ROBUST STATIC HEDGING OF BARRIER OPTIONS IN STOCHASTIC VOLATILITY MODELS

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Abstract. Static hedge portfolios for barrier options are extremely sensitive with respect to changes in the parameters of the underlying financial market model. In this paper we develop a new semi-infinite programming formulation of the static super-replication problem in stochastic volatility models which allows to incorporate the model parameter uncertainty in the sense of a worst case design. After proving existence of robust hedge portfolios and presenting an algorithm to numerically solve the underlying optimization problem, we apply the approach to a detailed real world example. Surprisingly, the optimal robust static hedge portfolios are only marginally more expensive than the barrier option itself although they offer protection for a broad range of model parameters.

Key Words. *Robust Optimization, Static Hedging, Barrier Options, Stochastic Volatility.*

AMS subject classification. *91B28, 90C30, 90C34*

1. Introduction. The growing demand for derivative products in the financial market industry has led to the development of fairly complex tools. Standard options like puts and calls, barrier options or auto trigger options are only some examples of the large variety of products offered on the markets today. The significant interest in these products is also reflected in the recent developments at the EUREX, the world's leading futures and options market for Euro denominated derivative instruments. In 2004, the trading volume in this market exceeded 1.07 billion contracts.

Along with this rapid growth of the option markets goes hand in hand an increasing risk of the banks issuing these products to the customers. This risk stems from the fact that the bank, by selling the product to the customer at time $t = 0$, enters the obligation of an insecure future payment at maturity $T > 0$. This payment is based on the future performance of a so-called underlying, usually the stock price, from time $t = 0$ to time $t = T$.

To reduce the risk of the insecure future payment traders immediately set up a portfolio of other financial instruments which in the sum matches the payoff of the sold product closely in all possible states of the market. Usually this *hedge portfolio* consists of stock and bond positions which are dynamically adjusted during the lifetime of the option according to the sensitivity of the sold product with respect to the stock price (*Delta-hedging*).

However, in the case of barrier options, this standard hedging strategy faces some serious problems if applied in practice. For example the delta of an up-and-out call is very sensitive to stock price movements when the underlying is close to the barrier and the option close to maturity. This sensitivity may cause huge losses for the bank selling the option, because traders cannot act fast enough to adapt the trading portfolio to changes in the stock price.

The problems mentioned above have lead to the development of so-called static trading strategies which avoid a continuous dynamic readjustment of the hedge portfolio. For example Carr, Ellis and Gupta show in [7] that a variety of barrier options can be perfectly replicated under the assumption of a volatility symmetry and zero cost of carry. Based on another approach, Derman, Ergener and Kani [9] derive a

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hedge portfolio of standard options which replicates the value of the barrier option at discrete points in time. However, the results of these approaches are only valid in very special cases and the possible extension to more advanced models seems rather difficult. For example, Toft and Xuan [25] show, that the portfolio derived by Derman, Ergener and Kani quickly deteriorates in a stochastic volatility environment. Although Fink [12] tries to improve this strategy by matching the barrier option on a time-volatility grid, the main problem of potentially significant losses for the bank is not solved.

To avoid these possible losses, Giese and Maruhn [14] focus on static trading strategies super-replicating the payoff of the target barrier option in all states of the market. As the authors show, these strategies can be computed by solving a stochastic optimization problem. The resulting super-replication portfolio then matches the delta and vega of the target option closely, consists only of a handful of standard options, and is only marginally more expensive than the barrier option itself. Moreover the approach also generalizes to more advanced models like Heston's stochastic volatility model.

However, as Maruhn and Sachs show in [21], this hedging strategy (as well as all other approaches mentioned above) is based on the crucial assumption that implied model parameters do not change over time. The authors show in case of the Black-Scholes model, that changing model parameters during the lifetime of the barrier option can still lead to significant hedging losses. To overcome these losses, Maruhn and Sachs propose a robust formulation of the hedging problem in the sense of a worst case design guaranteeing the super-replication property for a given set of model parameters. The resulting robust optimization problem is then solved by semi-infinite programming techniques. Surprisingly, the robust trading strategy is still only marginally more expensive than the barrier option itself.

The results obtained in the latter paper seem to be promising, but the question still remains if the findings also transfer in a similar way to more advanced models. Clearly, as the hedge portfolio consists of standard options, the model should have the ability to match a given set of market prices. In contrast to the Black-Scholes model stochastic volatility models are well known to deliver a good fit of a given volatility surface [2]. Hence the latter models seem to be suitable for the derivation of static hedging strategies consisting of standard options.

Accordingly, this paper generalizes the static hedging strategies derived in [21] to a stochastic volatility environment. The basis of our approach is the proof, that the stochastic constraints in the super-replication problem can be replaced by an infinite number of deterministic constraints. Based on this problem transformation the computation time for the solution of the problem as discussed in Giese and Maruhn [14] can be reduced from several hours to several seconds. This speed-up allows to incorporate additional constraints in the form of a worst-case design into the optimization problem taking the model-parameter uncertainty into account. Introducing such a robustified version of the problem is necessary as the model parameters enter into the constraints very nonlinearly by means of a parabolic differential equation. After proving existence of solutions of the robust optimization problem and convergence of an appropriate numerical method we present a detailed real world example for the case of Heston's stochastic volatility model. Surprisingly, the computed robust hedge portfolio is still very cheap and guarantees the super-replication property for a whole set of model parameters. Hence the proposed hedging technique may offer a solution to the problem of hedging barrier options.

The paper is organized as follows. In section 2 we describe the problem of static super-replication for general stochastic volatility models with fixed model parameters. Furthermore we prove the equivalence of the stochastic constraint to an infinite number of deterministic constraints. Section 3 is then devoted to the derivation of the robust optimization problem including a detailed existence proof. After discussing the numerical solution of the problem and the convergence of the proposed method we present a detailed numerical example in section 4.

Notation. Throughout this paper we define for two reals a, b that $(a)^+ := \max\{a, 0\}$ and $a \wedge b := \min\{a, b\}$. For a set $A \subset \mathbb{R}^n$, $A^c := \mathbb{R}^n \setminus A$ denotes its complement and $\bar{A}$ its closure. Furthermore (a.s.) is an abbreviation for an equation or inequality to hold almost surely with respect to a given measure. On a probability space $(\Omega, \mathcal{F}, P)$, with two given random variables X, Y the measure $P^{(X,Y)}$ denotes the joint distribution of X and Y. In case $(\Omega, \mathcal{F}) = (\mathbb{R}, \mathbb{B})$, where $\mathbb{B}$ denotes the Borel sigma algebra, $\mathrm{supp}(P)$ denotes the support of P, that is the smallest closed set with measure 1. Finally, if $B \in \mathcal{F}$, the measure $P|_B$ is defined by $P|_B(\cdot) := P(\cdot \cap B)/P(B)$.

2. Static Super-Replication. In this section we describe the problem of super-replicating an up-and-out call by a given set of standard options and a bond in a general stochastic volatility model. After posing the problem in Subsection 2.1 we prove that the stochastic optimization problem is equivalent to a deterministic linear semi-infinite optimization problem depending on the support of an appropriate distribution.

2.1. Description of the Problem. In order to state the optimization problem characterizing an optimal super-replicating strategy, we first have to describe the financial market model and the static trading strategies under consideration. As mentioned before, stochastic volatility models are particularly suited for the purpose of fitting a volatility surface. Hence we consider the following general form of such a model.

Assumption 2.1. *Let $\mathcal{M}$ be a stochastic volatility model with time index set $I = [0, T]$, probability space $(\Omega, \mathcal{F}, P)$, bond B and stock price S, where the dynamics of the bond and stock price process under an equivalent martingale measure Q are given by the stochastic differential equations*

$$\begin{aligned} dB_t &= rB_t dt, \quad B_0 \in (0, \infty), \; r > 0 \\ dS_t &= rS_t dt + \sigma(Y_t) S_t dW_t^1, \quad S_0 \in (0, \infty) \\ dY_t &= \alpha(Y_t) dt + \beta(Y_t) dW_t^2, \quad Y_0 \in \mathbb{R}. \end{aligned} \tag{2.1}$$

Here W^1, W^2 are two correlated Brownian motions with correlation coefficient ρ, $|\rho| < 1$, and the filtration $(\mathcal{F}_t)_{t \in [0,T]}$ is the augmented filtration generated by W^1, W^2. The process $(Y_t)_{t \in [0,T]}$ drives the volatility $\sigma(Y_t)$ of the stock price, where $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathcal{Y} \to [0, \infty)$ are assumed to be measurable and $\mathcal{Y} \subset \mathbb{R}$ is a given interval with $Y_t \in \mathcal{Y} \;\forall\; t$ (a.s.). Furthermore α and β are assumed to satisfy the usual regularity conditions such that a solution $(Y_t)_{t \in [0,T]}$ of the second stochastic differential equation exists. Finally assume that σ is a continuous or bounded function.

The market model described above in particular contains Heston's stochastic volatility model and the Stein-Stein model as special cases, but also the simple Black-Scholes model and time-dependent volatility models can be obtained by an appropriate choice of the corresponding functions. However, if these simple cases are excluded the market model $\mathcal{M}$ is arbitrage-free, but not complete (see e.g. [18]).

Our goal is to hedge an up-and-out call with strike K, barrier D and maturity $T > 0$ given by

$$C_{uo} = (S_T - K)^+ 1_{\{\tau > T\}}, \quad \tau := \inf\{t \in [0,T] \ : \ S_t = D\}$$

where τ denotes the stopping time of the first barrier hit and $\inf \emptyset := +\infty$. As shown in [7], [9], [14] and [21], standard calls C^i with strike K_i and payoff $C^i(T_i, S_{T_i}) = (S_{T_i} - K_i)^+$ at maturity T_i are particularly suited hedge instruments in order to replicate an up-and-out call as good as possible. Hence we seek to set up a hedge portfolio consisting of standard calls C^i, $i = 1, ..., n$ and the riskless bond B.

It is well known (see e.g. [13]), that under Assumption 2.1 the arbitrage-free value process $(C_t^i)_{t \in [0,T_i]}$ defined by

$$C_t^i := e^{-r(T_i - t)} E_Q((S_{T_i} - K_i)^+ | \mathcal{F}_t) \tag{2.2}$$

of a standard option C^i at time t only depends on time t, the value of the stock price S_t and the volatility Y_t, that is $C_t^i = C^i(t, S_t, Y_t)$ with pricing function

$$C^i(t, s, y) := e^{-r(T_i - t)} E_Q(S_{T_i - t}^{s,y} - K_i)^+ \tag{2.3}$$

where $(S_\nu^{s,y})_{\nu \in [0, T_i - t]}$ is the solution of the stochastic differential equations (2.1) with initial conditions $S_0 = s \in (0, \infty)$ and $Y_0 = y \in \mathcal{Y}$. Note that for $T_i < T$ we can extend the value process $(C^i(t, S_t, Y_t))_{t \in [0,T_i]}$ to the whole interval $[0,T]$ by setting

$$C^i(t, S_t, Y_t) := C^i(T_i, S_{T_i}, Y_{T_i}) \frac{B_t}{B_{T_i}}, \quad T_i < t \leq T \tag{2.4}$$

which simulates shifting the payoff of a call C^i in the hedge portfolio at time T_i to the bond position as soon as option C^i expires. Hence, if we assume that the standard options C^i are tradable, we can extend the market model $\mathcal{M}$ defined in Assumption 2.1 by the standard options C^i with value processes $(C^i(t, S_t, Y_t))_{t \in [0,T]}$. In general, the value of a hedge portfolio at time t consisting of the bond B and the standard options C^i, $i = 1, ..., n$ will be given by

$$\Pi_t = \Pi_t(\phi) := \phi_t^0 B_t + \sum_{i=1}^n \phi_t^i C^i(t, S_t, Y_t) \tag{2.5}$$

where $\phi = ((\phi_t^0, \phi_t^1, ..., \phi_t^n)^T)_{t \in [0,T]}$ denotes a general trading strategy. However, due to the problems associated with the dynamic hedging of barrier options, we are only interested in so-called static trading strategies as defined below.

