau_t}\prod_{u=0}^{t-2}\lambda_u(\bar{r}\cap\bar{c})\}\end{array}}$$

(15)

In equation (15) we have assumed that the grid points and time periods in the payoff tree (figure 2) and the premium tree (figure 4) are identical. This is does not have to be the case as shown in the appendix.

Equation (15) with $RR_c = 0$ and $\lambda^c > 0$ results in a default swap premium that is lower than the default swap premium without counterparty risk ($\lambda^c = 0$), which satisfies the no-arbitrage condition with respect to counterparty default risk. A high recovery rate of the counterparty RR_c can however result in a default swap premium s that is higher than the default swap premium excluding counterparty default risk. This is especially the case in scenario 1, because the swap premium payments cease but due to the high recovery rate of the counterparty the payoff will increase and with it the default swap value and consequently the premium s.

4. The model in combination with the Libor Market Model (LMM)

The model derived here can be easily combined with any interest rate term structure based model. We will show how it is combined with the Libor Market Model (LMM).

The Libor Market Model

The Libor Market Model[5] can be viewed as a generalization of the Heath-Jarrow-Morton (HJM) 1992 term structure model[6]. The main weakness of the HJM model is that interest rates are expressed as instantaneous rates, which are not

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[5] The Libor Market Model is credited to three groups of authors: Brace, A., D. Gatarek, and M. Musiela, (BGM) “The Market Model of Interest Rate Dynamics,” *Mathematical Finance*, 7, no.2 1997, p.127-155; F., Jamshidian, “Libor and Swap Market Models,” *Finance and Stochastics,* 1 (1997), p.293-330; and Miltersen, K., K. Sandmann, and D. Sondermann, “Closed Form Solutions for Term Structure Derivatives with LogNormal Interest Rates,” *Journal of Finance,* 52, no.1, March 1997, p.409-430
In the following, we will use the notation of Hull J., and A. White, “Forward Rate Volatilities, Swap Rate Volatilities, and Implementation of the LIBOR Market Model,” *Journal of Fixed Income,* Sep2000, Vol. 10 Issue 2, p.46-63 and J. Hull, “Options, Futures and Other Derivatives,” Prentice Hall, 2002

[6] Heath, D., R. Jarrow, and A. Morton, “Bond Pricing and the Term Structure of Interest Rates, A New Methodology,” *Econometrica,* 60, no.1, (1992), p.77-105

observable in the market. In the LMM model, interest rates can be conveniently expressed as discrete forward rates.

Hull and White (2000) show that a one-factor Libor Market Model can be expressed as

$$F_k(t_{j+1}) = F_k(t_j)\exp\left[\left(\sum_{i=j+1}^{k}\frac{\Delta\tau_i F_i(t_j)\Lambda_{i-j-1}\Lambda_{k-j-1}}{1+\Delta\tau_i F_i(t_j)} - \frac{\Lambda_{k-j-1}^2}{2}\right)\Delta\tau_j + \Lambda_{k-j-1}\,\varepsilon\sqrt{\Delta\tau_j}\right] \tag{16}$$

where

$F_k(t)$: Forward interest rate between time k and k+1, seen at time t, with compounding of $\Delta\tau_i$

$\Delta\tau_i$: time between horizontal nodes i and i+1, expressed in years

ε : random drawing from standard normal distribution

Λ_k : forward rate volatility term for time t_k to t_{k+1}. Assuming $\Delta\tau$ is constant, Λ can be derived iteratively using

$$\Lambda_{k-1} = \sqrt{k\,\sigma_k^2 - \sum_{v=0}^{k-2}\Lambda_v^2} \tag{17}$$

where σ_k is caplet's volatility between time t_k and t_{k+1}.

Valuing Default Swaps on the basis of the LMM model

The following 9 steps show how the value of the European-style default swap including reference asset – counterparty default correlation is derived using a Monte-Carlo implementation of the LMM model:

1. The default probability of the reference asset is simulated by a one-factor LMM model. Hence, in equations (14) and (15), the forward interests rates and their volatilities are replaced by the forward default probabilities of the reference asset and its volatilities.

2. The default probability of the counterparty is also simulated by a one-factor LMM model. Hence, in equations (16) and (17), the forward interests rates and their volatilities are replaced by the forward default probabilities of the counterparty and its volatilities.

3. The generated values for the default probability of the reference asset and the counterparty from point 1. and 2. are integrated into the trees in figure 2, as well as 3 and 4, via equations (1), (3), (4), and (5).

4.1. The discrete correlation approach

A random number between 0 and 1 is generated to dictate which of the four different default scenarios occurs: No default $\lambda(\bar{r} \cap \bar{c})$, only reference asset default $\lambda(r \cap \bar{c})$, only counterparty default $\lambda(c \cap \bar{r})$, or both reference asset and counterparty default $\lambda(r \cap c)$. If either the reference asset or the counterparty or both default, a random number between 0 and 1 is generated and multiplied with the length of the time step to simulate the exact default time in the specific time step.

