On the large investor's utility maximization problem in models with different liquidity effects

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What are illiquid markets ?

- Buy and sell orders cannot be executed at a fixed price
- Example: bid-ask spread
- In general: trading strategy in an asset has an impact on the price process of the asset (" feedback effects"): purchases let prices rise, sales let prices going down
- Prices are determined by market clearing condition **supply(price)** $\overset{!}{=}$ **demand(price)** auctions, continuous trading (electronic trading system)
- How to bring this idea together with continuous time models ?

Illiquid markets

- Model: a single large investor and many (accumulated) small ones. **Aim:** to model dynamic trading opportunities of large investor

- Small investors are *price takers*: they don't anticipate their own influence on the prices & have *constant risk aversion*.

- Large investor has complete information about the order book (e.g. satisfied for Xetra-Handel)

- $M \in \mathbb{R}_+$ total number of shares
  $(\beta_t, \theta_t)_{t\in[0,T]}$: dynamic trading strategy of large investor
  β_t number of liquid bonds with price process 1 at time t
  θ_t number of illiquid (risky) assets at time t

One model: Jarrow (1992, 1994), Bank & Baum (2004)

- *market clearing condition* at time t

$$\underbrace{\theta_t - \theta_{t-\triangle t}}_{\text{large investor}} + \underbrace{f(p,t,\omega) - (M - \theta_{t-\triangle t})}_{\text{small traders}} \overset{!}{=} 0,$$

where p denotes the equilibrium price to be determined

$$p = f^{-1}(\cdot, t, \omega)(M - \theta_t)$$

$\rightsquigarrow$ asset price is a function of **current** number of shares θ_t the large investor owns (not of $\theta_t - \theta_{t-\triangle t}$!)

- modelling: $(S(\vartheta, \cdot))_{\vartheta \in \mathbb{R}}$ family of semimartingales,
  $S(\vartheta, t)$: market price if large investor holds ϑ shares at time t
  $S(\vartheta, t) = S(\vartheta, t) + M(\vartheta, t) + A(\vartheta, t)$
  $M(\vartheta, \cdot)$ P-local martingale, $A(\vartheta, \cdot)$ process of finite variation

Relaxation of the assumption that all small agents are permanently updating their positions

$\rightsquigarrow$ liquidity can also be solely **temporary** imbalance of demand and supply $\rightsquigarrow$ quick transactions are expensive

Models with solely short-term price impact: Rogers, Singh (2006), Çetin, Jarrow, Protter (2004).
Unified approach K. (2006)

$\mu : \mathcal{B}([0,T]) \to [0,1]$ continuous, monotone **subadditive** set function with $\mu(\emptyset) = 0$ and $\mu([0,T]) = 1$.
Interpretation: $\mu(A)$ is fraction of small traders updating their positions during $A \subset [0,T]$. Assume that we have $\forall A_1 \cap A_2 = \emptyset$

$$\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2) - \mu(A_1)\mu(A_2) \quad \text{memoryless}$$

Assume $\sum_{t\in[0,T]} \mu(\{t\}) < \infty$ (finite activity). Let $\tilde{\mu}$ be the smallest **measure** dominating μ. We have

$$\mu(A) = 1 - \prod_{t\in A} (1 - \tilde{\mu}(\{t\})\exp\left(-\tilde{\mu}(A) + \sum_{t\in A} \tilde{\mu}(\{t\})\right)$$

Example 1: $\mu((t_1, t_2]) = 1 - \exp(-\lambda(t_2 - t_1)) \quad \forall 0 \leq t_1 \leq t_2 \leq T,\ \ \lambda > 0$
$\rightsquigarrow \tilde{\mu}((t_1, t_2]) = \lambda(t_2 - t_1)$ ”continuous trading”
I.e., the amount of orders during a time interval $(t_1, t_2]$ (against which the orders of the large investor can be executed) is of order $t_2 - t_1$. Consequently, $|d\theta_t| \ll dt$

Example 2: $\mu((t_1, t_2]) = 1 - \exp\left(-\sum_{i,\ t_1 < T_i \leq t_2} \alpha_i\right),\ a_i \in (0, 1],\ 0 \leq T_1 < T_2 < \ldots \leq T$ ”auctions”
$\rightsquigarrow \tilde{\mu}((t_1, t_2]) = \sum_{i,\ t_1 < T_i \leq t_2} \alpha_i.$

The model of Bank and Baum (2004) corresponds to $\mu(A) = 1$ $\forall A \neq \emptyset$ (which does not satisfy $\sum_{t \in [0,T]} \mu(\{t\}) < \infty$).

Construction of the model

Given: time continuous process $\theta = (\theta_t)_{t \in [0,T]}$ with $\theta_0 = 0$ and initial endowment β_0 in bonds
To determine: evolution of number of bonds $\beta = (\beta_t)_{t \in [0,T]}$ i.e. find β s.t. the strategy (β, θ) is self-financing (this characterizes the model)
Problem: simple strategies (e.g. $\theta = 1_{A \times (t_1, t_2]}, \;\; A \in \mathcal{F}_{t_1}$) may not be feasible $\rightsquigarrow$ usual approach does not work. Therefore: two-stage process
First step: take a sequence of grids, e.g. $T^n_k = \frac{k}{2^n}$ Discrete-time *market clearing condition* at time $\frac{k}{2^n}$

$$\underbrace{\theta_{\frac{k}{2^n}} - \theta_{\frac{k-1}{2^n}}}_{\text{large investor}} + \underbrace{\mu\big(\big(\frac{k-1}{2^n}, \frac{k}{2^n}\big]\big) \left[f\big(p, \frac{k}{2^n}, \omega\big) - \big(M - \theta_{\frac{k-1}{2^n}}\big)\right]}_{\text{small traders}} \overset{!}{=} 0,$$

where p denotes the equilibrium price to be determined.