Definition 2.2. *Assume that market model $\mathcal{M}$ is given as stated in Assumption 2.1, extended by the tradable standard calls C^i, $i = 1, ..., n$ with strikes K_i and maturities $0 < T_i \leq T$ satisfying $K_i \neq K_j$ for $T_i = T_j$, $i \neq j$, and arbitrage-free value processes $C_t^i = C^i(t, S_t, Y_t)$ as defined in (2.2) and (2.4). Further let an up-and-out call with strike K, barrier D, $0 \leq K < D$ and maturity $T > 0$ be given by $C_{uo} = (S_T - K)^+ 1_{\{\tau > T\}}$, where $\tau := \inf\{t \in [0,T] : S_t = D\}$ denotes the stopping time of the first barrier hit and $\inf \emptyset := +\infty$.*

Let $\alpha_0 \in \mathbb{R}$ and $\alpha_1, ..., \alpha_n \in \mathbb{R}$ be the initial portfolio positions of the bond B and the standard calls C^i. Then the trading strategy $\phi(\alpha) := ((\phi_t^0, \phi_t^1, ..., \phi_t^n)^T)_{t \in [0,T]}$ defined by

$$\phi_t^0 := \alpha_0 + \frac{\sum_{i=1}^n \alpha_i C^i(\tau, S_\tau, Y_\tau)}{B_\tau} 1_{\{\tau \leq t\}}$$
$$\phi_t^i := \alpha_i 1_{\{\tau > t\}}, \quad i = 1, ..., n$$

is called a "static knock-out trading strategy".

By definition, a static knock-out trading strategy is a constant strategy consisting of the portfolio positions $\alpha_0, ..., \alpha_n$ in case the stock price S_t does not hit the barrier D until maturity T ($\tau = +\infty$). In case the barrier is hit at some time $\tau \leq T$ (knock-out time), the value of the standard calls in the portfolio at that time is transfered to the bond position where it remains until terminal time T. Hence such a static trading strategy consists of buying/selling certain quantities of the bond B and the tradable calls $C^1, ..., C^n$ at the beginning and only changing the hedge portfolio once in case the first barrier hit occurs before maturity of the barrier option. This behavior is illustrated in Figure 2.1 for two possible stock price paths.

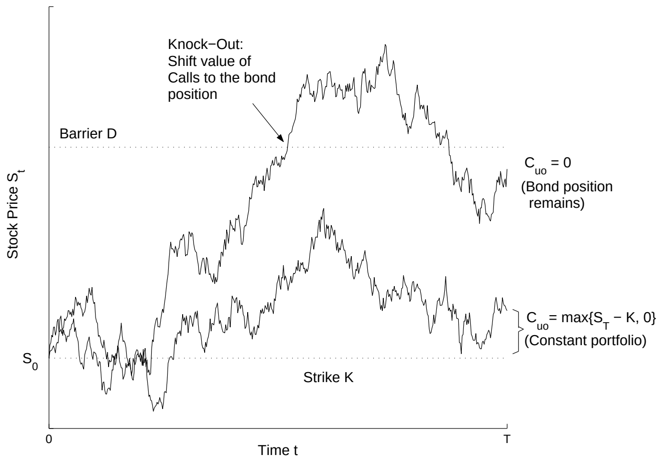


FIG. 2.1. *Portfolio positions and payoff of the up–and–out call for two possible stock–price paths*

Since a static knock-out trading strategy $\phi(\alpha)$ is uniquely identified by the associated vector $\alpha \in \mathbb{R}^{n+1}$, the set of these strategies is isomorphic to $\mathbb{R}^{n+1}$. Furthermore, the value of the hedge portfolio Π_t as given in (2.5) satisfies

$$\Pi_t = \Pi_t(\alpha) = \alpha_0 B_t + \frac{B_t}{B_{t\wedge\tau}} \sum_{i=1}^{n} \alpha_i C^i(t \wedge \tau, S_{t\wedge\tau}, Y_{t\wedge\tau}). \tag{2.6}$$

The goal of a bank is to find a hedge portfolio α with value $\Pi_T(\alpha)$ which matches the value of the up-and-out call as good as possible in all possible future states of the market $\omega \in \Omega$. To prevent potential losses, we seek the cheapest portfolio whose value is greater or equal to the value of the barrier option C_{uo}. Equipped with the notation introduced above, we can define such a super-replication strategy as follows.

DEFINITION 2.3. *Let $\mathcal{M}$ be a financial market model satisfying Assumption 2.1. Consider an up-and-out call C_{uo}, standard options C^i, $i = 1, ..., n$, and the corresponding static knock-out trading strategies $\phi = \phi(\alpha)$, $\alpha \in \mathbb{R}^{n+1}$, as given in Definition 2.2. Further let $\Pi_t(\phi) = \phi_t^0 B_t + \sum_{i=1}^n \phi_t^i C^i(t, S_t, Y_t)$ be the value of the hedge portfolio $\phi(\alpha)$ at time $t \in [0, T]$. Then a "cost-optimal static super-replication strategy" (COSS) is defined as a solution (if it exists) of the stochastic optimization problem*

$$\begin{aligned} &\min_{\alpha \in \mathbb{R}^{n+1}} \Pi_0(\phi(\alpha)) \\ &s.t. \quad \Pi_T(\phi(\alpha)) \geq C_{uo} \quad (a.s.) \end{aligned} \tag{2.7}$$

The problem of finding the cheapest trading strategy super-replicating the value of another instrument has first been introduced by El Karoui and Quenez in [11] in the case of dynamic trading strategies. The authors prove the existence of a cost-optimal super-replication strategy by stochastic control methods. But the static trading strategies given in Definition 2.2 only form a subspace of the space of all dynamic trading strategies considered by El Karoui and Quenez, hence the existence of a solution of optimization problem (2.7) is not obvious. However, for this case Giese and Maruhn [14] give an existence proof which is solely based on convex analysis. The result is stated in the following theorem.

THEOREM 2.4. *Let $\mathcal{M}$ be a market model satisfying Assumption 2.1. Then a cost-optimal static super-replication strategy exists, i.e. the stochastic optimization problem (2.7) has a solution.*

Proof. First we observe that the feasible set of optimization problem (2.7) is non-empty, because the constant strategy $\alpha_0 = (D-K)^+/B_T$, $\alpha_1 = ... = \alpha_n = 0$ delivers a payoff $\Pi_T(\alpha) = \alpha_0 B_T = (D-K)^+ \geq C_{uo}$ (a.s.) due to (2.6). Further note that the financial market model $\mathcal{M}$, extended by the value processes for the standard options C^i, $i = 1, ..., n$, is arbitrage-free. In particular it is arbitrage-free in the set of static trading strategies as defined in Definition 2.2. Hence, applying the general existence Theorem 3 in [14], the claim immediately follows. ☐

Besides the trivial super-replicating strategy only consisting of a bond position, another simple feasible strategy is easily obtained if the standard call with the same strike and maturity as the barrier option is included in the set of calls $\{C^i : i = 1, ..., n\}$ since $(S_T - K)^+ \geq (S_T - K)^+ 1_{\{\tau > T\}}$. This motivates that other cheaper static hedge portfolios might exist which also include calls C^i with strikes $K_i \neq K$ or maturities $T_i < T$.

So far Theorem 2.4 still does not answer the question how to compute a solution of the optimization problem (2.7). In general an analytic derivation of the solution will not be possible such that the problem has to be solved numerically. Therefore we use (2.6) to rewrite problem (2.7) as follows.

$$\begin{aligned}
&\min_{\alpha \in \mathbb{R}^{n+1}} \alpha_0 B_0 + \sum_{i=1}^{n} \alpha_i C^i(0, S_0, Y_0) \\
\text{s.t.} \quad & \alpha_0 B_T + \frac{B_T}{B_{T\wedge\tau}} \sum_{i=1}^{n} \alpha_i C^i(T\wedge\tau, S_{T\wedge\tau}, Y_{T\wedge\tau}) \geq C_{uo} \quad \text{(a.s.)} \qquad (2.8)\\
& C^i(t, S_t, Y_t) = \begin{cases} e^{-r(T_i - t)} E_Q(S_{T_i}^{S_t, Y_t} - K_i)^+ & , \ 0 \leq t \leq T_i \\ C^i(T_i, S_{T_i}, Y_{T_i}) \frac{B_t}{B_{T_i}} & , \ T_i < t \leq T \end{cases} \\
& dS_t = r S_t dt + \sigma(Y_t) S_t dW_t^1 \\
& dY_t = \alpha(Y_t) dt + \beta(Y_t) dW_t^2 \\
& dB_t = r B_t dt
\end{aligned}$$

This formulation shows one possibility of how to solve the optimization problem numerically. First we simulate $M \in \mathbb{N}$ samples of the stochastic differential equations describing S_t and Y_t. For each of these sample paths we compute the value of the stopping time τ, the value $C^i(T\wedge\tau, S_{T\wedge\tau}, Y_{T\wedge\tau})$ of the standard options as well as the value of the up-and-out call C_{uo}. Replacing the stochastic constraint by the resulting M linear constraints, we obtain a large scale linear programming problem as a discrete approximation of the underlying problem (2.7). This approach was taken by Giese and Maruhn in [14].

However, this Monte Carlo based method has the significant drawback, that the number of samples M has to be chosen quite large to obtain a good numerical approximation. As for each of the M samples the option prices $C^i(t \wedge \tau, S_{t\wedge\tau}, Y_{t\wedge\tau})$ and hence the conditional expectations have to be evaluated, the overall computation time for the solution of the problem can easily amount to several hours - even if closed form solutions for the conditional expectations exist. In the next subsection we will show, that the simulation of the sample paths of the stochastic differential equations is not necessary. Instead we are able to replace the stochastic constraint by an infinite number of deterministic constraints which, in practice, reduces computation time from several hours to several seconds. In the later sections, this computational advantage will allow us to also solve robustified versions of problem (2.7), which is not possible for a Monte-Carlo based procedure.

2.2. Semi-Infinite Equivalence. To reduce the computation time for the numerical solution of problem (2.7), we have to avoid the simulation of the sample paths of the stochastic differential equations for S_t and Y_t. In general the value of a standard call C^i with strike $0 < K_i < D$ and maturity $T_i < T$ at time T_i will depend on the stock price S_{T_i} at that time. Hence the value of an arbitrary trading strategy containing such a call will depend on the stochastic evolvement of the stock price which only allows a Monte Carlo procedure as a solution method.

However, as it turns out, if we exclude certain inefficient hedge instruments and only focus on the static knock-out trading strategies as given in Definition 2.2, the stochastic constraint in (2.7) can be replaced by an infinite number of deterministic constraints avoiding the numerically expensive simulations. To achieve this, we focus on the set of specific calls given below.