4.2: The Gaussian copula approach

For each time step a set of random numbers from a bivariate normal distribution is derived. The process is to perform a Cholesky decomposition of the correlation matrix, then generate two random numbers from the normal distribution and then matrix multiply the result of the Cholesky decomposition with a vector containing the two normally distributed random numbers. Next the result is put through an inverse cumulative normal distribution operation which produces two random numbers that are linked through a Gaussian copula (through the bivariate normal distribution).

The random numbers are then used to determine whether neither has defaulted $\lambda(\bar{r} \cap \bar{c})$, only the reference asset has defaulted $\lambda(r \cap \bar{c})$, only the counterparty has defaulted $\lambda(c \cap \bar{r})$ or both have defaulted $\lambda(r \cap c)$. This is done in two steps. The first breaks down the options in if statements and uses the direct result of the bivariate normal random number generator to make the decisions. The second converts the output of the bivariate normal random number generator into two binomial distributions (the full Gaussian copula model with binomial marginal distributions) and then uses the resulting binomially distributed random numbers (which are related through the Gaussian copula) to determine whether the reference asset, counter party, both, or neither has defaulted.

As in the discrete correlation approach, if either the reference asset or the counterparty or both default, a random number between 0 and 1 is generated and multiplied with the length of the time step to simulate the exact default time in the specific time step.

5. The payoff and the premium payment of the default swap are calculated taking into account the specific default scenario. Table 2 presents the payoffs and in arrears premium payments.[7]

Default Scenario	Payoff	Premium Payment	Paid at	How to Proceed
$\lambda(\bar{r} \cap \bar{c})$	0	$s\ N\ \Delta\tau_t$	t_{k+1}	Continue to next time step (if not maturity)
$\lambda(r \cap \bar{c})$	$N(1-RR_r-RR_r c\ Td_t)$	$s\ N\ \Delta\tau_t\ Td_t$	t_{k+1}	Stop trial
$\lambda(c \cap \bar{r})$	$S_f(Td_t)\ RR_c$	$\mathrm{Min}(s\ N\ \Delta\tau_t\ Td_t,$ $S_f(T_d)\ RR_c)$	t_{k+1}	Stop trial
$\lambda(c \cap r)$	$N(1-RR_r-RR_r c\ Td_t)$ RR_c	$\mathrm{Min}[s\ N\ \Delta\tau_t\ Td_t,$ $N(1-RR_r-RR_r c T_d)\ RR_c\ ]$	t_{k+1}	Stop trial

Table 2: Payoffs and in arrears premium payments for the four default scenarios

where

RR_c : Exogenous recovery rate of the counterparty

RR_r : Exogenous recovery rate of the reference entity

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[7] Calculations for premium payments and payoffs are more complicated when the observed time step is different than the time interval of premium payments or the time interval of coupon payments of the reference asset. The reason is that it is required to keep track of the accrued interest and the accumulated premium payment.

Td_t : Randomly simulated default time between the node t-1 and default, expressed in years

$S_f(T_d)$: Fair value of the default swap from the time the default swap was issued until the time of reference asset default without the possibility of counterparty default. $S_f(T_d)$ includes the notional amount N.

a : Accrued interest on the reference asset from the last coupon date until the default date, hence $a = c\,T_d$, where c : coupon of the reference asset

s : Default swap premium

N : Notional amount of the swap

6. The payoff (if other than zero) is discounted back to time zero using the interest rate term structure (that is modeled using LMM). The premium payment of the default swap is also discounted back to time zero using the interest rate term structure (that is modeled using LMM) and added to the previously discounted premium payments.

7. If there is no default during the time step and the maturity of the default swap is not reached, steps 1 to 6 are repeated until maturity or default is encountered. (See table 2 column "How to Proceed".)

8. The accumulated discounted payoffs are divided by the accumulated discounted premium payments to derive the swap premium s, following equation (13) for upfront premium payments or equation (15) for in arrears premium payments. Equation (13) and (15) include the default correlation between the reference asset and the counterparty of equations (1) and (3) to (5) for discrete correlation and equation (7) for the Gaussian copula. The swap premium s is derived as the average of all trials.

9. The steps 1 to 8 are repeated until the desired accuracy is achieved (recommended at least 100,000 times).

A visual basic program that follows the previously mentioned nine steps is available at www.dersoft.com/dslmmkkmcopula.xls. The program requires the inputs

- Caplet volatilities
- Reference asset forward default volatilities (derived by calibration)

- Counterparty forward default volatilities (derived by calibration)
- Default correlation between the reference asset and counterparty
- Recovery rate of the reference asset
- Recovery rate of the counterparty
- Maturity of the default swap
- Coupon of the reference asset and coupon payment frequency
- Default swap premium payment frequency
- Length of time step (0.25, 0.5 and 1 year are currently available)

The program provides a 95% confidence interval for the simulated result so that the accuracy of the result can be evaluated.