Assumption 2.5. *Assume that the calls C^i, $i = 1, ..., n$, with strikes K_i and maturities $T_i \leq T$ satisfy $K_i \geq D$ in case $T_i < T$, where D is the barrier of the up-and-out call. The set of these calls shall be denoted by $\mathcal{C} = \{C^1, ..., C^n\}$.*

Hence a set of calls under consideration may consist of options which expire before the up-and-out call does, but with strikes greater or equal to the barrier, and other calls with the same maturity as the barrier option but arbitrary strikes $K_i \geq 0$. Intuitively it is clear that calls with strikes $K_i < D$ for $T_i < T$ are inefficient hedge instruments as they may result in large positive payoffs at time T_i while the up-and-out call expires worthless. By solving optimization problem (2.7) with the Monte Carlo based procedure described above, we were also able to confirm numerically, that these inefficient calls are not part of a cost-optimal static hedge portfolio. In addition Assumption 2.5 is justified by the analytic static hedging results derived so far. For instance Derman, Ergener and Kani [9] as well as Carr, Ellis and Gupta [7] only consider hedge instruments (including binary calls) satisfying $K_i \geq D$ for $T_i < T$.

Restricting the calls in the portfolio to those described in Assumption 2.5, we can show that the value of the static hedge portfolio defined in Definition 2.2 is independent of the calls with maturities $T_i < t$ if the barrier has not been hit before time t.

Lemma 2.6. *Consider a financial market model $\mathcal{M}$ satisfying Assumption 2.1 and the static trading strategies $\alpha \in \mathbb{R}^{n+1}$ as stated in Definition 2.2 with corresponding portfolio value $\Pi_t(\alpha)$. Furthermore let $\omega \in \Omega$ and $t \in [0, T]$ such that $\tau(\omega) \geq t$. Then the value of a static hedge portfolio which consists of calls satisfying Assumption 2.5* *is given by*

$$\Pi_t(\alpha)(\omega) = \alpha_0 B_t + \sum_{C^i \in \mathcal{C}, T_i \geq t} \alpha_i C^i(t, S_t(\omega), Y_t(\omega)).$$

Proof. By assumption we have $\tau(\omega) \geq t$ such that $S_\nu(\omega) < D$ $\forall \nu \in [0, t)$. In particular this implies $S_{T_i}(\omega) < D$ for all calls $C^i \in \mathcal{C}$ with $T_i < t$. But due to Assumption 2.5 all these calls have strikes $K_i \geq D$ and hence must have expired worthless, because

$$C^i(T_i, S_{T_i}(\omega), Y_{T_i}(\omega)) = (S_{T_i}(\omega) - K_i)^+ \leq (D - K_i)^+ = 0.$$

This in turn implies $C^i(\nu, S_\nu(\omega), Y_\nu(\omega)) = 0$ $\forall \nu \in [T_i, t]$ due to (2.4). Thus, by (2.6) we obtain

$$\begin{aligned}\Pi_t(\alpha)(\omega) &= \alpha_0 B_t + \frac{B_t}{B_t} \sum_{i=1}^{n} \alpha_i C^i(t, S_t(\omega), Y_t(\omega)) \\ &= \alpha_0 B_t + \sum_{C^i \in \mathcal{C}, T_i \geq t} \alpha_i C^i(t, S_t(\omega), Y_t(\omega))\end{aligned}$$

which concludes the proof. □

Based on the representation of the hedge portfolio value in the previous lemma, we can now show that the stochastic constraint in problem (2.7) is equivalent to an infinite set of deterministic constraints.

THEOREM 2.7. *Consider the problem of finding the cost-optimal static super-replication strategy as stated in Definition 2.3 for a set of standard calls satisfying Assumption 2.5. If the mappings $(t, y) \mapsto C^i(t, D, y)$ with C^i as defined in (2.3) are continuous on supp $\left(Q^{(\tau, Y_{\tau\wedge T})}\right) \cap \left([0, T_i] \times \bar{\mathcal{Y}}\right)$, then the stochastic optimization problem (2.7) is equivalent to the deterministic linear semi-infinite optimization problem*

$$\begin{aligned} &\min_{\alpha \in \mathbb{R}^{n+1}} \alpha_0 B_0 + \sum_{i=1}^{n} \alpha_i C^i(0, S_0, Y_0) \\ s.t. \quad &\alpha_0 B_t + \sum_{C^i \in \mathcal{C}, T_i \geq t} \alpha_i C^i(t, D, y) \geq 0 \quad \forall\ (t, y) \in \Theta_1 \\ &\alpha_0 B_T + \sum_{C^i \in \mathcal{C}, T_i = T} \alpha_i (s - K_i)^+ \geq (s - K)^+ \quad \forall\ s \in \Theta_2 \end{aligned} \tag{2.9}$$

with $\Theta_1 :=$ *supp* $\left(Q^{(\tau, Y_{\tau\wedge T})}\right) \cap \left([0, T] \times \bar{\mathcal{Y}}\right)$, $\Theta_2 := \left\{s \in [0, D] : (\infty, s) \in \text{supp}\left(Q^{(\tau, S_T)}\right)\right\}$.

Proof. It is sufficient to show, that the stochastic constraint in (2.7) is equivalent to the infinite number of deterministic constraints in (2.9). We now distinguish the two cases of a barrier hit ($\tau \leq T$) and no barrier hit ($\tau = \infty$) until maturity T of the barrier option

$$\Pi_T(\phi(\alpha)) \geq C_{uo} \text{ (a.s.)} \quad \Longleftrightarrow \quad \begin{cases} \Pi_{\tau\wedge T}(\phi(\alpha)) 1_{\{\tau \leq T\}} \geq 0 \text{ (a.s.)} \\ \Pi_T(\phi(\alpha)) 1_{\{\tau = \infty\}} \geq C_{uo} 1_{\{\tau = \infty\}} \text{ (a.s.)} \end{cases}$$

where we used for the case $\tau \leq T$, that due to (2.6) $\Pi_T(\phi(\alpha)) = \Pi_{\tau\wedge T}(\phi(\alpha)) \cdot$

$B_T/B_{\tau\wedge T}$. Applying Lemma 2.6 to both cases then yields

$$\Pi_T(\phi(\alpha)) \geq C_{uo} \text{ (a.s.)} \iff$$
$$\iff \begin{cases} \left(\alpha_0 B_{\tau\wedge T} + \sum_{C^i\in\mathcal{C}, T_i\geq \tau\wedge T} \alpha_i C^i(\tau\wedge T, S_{\tau\wedge T}, Y_{\tau\wedge T})\right) 1_{\{\tau\leq T\}} \geq 0 \text{ (a.s.)} \\ \left(\alpha_0 B_T + \sum_{C^i\in\mathcal{C}, T_i=T} \alpha_i C^i(T, S_T, Y_T)\right) 1_{\{\tau=\infty\}} \geq (S_T-K)^+ 1_{\{\tau=\infty\}} \text{ (a.s.)} \end{cases}$$

Note that for $\tau \leq T$ we have $S_{\tau\wedge T} = S_\tau = D$ and hence $C^i(\tau\wedge T, S_{\tau\wedge T}, Y_{\tau\wedge T}) = C^i(\tau\wedge T, D, Y_{\tau\wedge T})$. Furthermore, for $T_i = T$ the value of call C^i at time T is given by $C^i(T, S_T, Y_T) = (S_T - K_i)^+$. Hence we obtain

$$\Pi_T(\phi(\alpha)) \geq C_{uo} \text{ (a.s.)} \iff$$
$$\iff \begin{cases} \left(\alpha_0 B_{t\wedge T} + \sum_{C^i\in\mathcal{C}, T_i\geq t\wedge T} \alpha_i C^i(t\wedge T, D, y)\right) 1_{\{t\leq T\}} \geq 0 \ \left(Q^{(\tau, Y_{\tau\wedge T})} - \text{a.s.}\right) \\ \left(\alpha_0 B_T + \sum_{C^i\in\mathcal{C}, T_i=T} \alpha_i (s-K_i)^+\right) 1_{\{t=\infty\}} \geq (s-K)^+ 1_{\{t=\infty\}} \ \left(Q^{(\tau, S_T)} - \text{a.s.}\right) \end{cases}$$

We now focus on the first case $t \leq T$. The proof of the second case $t = \infty$ follows in analogy. It remains to show that

$$\left(\alpha_0 B_{t\wedge T} + \sum_{C^i\in\mathcal{C}, T_i\geq t\wedge T} \alpha_i C^i(t\wedge T, D, y)\right) 1_{\{t\leq T\}} \geq 0 \ \left(Q^{(\tau, Y_{\tau\wedge T})} - \text{a.s.}\right) \iff$$
$$\alpha_0 B_t + \sum_{C^i\in\mathcal{C}, T_i\geq t} \alpha_i C^i(t, D, y) \geq 0 \quad \forall\ (t,y) \in \operatorname{supp}\left(Q^{(\tau, Y_{\tau\wedge T})}\right) \cap \left([0,T]\times\bar{\mathcal{Y}}\right)$$

The direction "$\Leftarrow$" immediately follows by the definition of the support of a measure. For the proof of the other direction "$\Rightarrow$" we first observe that

$$\left(\alpha_0 B_{t\wedge T} + \sum_{C^i\in\mathcal{C}, T_i\geq t\wedge T} \alpha_i C^i(t\wedge T, D, y)\right) 1_{\{t\leq T\}} \geq 0 \ \left(Q^{(\tau, Y_{\tau\wedge T})} - \text{a.s.}\right) \iff$$
$$\alpha_0 B_t + \sum_{C^i\in\mathcal{C}, T_i\geq t} \alpha_i C^i(t, D, y) \geq 0 \qquad \left(Q^{(\tau, Y_{\tau\wedge T})}|_{[0,T]\times\bar{\mathcal{Y}}} - \text{a.s.}\right)$$

This implies that there exists a set N with $Q^{(\tau, Y_{\tau\wedge T})}|_{[0,T]\times\bar{\mathcal{Y}}}(N) = 0$ such that

$$g(t,y) := \alpha_0 B_t + \sum_{C^i\in\mathcal{C}, T_i\geq t} \alpha_i C^i(t, D, y) \geq 0 \quad \forall\ (t,y) \in N^c.$$

If we define $M := N^c \cap \operatorname{supp}(Q^{(\tau, Y_{\tau\wedge T})}|_{[0,T]\times\bar{\mathcal{Y}}})$ the latter inequality also holds on M which is a set of measure one. If g is continuous on $\operatorname{supp}(Q^{(\tau, Y_{\tau\wedge T})}|_{[0,T]\times\bar{\mathcal{Y}}})$, then $g(t,y) \geq 0$ on the closure of M denoted by $\bar{M}$. As $\bar{M}$ is a closed set with measure one, $g(t,y) \geq 0$ must also hold on the smallest closed set with measure one which is by definition $\operatorname{supp}(Q^{(\tau, Y_{\tau\wedge T})}|_{[0,T]\times\bar{\mathcal{Y}}})$. Finally it is easy to show that

$$\operatorname{supp}(Q^{(\tau, Y_{\tau\wedge T})}|_{[0,T]\times\bar{\mathcal{Y}}}) = \operatorname{supp}(Q^{(\tau, Y_{\tau\wedge T})}) \cap \left([0,T]\times\bar{\mathcal{Y}}\right)$$

which proves the theorem.