5. Properties of the Model

5.1 The correlation properties

Figure 5 shows the result of the two different correlation approaches.

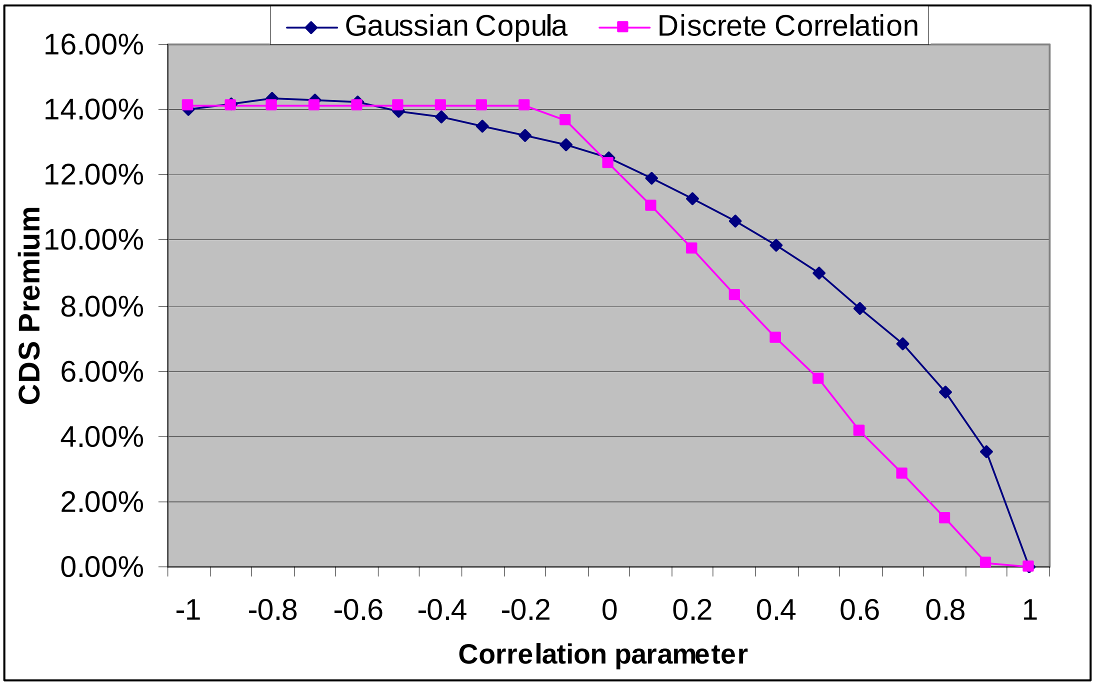

Figure 5: Credit default swap premium for different degrees of default correlation

As seen in figure 5, for the rather unrealistic case of negative default correlation between the reference asset and the counterparty[8], the Gaussian copula and the linear correlation approach give similar results. In this scenario the default swap premium is rather insensitive to correlation parameter changes. In the case of zero correlation, the models give, by definition, identical default swap premiums. In the case of positive correlation, in the copula model the default swap premium decreases asymptotically, whereas in the discrete correlation model the default swap premium decreases linearly. Importantly, for a positive correlation, the copula models results in a higher default swap premium, since the bivariate normal distribution function

$M\ [N^{-1}(\lambda^r),\ N^{-1}(\lambda^c);\ \bar{\rho}(\lambda^r,\lambda^c)]$ in equation (7) produces lower joint default probabilities.

5.2 Further properties

As to be expected, the default swap premium s has a positive dependence on the default probability of the reference asset and a negative dependence on the default probability of the counterparty, i.e., $\partial s/\partial\lambda^r \geq 0$ and $\partial s/\partial\lambda^c \leq 0$. Also, the higher the default correlation between the reference asset and the counterparty, the lower the default swap premium, i.e., $\partial s/\partial\rho \leq 0$. Naturally, for $\lambda^r = 0$ or $[\lambda^c = 1$ and $RR_c = 0]$ $\Rightarrow s = 0$.

If the default swap premium s is paid upfront, for $\lambda^r = 1$ and $RR_r = 0 \ \Rightarrow s = 100\%$ of the notional amount. In case the default swap premium is paid in arrears, the default swap premium can take values bigger than 100% of the notional amount. This is because default is possible in the first period and in this case only the accrued on the default swap premium, i.e. $s\ N\ \Delta\tau_t\ Td_t$ is due from the default swap buyer. (For default after one day, $Td_t = 1/365$, hence $s\ N\ \Delta\tau_t\ Td_t$ is very small.) However, the default swap seller has to pay the entire payoff $N(1\text{-}RR_r\text{-}RR_r a)$.