Regarding the continuity of g note that by assumption the maps $(t,y) \mapsto C^i(t,D,y)$ are continuous on supp $\left(Q^{(\tau,Y_{\tau\wedge T})}\right) \cap \left([0,T_i] \times \bar{\mathcal{Y}}\right)$. This immediately implies the continuity of g in points (t,y) with $t \neq T_i < T$. However, for $t = T_i < T$ the index set of the summation differs to the left and right of T_i which might destroy the continuity in these points. But note that for a sequence $(t_k, y_k)_k$ converging to (T_i, y) with $t_k < T_i$ $\forall k \in \mathbb{N}$ it follows that

$$C^i(t_k, D, y_k) \to_{k\to\infty} C^i(T_i, D, y) = (D - K_i)^+ = 0$$

due to $K_i \geq D$. Thus g is also continuous in points (T_i, y) and hence on the entire set $\mathrm{supp}(Q^{(\tau,Y_{\tau\wedge T})}) \cap \left([0,T] \times \bar{\mathcal{Y}}\right) = \mathrm{supp}(Q^{(\tau,Y_{\tau\wedge T})}|_{[0,T]\times\bar{\mathcal{Y}}})$. □

Although Theorem 2.7 looks complicated on the first sight, the result allows an easy and intuitive interpretation. The original stochastic constraint $\Pi_T(\phi(\alpha)) \geq C_{uo}$ (a.s.) of problem (2.7) requires the hedge portfolio to super-replicate the up-and-out call in all possible states of the economy. As the underlying financial market model is a stochastic volatility model, the states of the economy are described by the two stochastic differential equations (2.1) for S_t and Y_t. These states are now split into those cases in which the barrier is hit until maturity and the cases in which the barrier is not hit at all. Of course the super-replication property has to hold for both cases.

On the one hand, in case the barrier is hit ($\tau \leq T$), the hedge portfolio only consists of calls with maturity greater or equal to the time of the barrier hit as all calls with shorter maturities T_i have expired worthless (Lemma 2.6). The remaining calls have to super-replicate the value of the barrier option $C_{uo}1_{\{\tau\leq T\}} = 0$ in all possible states of the model economy for which the barrier is hit. Hence the super-replication property has to be guaranteed for all time-volatility combinations $(\tau, Y_{\tau\wedge T})$ with $\tau \leq T$. These combinations are given by the set $\Theta_1 = \mathrm{supp}\left(Q^{(\tau,Y_{\tau\wedge T})}\right) \cap \left([0,T] \times \bar{\mathcal{Y}}\right)$.

On the other hand, in case the barrier is not hit at all ($\tau = \infty$), the super-replication property requires the payoff of all remaining calls in the portfolio at time T to be greater or equal to the value of the up-and-out call $C_{uo}1_{\{\tau=\infty\}} = (S_T - K)^+$. These calls have strikes $T_i = T$ and hence their payoff at time T does not depend upon the volatility at time T, but only on the stock price S_T at terminal time. Accordingly, the calls have to super-replicate the barrier option for all possible stock prices S_T which can be attained if the barrier is not hit. The set of these states is given by $\Theta_2 = \left\{s \in [0,D] : (\infty, s) \in \mathrm{supp}\left(Q^{(\tau,S_T)}\right)\right\}$.

Due to Theorem 2.7 a Monte-Carlo type simulation of the stochastic differential equations for S_t and Y_t is not necessary for the solution of optimization problem (2.7). The states for which the super-replication property has to hold are known in advance and given by Θ_1 and Θ_2. In general these sets can describe very complicated subsets of $\mathbb{R}^2$ and $\mathbb{R}$, respectively. However, for most stochastic volatility models Θ_1 and Θ_2 have a rather simple interval-type structure as the following example shows.

EXAMPLE 2.8. *We now state the sets Θ_1 and Θ_2 for four famous models which are special cases of the general model defined in Assumption 2.1, namely the Black-Scholes model, time-dependent volatility models, Heston's stochastic volatility model and the Stein-Stein model. We omit detailed proofs at this point and rather focus on the explanation of the underlying intuitive idea.*

i) The Black-Scholes model is given by the model equations

$$\begin{aligned} dS_t &= rS_t dt + \sigma(Y_t) S_t dW_t^1, \quad S_0 \in (0,\infty) \\ dY_t &= 0dt + 0dW_t^2, \quad Y_0 \in \mathbb{R} \end{aligned}$$

which is a special case of the general market model (2.1) if we choose $\alpha \equiv 0$ *and* $\beta \equiv 0$. *Hence* $Y_t \equiv Y_0 = const$ *such that the volatility* $\sigma(Y_t) = \sigma(Y_0) =: \sigma_0$ *of the stock price is constant. Clearly the stock price can hit the barrier at any time* $t \in (0, T]$ *such that the support of the stopping time* $\tau \wedge T$ *is* $[0, T]$. *As the process* $Y_t = Y_0$ *driving the volatility is constant it also equals* Y_0 *in case the barrier is hit. Thus* $Y_{\tau\wedge T} = Y_0$ *and hence* $\Theta_1 = supp\left(Q^{(\tau, Y_{\tau\wedge T})}\right) \cap \left([0,T] \times \bar{\mathcal{Y}}\right) = [0,T] \times \{Y_0\}$.

ii) In time-dependent volatility models the assumption of a constant volatility is relaxed to a deterministic volatility as described by the model equations

$$\begin{aligned} dS_t &= rS_t dt + \sigma(Y_t) S_t dW_t^1, \quad S_0 \in (0, \infty) \\ dY_t &= \alpha(Y_t) dt + 0 dW_t^2, \quad Y_0 \in \mathbb{R} \end{aligned}$$

Now the process $Y_t = Y(t)$ *is the solution of a deterministic differential equation such that* $\sigma(Y_t) = \sigma(t)$ *is also a deterministic function. Still the barrier can be hit any time* $t \in (0, T]$ *with the corresponding source of volatility* $Y(t)$. *Thus we have* $\Theta_1 = supp\left(Q^{(\tau, Y_{\tau\wedge T})}\right) \cap \left([0,T] \times \bar{\mathcal{Y}}\right) = \{(t, Y(t)) : t \in [0,T]\}$.

iii) To account for the stochastic nature of the volatility of the stock price, Heston [16] incorporated an additional source of uncertainty in the process driving the volatility function

$$\begin{aligned} dS_t &= rS_t dt + \sqrt{Y_t} S_t dW_t^1, \quad S_0 \in (0, \infty) \\ dY_t &= \kappa(\theta - Y_t) dt + \xi \sqrt{Y_t} dW_t^2, \quad Y_0 \in \mathbb{R} \end{aligned}$$

where κ, θ *and* ξ *are positive constants satisfying* $\kappa\theta - \xi^2/2 \geq 0$. *As Cox, Ingersoll and Ross [8] note, the distribution of* Y_t *given* Y_s *for some* $s < t$ *is, up to a scale factor, a noncentral chi-squared distribution and hence has a support of* $[0, \infty)$. *Furthermore Broadie and Kaya [4] prove, that the conditional distribution of* $log(S_t)$ *given* $(Y_s)_{0\leq s\leq T}$ *is normal. Hence the barrier can be hit any time* $t \in (0, T]$ *and at that time the process* Y_t *can attain any value* $Y_t \in (0, \infty)$. *This implies that for Heston's model* $\Theta_1 = supp\left(Q^{(\tau, Y_{\tau\wedge T})}\right) \cap \left([0,T] \times \bar{\mathcal{Y}}\right) = [0,T] \times [0, \infty)$.

iv) The Stein-Stein model (see e.g. [13]) is another well known stochastic volatility model driven by the model equations

$$\begin{aligned} dS_t &= rS_t dt + |Y_t| S_t dW_t^1, \quad S_0 \in (0, \infty) \\ dY_t &= \kappa(\theta - Y_t) dt + \xi dW_t^2, \quad Y_0 \in \mathbb{R} \end{aligned}$$

with constants $\kappa, \theta, \xi > 0$. *Here* Y_t *is a Gaussian process such that* Y_t *can attain any value in* $\mathbb{R}$ *in case the barrier is hit. Accordingly, we obtain in this model* $\Theta_1 = supp\left(Q^{(\tau, Y_{\tau\wedge T})}\right) \cap \left([0,T] \times \bar{\mathcal{Y}}\right) = [0,T] \times (-\infty, \infty)$.

As a summary, Figure 2.2 illustrates the time-volatility combinations $(t, y) \in \Theta_1$ *for which the hedge portfolio has to super-replicate the up-and-out call. Regarding the set* $\Theta_2 = \left\{s \in [0, D] : (\infty, s) \in supp\left(Q^{(\tau, S_T)}\right)\right\}$, *that is the set of all possible stock prices* S_T *in case the barrier is not hit at all, we obtain* $\Theta_2 = [0, D]$ *for all four models.*

Based on the derivations above it is clear that a cost-optimal super-replication strategy will crucially depend on the support of the measures $Q^{(\tau, Y_{\tau\wedge T})}$ and $Q^{(\tau, S_T)}$. Other static hedging approaches like the minimization of the L^2 hedge error (see e.g. [10]) by definition focus on expected value functionals which are computationally easier to handle but neglect low probability events. This leads to possibly huge hedge errors for the unlikely but dangerous cases of barrier hits close to maturity of the

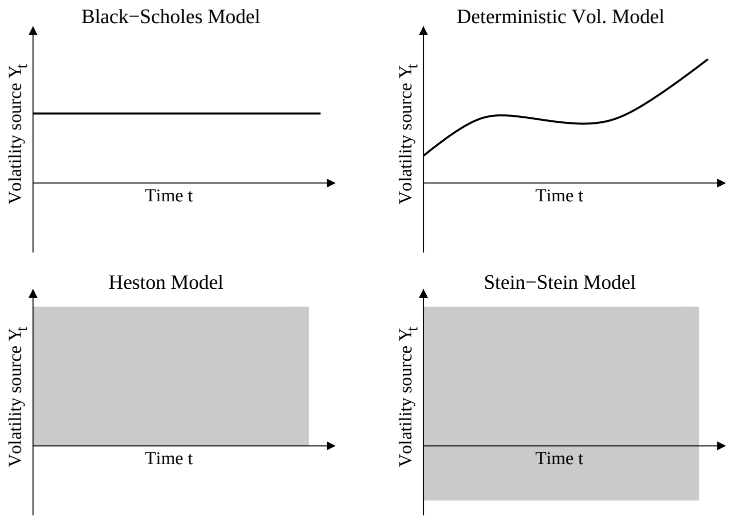


FIG. 2.2. *Sketch of the set* $\Theta_1 := supp\left(Q^{(\tau, Y_{\tau\wedge T})}\right) \cap \left([0,T] \times \bar{\mathcal{Y}}\right)$ *for typical market models*

barrier option. In contrast to this the support-based approach pursued in this paper also guarantees a good hedge performance for unlikely but critical events.

Although this implies, that the super-replication property is guaranteed for all states of the model economy, the robustness of the resulting hedge portfolio is still limited to the model parameters chosen at time $t = 0$. To further robustify the portfolio with respect to changes in the model parameters, the next section will introduce a robust counterpart of optimization problem (2.9). Furthermore, we present an algorithm for the numerical solution of the robust optimization problem which can also be used to solve problem (2.9).

3. Robustification of the Hedging Strategy. Even though the static hedging strategy derived in the previous section already has some attractive properties, the super-replication property of the portfolio is based on the crucial assumption that the standard calls can be sold for the model prices $C^i(\tau, S_\tau, Y_\tau)$ at the time $\tau \leq T$ of a barrier hit. These prices in turn depend on the model parameters chosen at time $t = 0$, for example by calibrating the model to a given volatility surface. But as market data and hence implied model parameters change over time, the model and market prices of the calls might differ significantly at the future time of a barrier hit which leads to potentially huge hedging losses.

The negative effect of this so-called model parameter uncertainty was for example observed by Nalholm and Poulsen [23] for the static hedging strategies of Derman, Egener and Kani [9] and Carr, Ellis and Gupta [7]. To avoid model parameter uncertainty, Brown, Hobson and Rogers [5] derive upper and lower bounds for barrier options as well as a simple hedging strategy which is independent of the dynamics of the stock price and hence the financial market model. However, by taking this extreme point of view the authors loose a lot of information and thus only obtain very conservative bounds for the price of a barrier option which are too rough for practical applications. To summarize, model parameter uncertainty in static hedging of barrier options cannot be neglected, but so far no method is known that allows to derive practical hedging strategies taking this uncertainty into account.

As we will see in this section, the static super-replication approach allows us to incorporate the desired robustness against changing model parameters into the description of the optimization problem. After proving the existence of such a robust static hedging strategy in Subsection 3.1, we focus on the numerical solution of the problem including a convergence proof for the presented algorithm.

3.1. Definition of the Robust Problem. To derive a practical static hedging strategy we have to guarantee, that the value of the hedge portfolio is always greater or equal to the value of the up-and-out call. As derived in Theorem 2.7 we have to consider the two separate cases of a barrier hit and no barrier hit before maturity T.

In case the barrier is not hit at all from $t = 0$ to $t = T$, the super-replication property requires the calls with maturity T to provide a payoff greater or equal to the payoff of the up-and-out call for all possible stock prices S_T that can be attained without a barrier hit. In Theorem 2.7 this is reflected by the constraints

$$\alpha_0 B_T + \sum_{C^i \in \mathcal{C}, T_i = T} \alpha_i (s - K_i)^+ \geq (s - K)^+ \quad \forall\, s \in \Theta_2$$

where $\Theta_2 = [0, D]$ for most models of practical interest as noted in Example 2.8. Hence these constraints are independent of the model parameters chosen at time $t = 0$ such that the model parameter uncertainty only affects the case of a barrier hit.

The cases of a barrier hit are reflected by the first constraints in optimization problem (2.9) guaranteeing that the sum of the calls in the portfolio provide a payoff greater or equal to zero:

$$\alpha_0 B_t + \sum_{C^i \in \mathcal{C}, T_i \geq t} \alpha_i C^i(t, D, y) \geq 0 \qquad \forall\, (t, y) \in \Theta_1$$

However, in these constraints the values of the call prices at the time of a barrier hit $C^i(t, D, y)$ crucially depend on the model parameter p_0 chosen at time $t = 0$. Hence the performance of the resulting hedge portfolio will stand and fall with the sensitivity of the call prices with respect to model parameters.

Observing this, it is an intuitive idea to ask the latter super-replication property in case of a barrier hit to hold for a whole set of model parameters $p \in P$. This leads to the following definition of a robust static hedging strategy.

DEFINITION 3.1. *Assume that the Assumptions of Theorem 2.7 are satisfied such that the problem of finding the cost-optimal static super-replication strategy as stated in Definition 2.3 is equivalent to problem (2.9). Further assume that the functions* α, β *and* σ *describing the financial market model (2.1) depend on model parameters* $x \in \mathbb{R}^k$ *such that* $\alpha = \alpha(y, x)$, $\beta = \beta(y, x)$ *and* $\sigma = \sigma(y, x)$. *Accordingly the call prices* $C^i(t, D, y)$ *also depend on* x *and in addition the correlation coefficient* ρ *of the Brownian motions such that* $C^i(t, D, y) = C^i(t, D, y, p)$ *with* $p := (x^T, \rho)^T \in \mathbb{R}^{k+1}$. *Let* $P \subset \mathbb{R}^{k+1}$ *be a compact set of model parameters and fix* $p_0 \in P$. *Finally assume that the sets* Θ_1 *and* Θ_2 *in Theorem 2.7 are independent of* $p \in P$. *Then a "cost-optimal robust static super-replication strategy" is defined as a solution of the robust* *optimization problem*

$$\begin{aligned} &\min_{\alpha \in \mathbb{R}^{n+1}} \alpha_0 B_0 + \sum_{i=1}^{n} \alpha_i C^i(0, S_0, Y_0, \mathbf{p_0}) \\ s.t. \quad &\alpha_0 B_t + \sum_{C^i \in \mathcal{C}, T_i \geq t} \alpha_i C^i(t, D, y, \mathbf{p}) \geq 0 \quad \forall\ (t, y, \mathbf{p}) \in \Theta_1 \times \mathbf{P} \\ &\alpha_0 B_T + \sum_{C^i \in \mathcal{C}, T_i = T} \alpha_i (s - K_i)^+ \geq (s - K)^+ \quad \forall\ s \in \Theta_2. \end{aligned} \tag{3.1}$$

Clearly, the robust optimization problem (3.1) is a natural extension of the non-robust problem (2.9) taking the model parameter uncertainty into account in the sense of a worst case design. While problem (2.9) only requires the super-replication property to hold for the model parameter p_0 chosen at time $t = 0$, the robust static hedging strategy defined above guarantees this property for a whole set of model parameters $p \in P$. From a financial point of view this means that the hedge portfolio super-replicates the up-and-out call for all possible future market prices which correspond to or are close to volatility surfaces associated with some model parameter $p \in P$.

Although the robust optimization problem (3.1) has attractive properties, it is still unclear if a solution of this problem exists at all. In the remainder of this subsection we will prove the existence of a solution under the following mild conditions.

ASSUMPTION 3.2. *Assume that the coefficient functions α, β and σ of financial market model (2.1) are continuous. Further assume that there exists an open set $\mathcal{O} \subset \mathcal{Y}$ such that α, β and σ are infinitely often differentiable on $\mathcal{O}$ and $\sigma(y), \beta(y) \neq 0$ for all $y \in \mathcal{O}$. Let the support of the joint distribution of (S_t, Y_t) under the equivalent martingale measure Q be equal to $[0, \infty) \times \bar{\mathcal{Y}}$ for all $t \in (0, T]$ with some interval $\mathcal{Y} \subset \mathbb{R}$. In addition assume that the pricing functions $C^i(t, s, y)$ defined in (2.3) are continuously differentiable with respect to t and twice continuously differentiable with respect to s and y.*

Clearly these assumptions are satisfied by the usual stochastic volatility models like Heston's model or the Stein-Stein model. Furthermore it is well known, that under Assumption 3.2 the pricing function $C^i : (0, T_i) \times (0, \infty) \times \mathcal{Y} \to \mathbb{R}$ satisfies the parabolic differential equation

$$\begin{aligned} \frac{1}{2} s^2 \sigma(y)^2 \frac{\partial^2 C^i}{\partial s^2} + \rho s \sigma(y) \beta(y) \frac{\partial^2 C^i}{\partial s \partial y} + \frac{1}{2} \beta(y)^2 \frac{\partial^2 C^i}{\partial y^2} + \\ + rs \frac{\partial C^i}{\partial s} + \alpha(y) \frac{\partial C^i}{\partial y} + \frac{\partial C^i}{\partial t} = r C^i \\ (t, s, y) \in (0, T_i) \times (0, \infty) \times \mathcal{Y} \end{aligned} \tag{3.2}$$

with end condition $C^i(T_i, s, y) = (s - K_i)^+ \ \forall\ s \in (0, \infty)$ and appropriate boundary conditions. These findings will be used to show that the robust hedging problem (3.1) in fact has a solution.

THEOREM 3.3. *Consider the problem of finding a cost-optimal robust static super-replication strategy as stated in Definition 3.1. Assume that $\Theta_1 = [0, T] \times \bar{\mathcal{Y}}$ and $\Theta_2 = [0, D]$. If the financial market model satisfies Assumption 3.2, then a solution of the robust optimization problem (3.1) exists. Furthermore the set of solutions is convex and compact.*

Proof. In analogy to the proof of Theorem 2.4 we obtain, that the feasible set of optimization problem (3.1) is non-empty. Hence, due to well-known theorems in linear semi-infinite optimization (see for example [17]) it is sufficient to show that the objective function and the feasible set have no direction of recession in common. Assume that $d \in \mathbb{R}^{n+1}$ is such a direction of recession. In the following we will show that d must be equal to the zero vector which proves the theorem.

By definition d satisfies

$$\begin{aligned} d_0 B_0 + \sum_{i=1}^{n} d_i C^i(0, S_0, Y_0, p_0) &\leq 0 \\ d_0 B_t + \sum_{C^i \in \mathcal{C}, T_i \geq t} d_i C^i(t, D, y, p) &\geq 0 \quad \forall\ (t, y, p) \in \Theta_1 \times P \\ d_0 B_T + \sum_{C^i \in \mathcal{C}, T_i = T} d_i (s - K_i)^+ &\geq 0 \quad \forall\ s \in \Theta_2 \end{aligned} \tag{3.3}$$

In particular the second inequality holds for the parameter $p_0 \in P$. But then one can show in analogy to the proof of Theorem 2.7 that the set of constraints

$$\begin{aligned} d_0 B_t + \sum_{C^i \in \mathcal{C}, T_i \geq t} d_i C^i(t, D, y, \mathbf{p_0}) &\geq 0 \quad \forall\ (t, y) \in \Theta_1 = [0, T] \times \bar{\mathcal{Y}} \\ d_0 B_T + \sum_{C^i \in \mathcal{C}, T_i = T} d_i (s - K_i)^+ &\geq 0 \quad \forall\ s \in \Theta_2 = [0, D] \end{aligned}$$

is equivalent to $\Pi_T(d) \geq 0$ (a.s.), where $\Pi_t(d)$ denotes the value of the hedge portfolio d at time t. Hence (3.3) implies $\Pi_0(d) \leq 0$, $\Pi_T(d) \geq 0$ (a.s.).

If we assume that $\Pi_0(d) < 0$, then d would be an arbitrage strategy which contradicts the fact that the financial market model is arbitrage-free in the set of static trading strategies. Hence $\Pi_0(d) = 0$ and $\Pi_T(d) \geq 0$ (a.s.). By the same argument $\Pi_T(d)$ must be equal to zero almost surely such that $\Pi_0(d) = 0$, $\Pi_T(d) = 0$ (a.s.). This means that d is a static knock-out trading strategy consisting of a bond and standard options whose values can be matched in such a way that the portfolio payoff at terminal time T is always equal to zero. As the calls in the portfolio either have different strikes or maturities (see Definition 2.2), it is intuitively clear that this can only occur for the portfolio $d = 0$. However, the formal derivation of this result is not that simple and will cover the remainder of the proof.

An analogous argumentation to the proof of Theorem 2.7 shows that the equation $\Pi_T(d) = 0$ (a.s.) is equivalent to

$$\begin{aligned} d_0 B_t + \sum_{C^i \in \mathcal{C}, T_i \geq t} d_i C^i(t, D, y, p_0) &= 0 \quad \forall\ (t, y) \in [0, T] \times \bar{\mathcal{Y}} \\ d_0 B_T + \sum_{C^i \in \mathcal{C}, T_i = T} d_i (s - K_i)^+ &= 0 \quad \forall\ s \in [0, D]. \end{aligned} \tag{3.4}$$

By picking the call $C^i \in \mathcal{C}$ with maturity $T_i = T$ and the minimal strike $K_j := \min\{K_i : C^i \in \mathcal{C}, T_i = T\}$ we deduce from the latter equality that

$$d_j (s - K_j) = -d_0 B_T \quad \forall\ s \in [K_j, K_j + \epsilon]$$

for some $\epsilon > 0$ such that $K_j + \epsilon < K_i$ for all $C^i \in \mathcal{C}$, $T_i = T$, $i \neq j$. As $B_T > 0$ does not depend on s, this can only hold true if $d_j = d_0 = 0$. Similarly one can prove that

$d_i = 0$ for all $C^i \in \mathcal{C}$, $T_i = T$ with $K_i < D$ such that (3.4) is transformed to

$$\begin{aligned} \sum_{C^i \in \mathcal{C}, T_i \geq t, K_i \geq D} d_i C^i(t,s,y,p_0) = 0 \quad & \forall\, (t,s,y) \in [0,T] \times \{D\} \times \bar{\mathcal{Y}} \\ \sum_{C^i \in \mathcal{C}, T_i \geq t, K_i \geq D} d_i C^i(t,s,y,p_0) = 0 \quad & \forall\, (t,s,y) \in \{T\} \times [0,D] \times \bar{\mathcal{Y}}. \end{aligned} \tag{3.5}$$