A further expected result of the model is the zero value for the default swap premium s, if there is a perfect correlation between the default processes for the reference asset and the counterparty, i.e., the correlation coefficient $\rho(\lambda^r\ ,\ \lambda^c)$ in equation (1) is 1, and the probabilities and the default volatilities of the reference asset

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[8] Most empirical studies find a positive default correlation of creditors. See for example Servigny and Renault (2002), Chava and Jarrow (2004) or Li and Meissner (2004)

and the counterparty are identical. If additionally the recovery rates are zero, in this case, if the reference asset defaults, the counterparty will default, making the default swap useless.

It is also interesting to note that the default swap premium s is negatively correlated to the recovery rate of the reference entity, i.e. $\partial s / \partial RR_r \leq 0$. This is because the recovery rate is deducted from the payoff, which is N(1-RR_r-RR_ra). Hence with a higher recovery rate RR_r, the value of the default swap and with it the default swap premium s decreases. However, the swap premium is positively correlated with the recovery rate of the counterparty. i.e. $\partial s / \partial RR_c \geq 0$. This is because a default swap buyer is willing to pay a higher default swap premium s, if the payoff will be higher due to a higher recovery rate RR_c.

The user may verify these results at *www.dersoft.com/dslmmkkmcopula.xls*.

Limitations of the Model

As in most reduced form models, the direct economic reasons for default, i.e. the company's specific asset-liability structure or the company's liquidity are not part of the analysis.

Furthermore, the default correlation coefficient, as well as the recovery rates of the reference asset and the counterparty do not depend on the model variables, but are exogenous and assumed constant over the life of the default swap. Furthermore, the model does not incorporate the default risk of the default swap buyer in case of periodic default swap premium.[9] Also, the interest rate LMM process and the default LMM processes are uncorrelated.

6. Conclusion

The paper derives a model with a closed form solution for valuing default swaps including reference asset – counterparty default correlation. The default correlation between the reference asset and the counterparty is incorporated in two quadruple trees. One tree represents the default swap payoff of the default swap seller;

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[9] For an extension of the model including default swap buyer default risk, see Meissner, G., "Pricing Default Swaps – Which Default Probabilities, Which Default Correlations should be Included?" Hawaii Pacific University Working Paper

the other tree represents the default swap premium payments of the default swap buyer. Swap valuation techniques are then applied to derive the fair default swap price.

The model incorporates two correlation approaches used in today's credit practice. The Gaussian copula approach and the discrete correlation approach. The Gaussian copula results in a higher credit default swap premium than the discrete approach and in a non-linear relationship between the credit default swap premium and the correlation parameter.

The model is based on three LMM (Libor Market Model) processes. One LMM process simulates risk-free short interest rates. Two more LMM processes simulate the reference asset default process and the counterparty default process. Two Visual Basic open source code versions of the model are provided. In one model the default swap premium is paid upfront (at the beginning of each period); in the other model the default swap premium is paid in arrears (at the end of each period).

Further research will focus on the integration of a correlated process for market risk and credit risk (see e.g. Lando (1998), B´elanger, Shreve, and Wong (2001) and Jarrow and Yildirim (2002)) into the model. Also, two-sided default risk (see Duffie and Huang (1996)), i.e. the default risk of the default swap seller and buyer in case the premium is paid periodically, awaits integration into the model.

Appendix

In equation (15) it is assumed that the grid points and time periods in the payoff tree (figure 2) and the premium tree (figure 3) are identical. For different points in time and time periods we derive

$$s=\frac{\begin{array}{l}\sum_{t=1}^{T}\{[\lambda_{t-1}(r\cap c)\,N(1-RR_r-RR_r a)RR_c+\lambda_{t-1}(r\cap\bar{c})\,N(1-RR_r-RR_r a)+\\ \lambda_{t-1}(c\cap\bar{r})\,S_f(t)RR_c\,]e^{-r_{t-1}\tau_t}\prod_{u=0}^{t-2}\lambda_u(\bar{r}\cap\bar{c})\}\end{array}}{\begin{array}{l}\sum_{v=1}^{V}\{[\lambda_{v-1}(r\cap c)\ \min[x_v]/s+\lambda_{v-1}(\bar{r}\cap\bar{c})\,N\,\Delta\tau_v Td_v+\lambda_{v-1}(r\cap\bar{c})\,N\,\Delta\tau_v Td_v\\ \lambda_{v-1}(c\cap\bar{r})\min[y_v]/s\,]e^{-r_{v-1}\tau_v}\prod_{u=0}^{v-2}\lambda_u(\bar{r}\cap\bar{c})\}\end{array}}$$

where v are points in time and $\Delta\tau_v$ is the time between v-1 and v, expressed in years.

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