Hence the value of the static knock-out trading strategy d only consisting of standard calls vanishes for all $(t,s,y) \in ([0,T] \times \{D\} \times \bar{\mathcal{Y}}) \cup (\{T\} \times [0,D] \times \bar{\mathcal{Y}})$. For arbitrary $(t,s,y) \in [0,T] \times [0,D] \times \bar{\mathcal{Y}}$ it is easy to see, that $\sum_{C^i \in \mathcal{C}, T_i \geq t, K_i \geq D} d_i C^i(t,s,y,p_0)$ is the fair value of the static knock-out trading strategy d starting at time t given $S_t = s$ and $Y_t = y$. Due to (3.5) the payoff of this trading strategy is zero in both cases of a barrier hit and no barrier hit before maturity T. Thus, as the market model is arbitrage-free, the fair value of d at time t given $S_t = s$ and $Y_t = y$ must be equal to zero, which is the discounted expected future cash flow. This implies that

$$\sum_{C^i \in \mathcal{C}, T_i \geq t, K_i \geq D} d_i C^i(t,s,y,p_0) = 0 \quad \forall\, (t,s,y) \in [0,T] \times [0,D] \times \bar{\mathcal{Y}}. \tag{3.6}$$

Now we group the standard calls $C^i \in \mathcal{C}$ with $K_i \geq D$ into sets $I_1, ..., I_f$ with equal maturities $\bar{T}_1 < \bar{T}_2 < ... < \bar{T}_f$. By the superposition principle, the function $c(t,s,y) := \sum_{i \in I_f} d_i C^i(t,s,y,p_0)$ satisfies the parabolic differential equation (3.2) with end condition

$$c(\bar{T}_f, s, y) = \sum_{i \in I_f} d_i (s - K_i)^+ \quad \forall s \in (0, \infty).$$

Further (3.6) implies that c vanishes on $(\bar{T}_{f-1}, \bar{T}_f) \times (0,D) \times \mathcal{O}$. Thus we can conclude by Mizohata's uniqueness theorem [22] that $c(t,s,y) = 0$ for all $(t,s,y) \in (\bar{T}_{f-1}, \bar{T}_f) \times (0,\infty) \times \mathcal{O}$. In particular this implies $\sum_{i \in I_f} d_i (s - K_i)^+ = 0$ for all $s \in (0,\infty)$ and hence $d_i = 0$ for $i \in I_f$.

The argumentation above is repeated in analogy on the time strips $(\bar{T}_i, \bar{T}_{i+1}]$. Hence, proceeding recursively, all coefficients d_i have to vanish which proves the theorem. □

The theorem shows, that a solution of optimization problem (3.1) exists under mild conditions satisfied by stochastic volatility models commonly used in practical applications. In case a model does not satisfy Assumption 3.2, the existence can still be guaranteed by adding simple box constraints $\alpha_i^{lb} \leq \alpha_i \leq \alpha_i^{ub}$ to the description of optimization problem (3.1) because the minimization of a linear function on a closed convex and bounded set clearly has a solution.

However, in general an analytic derivation of the solution is not possible. Hence the next subsection will be devoted to the derivation of an algorithm for the numerical solution of the optimization problem.

3.2. Numerical Solution. To solve optimization problems (2.9) and (3.1) numerically, we first observe that these problem are linear semi-infinite optimization problems of the following form

$$\begin{aligned} & \min_{\alpha \in \mathbb{R}^{n+1}} c^T \alpha \\ \text{s.t.} \quad & a_1(t,y,p)^T \alpha \geq 0 \qquad \forall\, (t,y,p) \in S_1 \subseteq \Theta_1 \times P \\ & a_2(s)^T \alpha \geq b_2(s) \qquad \forall\, s \in S_2 = [0,D] \\ & \alpha_i^{lb} \leq \alpha_i \leq \alpha_i^{ub} \end{aligned} \tag{3.7}$$

where c, a_1, a_2 and b_2 denote suitable vectors and scalars, respectively. The additional box constraints allow to impose bounds on the hedge positions and guarantee the existence of a solution for arbitrary models not satisfying Assumption 3.2. Further note that the non-robust problem (2.9) can be obtained by setting $S_1 = \Theta_1 \times \{p_0\}$ and hence any algorithm for the solution of (3.7) can also be applied to solve the non-robust problem.

In general an algorithm solving problem (3.7) will replace the infinite number of constraints associated with the sets S_1 and S_2 by a discrete set of constraints. Let the discrete approximations of these index sets be denoted by $M_1 \subset S_1$ and $M_2 \subset S_2$, $|M_1|, |M_2| < \infty$. By neglecting the rest of the constraints, an optimal solution of the resulting discretized problem is in general not feasible for the original problem (3.7). To reduce this infeasibility one might employ a refinement of the meshs M_1, M_2 around nearly active constraints. However, for typical stochastic volatility models the set S_1 is six-dimensional which prevents a local mesh refinement because of the curse of dimensionality. In these cases cutting plane discretizations are more suitable methods as for example presented in Goberna and Lopez [15]. Applying these methods to problem (3.7) leads to the following algorithm.

ALGORITHM 3.4. *Let $M_1 \subset S_1$ and $M_2 \subset S_2$, $|M_1|, |M_2| < \infty$ be given initial grids and $(\epsilon_k)_{k\in\mathbb{N}}$ a sequence of non-negative numbers converging to zero. Further let $TOL > 0$ be a suitable convergence tolerance and set $k = 0$.*

(S1) Calculate an optimal solution α^k of the discretized problem

$$\begin{aligned} &\min_{\alpha\in\mathbb{R}^{n+1}} c^T\alpha \\ s.t.\quad & a_1(t,y,p)^T\alpha \geq 0 \qquad \forall\ (t,y,p) \in \mathbf{M_1} \\ & a_2(s)^T\alpha \geq b_2(s) \qquad \forall\ s \in \mathbf{M_2} \\ & \alpha_i^{lb} \leq \alpha_i \leq \alpha_i^{ub} \end{aligned} \tag{3.8}$$

(S2) Determine the constraint violation of α^k for problem (3.7) by minimizing the slack–functions at α^k:

$$\delta_1 = \min_{(t,y,p)\in S_1} a_1(t,y,p)^T\alpha^k, \qquad \delta_2 = \min_{s\in S_2} a_2(s)^T\alpha^k - b_2(s) \tag{3.9}$$

In the process of these minimizations identify non-empty finite sets Q_1, Q_2 of ϵ_k-minimizers satisfying

$$\begin{aligned} a_1(t,y,p)^T\alpha^k &\leq \delta_1 + \epsilon_k \quad \forall\ (t,y,p) \in Q_1 \\ a_2(s)^T\alpha^k - b_2(s) &\leq \delta_2 + \epsilon_k \quad \forall\ s \in Q_2. \end{aligned}$$

If $\min(\delta_1, \delta_2) \geq -TOL$ then STOP.

(S3) Add the ϵ_k-minimizers of the slack functions (the most violating constraints) to M_1, M_2 by setting $M_1 \leftarrow M_1 \cup Q_1$, $M_2 \leftarrow M_2 \cup Q_2$. Further set $k \leftarrow k+1$ and go to step (S1).

To solve the robust optimization problem, Algorithm 3.4 successively solves a sequence of linear optimization problems (S1) and nonlinear optimization problems (S2). From a financial point of view, the algorithm first calculates a hedge portfolio in step (S1) guaranteeing the super-replication property for the cases of a barrier hit $(t,y,p) \in M_1$ and the possible stock prices $s \in M_2$ in case of no barrier hit. In step (S2) the algorithm computes the worst case hedge error of the portfolio α^k for all possible states of the economy not considered by the sets M_1 and M_2. The most violating states are then added to the sets M_1 and M_2 leading to a more robust solution in the next iteration. This procedure is repeated recursively until the portfolio delivers a worst case hedge error smaller than TOL. The next theorem shows, that this iterative procedure converges to the desired robust hedge portfolio under suitable assumptions.

THEOREM 3.5. *Assume* $\exists\, M > 0$ *such that* $\|a_1(t,y,p)\| \le M \;\forall\; (t,y,p) \in S_1$. *If the feasible set of problem (3.7) is non-empty and* $\min_{(t,y,p)\in S_1} a_1(t,y,p)^T \alpha^k$ *exists at each iteration of Algorithm 3.4, then every limit point of the sequence* $(\alpha^k)_k$ *is an optimal solution of problem (3.7).*

Proof. Since the feasible set of problem (3.7) is non-empty, closed, convex and due to the box constraints compact, a solution of problem (3.7) exists. By the same argument, an optimal solution of the discretized problem (3.8) exists for arbitrary sets M_1 and M_2.

It is easy to see that $s \mapsto a_2(s)$ is a continuous function such that $\|a_2(s)\|$ is bounded for $s \in [0, D]$. The continuity also implies that $\min_{s\in[0,D]} a_2(s)^T \alpha^k - b_2(s)$ exists for arbitrary iterates α^k. Hence Algorithm 3.4 is well defined. Applying the general convergence theory of linear semi-infinite optimization (see e.g. Goberna and Lopez [15], Theorem 11.2), the theorem immediately follows. □

Note that the boundedness assumption of Theorem 3.5 as well as the existence of the minimum are trivially fulfilled if $S_1 \subseteq \Theta_1 \times P$ is a compact set. However, as illustrated in Example 2.8, the set Θ_1 is closed but for some models not bounded. Hence $\Theta_1 \times P$ is not necessarily bounded as well. In these cases, from a numerical point of view, it is necessary to restrict $\Theta_1 \times P$ to a compact set S_1 in order to carry out the minimizations is step (S2) of the algorithm. This restriction can also be justified from a financial perspective: An unboundedness of Θ_1 can only occur in the volatility direction y of Θ_1 leading to extreme volatility states like $+\infty$ which is of no practical interest. Thus it is natural to exclude extreme volatility states, for example one might choose in case of Heston's model $S_1 = [0,T] \times [0, 100\%] \times P$ instead of $[0,T] \times [0,\infty) \times P$.

Further note that the feasible set of problem (3.7) is non-empty if the box constraints or hedge instruments are chosen appropriately. For example the condition $\alpha_0^{lb} \le (D-K)^+/B_T \le \alpha_0^{ub}$ and $\alpha_i^{lb} \le 0 \le \alpha_i^{ub}$, $i = 1, ..., n$, assures the existence of a static super-replicating strategy (see the proof of Theorem 2.4). But also the simple hedging strategy of buying a regular call with the same strike and maturity as the barrier option is a feasible strategy if the box constraints do not exclude it. Hence the assumptions of Theorem 3.5 are fulfilled for all cases of practical interest such that a solution of problem (3.7) exists and any limit point of the sequence $(\alpha^k)_k$ is an optimal hedge portfolio.

To solve the linear and nonlinear optimization problems arising in steps (S1) and (S2) of Algorithm 3.4, standard optimization solvers can be applied. As it turns out, the computation of δ_1 in step (S2) is the most challenging minimization which also quantifies the model parameter uncertainty. However, for most models it is easy to see, that this optimization problem is smooth except some lines of non-differentiability in time-direction occuring at the maturities T_i of the calls included in the hedge portfolio. As these lines of non-differentiability are well-known in advance, the minimization can be carried out on smooth subregions by Newton-type methods.

4. A Real World Example. In this section we apply the previous findings to a real world example. As in Giese and Maruhn [14] our goal is to hedge an up-and-out call with strike $K = 2750$, barrier $D = 3300$ and a maturity of $T = 1$ year. The underlying of the barrier option is the EuroStoxx50 index with price $S_0 = 2750$ in

September 2004.

In order to solve the optimization problem we first have to select the financial market model under consideration. As mentioned before, the model should be able to price a given set of standard calls with varying strikes and maturities sufficiently well. Stochastic volatility models are well-suited for this task because they usually provide a good fit of a given volatility surface. In our case we choose Heston's stochastic volatility model (see Example 2.8, extended by a dividend yield δ) as the basis of our computations. Besides a good fit of the volatility surface this model has the particular advantage that a closed form solution for the price of standard calls is readily available (see Heston [16]) to significantly speed up computations. Following Toft and Xuan [25] as well as Fink [12] we choose the Heston model parameters as follows: The risk-free interest rate is defined to be $r = 5.5\%$, the dividend yield $\delta = 2.5\%$, the start variance $Y_0 = 0.04$, the long run mean of the variance $\theta = 0.04$, the mean reversion speed $\kappa = 1.5$, the volatility of volatility $\xi = 0.2$ and the correlation $\rho = -0.5$. For this set of parameters, the fair value of the considered up-and-out call is 1.60% in percent of the underlying S_0.

The next step to set up a static hedging strategy is to select the hedge instruments which should be included in the hedge portfolio. In contrast to other static hedging approaches the optimization framework of the cost-optimal static super-replication concept allows us to choose hedge instruments that are really traded in the market. To identify the most suitable instruments one might first feed the optimizer with a large number of available calls. Based on the optimal portfolio weights for this huge portfolio it is easy to reduce the number of calls to the most efficient hedge instruments. Following this approach we identified the set of standard calls listed in Table 4.1 which were traded on the EUREX in September 2004.

TABLE 4.1
Standard calls C_i included in the hedge portfolio

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}
T_i	0.50	0.50	0.75	0.75	0.75	1.00	1.00	1.00	1.00	1.00
K_i	3300	3500	3300	3400	3600	2750	3300	3350	3450	3600

Obviously, the set of calls $\{C_1, ..., C_{10}\}$ satisfies Assumption 2.5. Hence, by Theorem 2.7, optimization problem (2.7) is equivalent to the semi-infinite optimization problem (2.9) with $\Theta_1 = [0, T] \times [0, \infty)$ and $\Theta_2 = [0, D]$ (see Example 2.8). If we define the parameter vector $p = (\kappa, \theta, \xi, \rho)^T$, it is easy to see that Heston's model satisfies the assumptions of Definition 3.1 as well as Assumption 3.2. Accordingly Theorem 3.3 implies that the robust optimization problem (3.1) has a solution and that the set of optimal solutions is convex and compact.

To ensure that the discretized subproblems (3.8) also have a solution in each iteration of Algorithm 3.4 we impose the simple bounds $\alpha_i^{lb} = -50$ and $\alpha_i^{ub} = 50$ on the portfolio positions. These bounds further guarantee that the trading strategy $\alpha_6 = 1$, $\alpha_i = 0$, $i \neq 6$, solely consisting of the call C_6 with the same strike and maturity as the barrier option is feasible for problem (3.7). Thus Theorem 3.5 implies for compact sets $S_1 \subseteq \Theta_1 \times P$ that every limit point of the sequence generated by Algorithm 3.4 is an optimal solution of optimization problem (3.7).

4.1. The Non-Robust Static Hedging Strategy. The non-robust optimization problem is a special case of problem (3.7) by setting $P = \{p_0\}$ with $p_0 = (\kappa_0, \theta_0, \xi_0, \rho_0) := (1.5, 0.04, 0.2, -0.5)^T$. In order to solve the resulting problem numerically, we have to restrict the unbounded set $\Theta_1 = [0,T] \times [0,\infty)$ to a suitable compact subset. From a financial point of view it is reasonable to bound the variance in case of a barrier hit by 100% such that we replace $\Theta_1 \times \{p_0\}$ by $S_1 := [0,T] \times [0,1] \times \{p_0\}$.

In terms of Algorithm 3.4 we choose a first discretization M_1, M_2 of S_1, S_2 consisting of a total of 252 grid points. For this setup Table 4.2 shows, that the algorithm terminates after 13 iterations with a worst case hedge error $\min(\delta_1, \delta_2)$ satisfying the prespecified tolerance TOL $= 10^{-7}$. Here the cost as well as the worst case hedge error of the portfolio are listed in percent of the underlying S_0. Obviously the cost of the hedge portfolio converges to approximately 1.80% which is not much more expensive than the fair value of the barrier option (1.60%). In particular the price difference is less than half of the typical bid-ask spread for barrier options in the OTC market.

Table 4.2
Iteration Process for the Non-Robust Problem

Iteration	WC-Hedge Error	$\text{cost}(\alpha^k)$	$\lvert M_1\rvert + \lvert M_2\rvert$	ϵ_k
0	-2.037945e-001	1.419938e-002	252	1.00000e-002
1	-6.515309e-003	1.768397e-002	266	1.00000e-003
2	-1.378396e-003	1.791617e-002	273	1.00000e-004
3	-2.007605e-003	1.792881e-002	275	1.00000e-005
4	-7.201896e-004	1.796985e-002	285	1.00000e-006
5	-4.653767e-004	1.800102e-002	288	1.00000e-007
6	-9.036166e-005	1.800403e-002	298	1.00000e-008
7	-2.090027e-004	1.800496e-002	302	1.00000e-008
8	-1.616535e-005	1.800862e-002	304	1.00000e-008
9	-4.707759e-005	1.800986e-002	311	1.00000e-008
10	-1.125056e-005	1.801157e-002	313	1.00000e-008
11	-3.552461e-006	1.801166e-002	322	1.00000e-008
12	-1.999220e-006	1.801171e-002	329	1.00000e-008
13	-3.458727e-008	1.801171e-002	334	1.00000e-008

The optimal solution of the problem is presented in Table 4.3. The optimal hedge portfolio is very close to the one found by Giese and Maruhn in [14] by a Monte-Carlo based discretization of problem (2.8). However, due to the semi-infinite equivalence and by exploiting this structure in Algorithm 3.4, we are now able to compute the same hedge portfolio in several seconds instead of several hours.

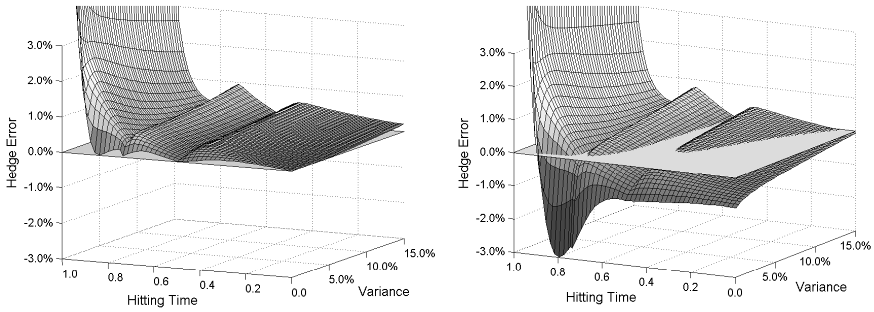
Table 4.3
Optimal Portfolio Weights α_i for the Non-Robust Problem

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}
T_i	0.50	0.50	0.75	0.75	0.75	1.00	1.00	1.00	1.00	1.00
K_i	3300	3500	3300	3400	3600	2750	3300	3350	3450	3600
α_i	0.23	-0.23	0.35	-0.26	-0.42	1.00	-25.29	29.28	-5.97	1.42

Figure 4.1 (a) gives some further insight into the structure of the hedge error in case of a barrier hit. Clearly the optimal strategy is a super-replication strategy guaranteeing a payoff greater or equal to zero on the barrier. Shortly before the maturity $T = 1$ of the barrier option the value of the hedge portfolio bends upwards to $(D-K)/S_0 = 20\%$ to super-replicate the barrier option in case of no barrier hit. Furthermore it is easy to recognize the lines of non-differentiability along the maturities of the standard calls in the portfolio ($t = 0.5$ and $t = 0.75$). In summary, the optimal

hedge portfolio presented in Table 4.3 behaves as expected offering protection for a wide range of volatility states and hitting times.

FIG. 4.1. *Hedge Error on the Barrier for the non-robust Portfolio*



(a) Parameter $p_0 = (1.5, 0.04, 0.2, -0.5)^T$ (b) Perturbed $p = (1.4, 0.025, 0.25, -0.6)^T$

However, this well-behaved evolvement of the hedge error completely changes if the model parameters p in case of a barrier hit differ from those used for the computation of the hedge portfolio ($p_0 = (1.5, 0.04, 0.2, -0.5)^T$). Figure 4.1 (b) illustrates graphically that the value of the hedge portfolio and hence the hedge error on the barrier is extremely sensitive to changes in the model parameters. The perturbation from p_0 to $p = (1.4, 0.025, 0.25, -0.6)^T$ leads to possibly huge hedging losses of up to 3% which is clearly unacceptable in comparison to the price of the barrier option (1.6%). This model parameter uncertainty is due to the complex nonlinearity of the call prices and hence the function $a_1(t, y, p)$ in problem (3.7).

To better quantify the effect of the model parameter uncertainty, we analyze the worst case hedging error δ_1 (see (3.9) in Algorithm 3.4) of the non-robust hedge portfolio for varying model parameter sets P. Starting with the non-robust case $P = \{p_0\}$ we successively increase P to capture the risk of more wildly moving model parameters. For this purpose it is sufficient to model P as multi-dimensional intervals $[\kappa_{\min}, \kappa_{\max}] \times [\sqrt{\theta_{\min}}, \sqrt{\theta_{\max}}] \times [\xi_{\min}, \xi_{\max}] \times [\rho_{\min}, \rho_{\max}]$ around $(\kappa_0, \sqrt{\theta_0}, \xi_0, \rho_0)^T$ with increasing diameter Δ. Of course, in addition the points in P must satisfy the Heston cone constraint $\kappa\theta - \xi^2/2 \geq 0$ guaranteeing the positivity of the variance process $(Y_t)_t$. The resulting worst case hedge errors are shown in Table 4.4.

TABLE 4.4
Worst case hedge error δ_1 of the non-robust portfolio for varying model parameter sets P.

Diameter Δ	0%	5%	10%	15%	20%	30%	40%
WC-Error	0.0%	-1.20%	-2.90%	-3.90%	-4.83%	-5.82%	-6.91%

The numbers shown in Table 4.4 precisely quantify the model parameter uncertainty of the non-robust hedge portfolio. For a fixed model parameter p_0 ($\Delta = 0\%$) the portfolio offers perfect protection over the lifetime of the barrier option. However, if model parameters only change slightly the nonlinearity of the call option prices leads to hedging losses that can be multiples of the fair value of the barrier option. As implied model parameters p change daily, the model parameter uncertainty cannot be neglected and must be included in the design of static hedge portfolios. This will be the focus of the next subsection.

4.2. Adding Robustness to the Hedge Portfolio. Due to the strong effect of model parameter uncertainty presented in the previous subsection we now aim at adding robustness to the static hedge portfolio. By definition the robust optimization problems (3.1) and (3.7) allow to easily take model parameter uncertainty into account by means of a set P in which the parameters p are allowed to vary without loosing the super-replication property.

Hence the question arises of how to choose a suitable set P. Of course the set should be chosen such that the implied model parameters p stay within P over the lifetime of the barrier option. At the first sight this does not seem like an easy task, but a quick analysis of the evolvement of implied model parameters over time (see Figure 4.2 showing implied model parameters over a time span of 16 months) reveals the desired information. Although the parameters $p = (\kappa, \theta, \xi, \rho)^T$ vary stochastically over time, the movement is not arbitrary. Instead the parameters stay within certain bounds which allows us to capture them in appropriate robustness intervals. Thus it is reasonable to choose P as a multidimensional interval of the form $P = [\kappa_{\min}, \kappa_{\max}] \times [\sqrt{\theta_{\min}}, \sqrt{\theta_{\max}}] \times [\xi_{\min}, \xi_{\max}] \times [\rho_{\min}, \rho_{\max}]$.

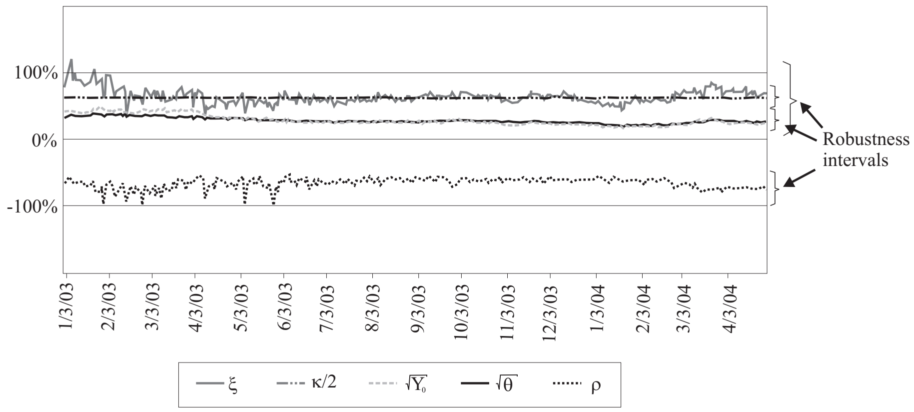


FIG. 4.2. *Implied Heston Parameters over time for the EuroStoxx50 index resulting from daily calibrations with soft penalty. Source: Hans Buehler [6], Deutsche Bank AG.*

In addition Figure 4.2 gives another surprising insight. The implied short term volatility $\sqrt{Y_0}$ over time does not take arbitrary values from 0% to 100% either. Hence it is not necessary to require the super-replication property to hold for the whole variance set $[0\%, 100\%]$ as we did in Subsection 4.1 for the non-robust problem. In contrast to this we will now choose $S_1 := [0, T] \times [Y_{\min}, Y_{\max}] \times P \subsetneq \Theta_1 \times P$ in problem (3.7). By doing this we give up the theoretical super-replication property $\Pi_T(\alpha) \geq C_{uo}$ (a.s.) for the Heston model with fixed model parameter p_0 (see the equivalence Theorem 2.7), but focus on what is really important for the design of a good robust static hedge portfolio: The value of the hedge portfolio must be greater or equal to zero in case of a barrier hit. As the value of the hedge portfolio at the time of a barrier hit is given by the future call prices (which are part of the future volatility surface), it is sufficient to require the super-replication property to hold for a set of possible future implied model parameters. And this property is not reflected by an almost-surely constraint but precisely by the semi-infinite constraints in problem (3.7) with an interval-type set $S_1 \subsetneq \Theta_1 \times P$.

For our computations we choose intervals $[\sqrt{Y_{\min}}, \sqrt{Y_{\max}}]$ and P with diameter

Δ around the implied model parameters $\sqrt{Y_0}$ and $(\kappa_0, \sqrt{\theta_0}, \xi_0, \rho_0)$ at time $t = 0$. The points in P of course again have to satisfy the Heston constraint $\kappa\theta - \xi^2/2 \geq 0$ which simply can be included as an additional nonlinear constraint in the computation of the worst case hedging error (3.9). We start Algorithm 3.4 with a first discretization M_1, M_2 of the sets S_1, S_2 consisting of a total of 5496 grid points. For robustness intervals with a diameter of $\Delta = 15\%$ Table 4.5 shows the resulting iteration process.

TABLE 4.5
Iteration Process for the Robust Problem with diameter $\Delta = 15\%$ of P

Iteration	WC-Hedge Error	$\text{cost}(\alpha^k)$	$\|M_1\| + \|M_2\|$	ϵ_k
0	-3.214171e-002	1.743007e-002	5496	1.00000e-002
1	-3.088605e-003	1.806816e-002	5506	1.00000e-003
2	-1.520504e-003	1.816625e-002	5511	1.00000e-004
3	-1.460408e-003	1.827471e-002	5520	1.00000e-005
4	-3.307436e-004	1.842930e-002	5529	1.00000e-006
5	-9.002903e-005	1.845258e-002	5534	1.00000e-007
6	-2.529412e-005	1.845378e-002	5540	1.00000e-008
7	-5.717408e-006	1.845461e-002	5544	1.00000e-008
8	-6.045258e-006	1.845462e-002	5547	1.00000e-008
9	-1.400333e-005	1.845472e-002	5556	1.00000e-008
10	-3.262932e-007	1.845530e-002	5562	1.00000e-008
11	-7.967239e-008	1.845530e-002	5566	1.00000e-008

The Algorithm terminates after 11 iterations with a worst case hedging loss less than TOL $= 10^{-7}$. The cost of the portfolio converges to 1.84% which is still surprisingly cheap for a super-replication portfolio with a 15%-robustness against model parameter uncertainty. The optimal solution listed in Table 4.6 differs significantly from the solution of the non-robust problem and offers a lot more protection against hedging errors.

TABLE 4.6
Optimal Portfolio Weights α_i for the Robust Problem with diameter $\Delta = 15\%$

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}
T_i	0.50	0.50	0.75	0.75	0.75	1.00	1.00	1.00	1.00	1.00
K_i	3300	3500	3300	3400	3600	2750	3300	3350	3450	3600
α_i	0.01	0.10	0.04	0.31	-0.29	1.00	-30.42	37.95	-9.13	0.47

Of course the robustness of the hedge portfolio listed in Table 4.6 is limited to model parameter changes within the prespecified intervals with a diameter of 15% around p_0. However, traders might prefer more conservative hedge portfolios offering protection for an even wider range of model parameters. For this purpose, Table 4.7 illustrates the cost of static hedge portfolios with varying degree of robustness. As expected the cost increases with an increasing size of the robustness intervals. But still the cost is surprisingly low in comparison to the potential hedging losses if the model parameter uncertainty is not taken into account (see Table 4.4). In particular the price difference of a 20% robust portfolio and the fair value of the barrier option (1.60%) is still within the typical bid-ask spread for barrier options in the OTC market.

5. Conclusions. In this paper we extended the robust static super-replication approach developed by Maruhn and Sachs in [21], which is based on the work of Giese and Maruhn in [14], to general stochastic volatility models. As it turned out, the

Table 4.7
Cost of optimal hedge portfolios with varying degree of robustness.

Diameter Δ	5%	10%	15%	20%	30%	40%
$\sqrt{Y_{\min}} = \sqrt{\theta_{\min}} = \xi_{\min}$	17.5%	15.0%	12.5%	10.0%	10.0%	10.0%
$\sqrt{Y_{\max}} = \sqrt{\theta_{\max}} = \xi_{\max}$	22.5%	25.0%	27.5%	30.0%	40.0%	50.0%
$\kappa_{\min}$	147.5%	145.0%	142.5%	140.0%	130.0%	120.0%
$\kappa_{\max}$	152.5%	155.0%	157.5%	160.0%	160.0%	160.0%
$\rho_{\min}$	-52.5%	-55.0%	-57.5%	-60.0%	-70.0%	-80.0%
$\rho_{\max}$	-47.5%	-45.0%	-42.5%	-40.0%	-40.0%	-40.0%
Cost	**1.62%**	**1.73%**	**1.84%**	**1.95%**	**2.07%**	**2.17%**

resulting stochastic optimization problem can be proven to be equivalent to a linear semi-infinite optimization problem. By exploiting this structure, the computation time for the optimal static hedging strategy can be reduced from several hours to several seconds.

Although the non-robust super-replication strategy already has very attractive properties, the numerical results show that the model parameter uncertainty of static hedge portfolios for barrier options cannot be neglected. Due to the strong nonlinearity of the call option prices even small perturbations of the model parameters can lead to extreme losses amounting to multiples of the fair value of the barrier option.

However, as we have shown, the linear semi-infinite optimization problem allows to easily incorporate the model parameter uncertainty into the design of the static hedging strategy. After proving existence of such a robust hedging strategy we presented an algorithm to numerically solve the optimization problem. As a byproduct, the algorithm also allows to quantify the model parameter uncertainty by computing the worst case hedging error. Hence, by successively solving a sequence of linear and nonlinear optimization problems, the algorithm generates iterates converging to the optimal static hedge portfolio.

A detailed numerical example showed the applicability of our results. Although the model parameter uncertainty of static hedge portfolios for barrier options is huge, it can be eliminated by surprisingly low cost. In particular our method allows to compute robust static hedge portfolios with a price difference to the fair value of barrier options that is within the typical bid-ask-spread in the OTC market.

Furthermore, the presented method allows another surprising insight. It is not decisive for the performance of a static hedge portfolio for barrier options to super-replicate a barrier option almost surely within some mathematical model with fixed parameters. Instead the key issue is that the hedging strategy guarantees the super-replication property for the implied model parameters at the time of a barrier hit. As these parameters and the corresponding call prices are not known in advance, the super-replication property has to be guaranteed for a prespecified set of model parameters. This very intuitive requirement is exactly the definition of our robust static hedging strategy. Traders usually have a very good feeling of how implied model parameters change over time such that the robust parameter sets can easily be chosen.

By gaining these insights it is also possible to quantify what makes up a good model for static hedging of barrier options. First of all, it is important that the model calibrates well to volatility surfaces, because the fit implies how close the price of the hedge portfolio at the time of a barrier hit is to the real price in the market. Furthermore it is desired to choose the robust parameter sets as small as possible. Thus the implied model parameters over time should not vary too wildly. Hence a model is not a good model if it has thousands of parameters, but if it leads to a good fit with stable parameters over time.

As we showed Heston's stochastic volatility model combines both desirable properties such that it is well suited for the computation of static hedge portfolios. Based on the optimization framework it is possible to identify the best super-replication portfolio consisting of calls that are really traded in the market. As the resulting optimal portfolio super-replicates the barrier option in reality, the price is also an upper bound for the price of a barrier option. The numerical results showed, that in contrast to other static hedging approaches the upper bound we obtain is very sharp. Hence pricing of a barrier option can be tackled with the answer of the question: What is the cost of the optimal robust static hedging strategy?

In addition the optimization approach allows to easily incorporate some extensions which were not considered in this paper. First of all we assumed that the hedge portfolio can be liquidated exactly on the barrier D. However, a hedging delay can simply be included in the problem setup by asking the super-replication to hold for a stock price interval $[D, S_{\max}]$ instead of the simple stock price set $\{D\}$. This increases the dimension of the semi-infinite parameter space by 1, but does not make the problem more complicated. In particular the resulting semi-infinite programming problem can be solved by the algorithm presented in this paper.

Furthermore it is also possible to add an additional robustness against model errors. Although the hedging strategy might guarantee the super-replication property for the implied model parameters at the time of a barrier hit, the corresponding model option prices can still slightly differ from the market prices. To cope with this problem, Leibfritz and Maruhn [19] further robustify the hedge portfolio with respect to slight perturbations of the model option prices for each implied model parameter set. This leads to a semi-definite programming approach which can be tackled by standard optimization techniques.

Hence the approach presented in this paper adresses model parameter uncertainty and can even be extended to account for model errors. The derivation of cheap static hedge portfolios considering both uncertainties requires to exploit the relation of call prices for various strikes and maturities (the volatility surface) as much as possible. In our oppinion the semi-infinite programming method presented in this paper is more suitable for this task than a completely model-independent approach.

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