Solving for p yields

$$\begin{aligned} p &= f^{-1}\left(\cdot, \frac{k}{2^n}, \omega\right)\left(M - \theta_{\frac{k-1}{2^n}} - \frac{\theta_{\frac{k}{2^n}} - \theta_{\frac{k-1}{2^n}}}{\mu((\frac{k-1}{2^n}, \frac{k}{2^n}])}\right) \\ &= S\left(\vartheta, \frac{k}{2^n}\right), \quad \text{with } \vartheta = \theta_{\frac{k-1}{2^n}} + \frac{\theta_{\frac{k}{2^n}} - \theta_{\frac{k-1}{2^n}}}{\mu((\frac{k-1}{2^n}, \frac{k}{2^n}])}. \end{aligned}$$

The discrete-time self-financing condition becomes

$$\beta^n_{\frac{k}{2^n}} - \beta^n_{\frac{k-1}{2^n}} = -\left(\theta_{\frac{k}{2^n}} - \theta_{\frac{k-1}{2^n}}\right) S\left(\theta_{\frac{k-1}{2^n}} + \frac{\theta_{\frac{k}{2^n}} - \theta_{\frac{k-1}{2^n}}}{\mu((\frac{k-1}{2^n}, \frac{k}{2^n}])}, \frac{k}{2^n}\right), \; k = 1, \ldots$$

Theorem K.(2006) Assume that $\theta_t = \int_0^t y_s \, \tilde{\mu}_s$, $t \in [0, T]$ with $\int_0^T y_s^2 \, \tilde{\mu}_s < \infty$ (θ should be predictable). Then, $(\beta^n)_{n \in \mathbb{N}}$ converges for $n \to \infty$ to the process

$$\beta_t = \beta_0 - \int_0^t y_s S(\theta_{s-} + y_s, s) \, d\tilde{\mu}_s, \quad t \in [0, T].$$

uniformly in probability.

Merton's problem

Given $\beta_0 \in \mathbb{R}$, $\theta_0 = 0$

$E\left(u(\beta_T)\right)$ max! $\quad \theta = \int_0^{\cdot} y_s \, d\tilde{\mu}_s$, $\theta_T = 0$, $\quad (\beta, \theta)$ self-financing

$u : \mathbb{R} \to \mathbb{R}$, increasing, concave, $\{u > -\infty\} = \mathbb{R}$, ...

Theorem K. (2006) The problem possesses a unique maximizer $\theta_t = \int_0^t y_s \, d\tilde{\mu}_s$. Let $\tilde{\mu}$ be continuous and $\beta_t = \beta_0 - \int_0^t y_s S(\theta_{s-} + y_s, s) \, d\tilde{\mu}_s$ be the corresponding evolution of the riskless bank account. Assume that y is càdlàg. Then, under the measure Q with

$$\frac{dQ}{dP} = \frac{u'(\beta_T)}{Eu'(\beta_T)}.$$

the process $t \mapsto$

$$\underbrace{\int_0^t S(\theta_s, ds)}_{\text{exogenous price moves}} + \underbrace{S(\theta_t + y_t, t) - S(\theta_t, t) + y_t \partial_1 S(\theta_t + y_t, t)}_{\text{marginal transaction/liquidity costs}}$$

is a local martingale.

How to evaluate stockholdings within this model ?

General assumption: $S(\vartheta_1,\cdot) \le S(\vartheta_2,\cdot)$ for $\vartheta_1 \le \vartheta_2$
(otherwise arbitrage for large investor)

Consider the case $\mu(A) = 1,\ \forall A \neq \emptyset$, permanent updating

What can we achieve by selling y stocks – quickly but not sillily ?

silly: sell everything at the same time $\rightsquigarrow$ $yS(0,t)$

liquidation value $:= \underbrace{x \cdot 1}_{x\text{: number of bonds}} + L(y,t) := x + \int_0^y S(\vartheta,t)\,d\vartheta$

value process V = liquidation value process.

Property of liquidation value process $\beta_t + \int_0^{\theta_t} S(\vartheta, t)\, d\vartheta$:
continuous rebalancing of finite variation does not change $V \rightsquigarrow$ **free round trips** Itô-Wentzell-formula yields in case that θ is a (predictable) semimartingale)

$$\mathbf{V_t}(\theta) \;=\; \mathbf{v_0} + \underbrace{\int_{\mathbf{0}}^{\mathbf{t}} \mathbf{L}(\theta_{\mathbf{s}},\, \mathbf{ds})}_{\text{non-linear integral}} - \frac{1}{2}\int_0^t \partial_1 S(\theta_{s-}, s-)\, d[\theta, \theta]_s^c$$

$$+ \sum_{0<s\le t} \left\{ \int_{\theta_{s-}}^{\theta_s} S(\vartheta, s-)\, d\vartheta - \Delta\theta_s S(\theta_s, s-) \right\}$$

For the idealized model we drop the condition that θ is a semimartingale.
Idealization: $V_t(\theta) = v_0 + \int_0^t L(\theta_s,\, ds)$, i.e. value process only varies due to **exogenous** price movements

Utility maximization

Bank & Baum (2004): solution under the assumption that there is a **joint** equivalent martingale measure for all processes $S(\vartheta,\cdot)$, $\vartheta \in \mathbb{R} \rightsquigarrow$ in this case solution similar to liquid case (with only $S(0,\cdot)$ as the price process).

Typical situation: the P-drift rate of $S(\vartheta,\cdot)$ is decreasing in ϑ

Theorem Under some conditions ... e.g. the P-drift rate of $S(\vartheta,\cdot)$ is nonincreasing in ϑ there exists a unique optimal strategy $\widehat{\theta}$. We have $E\left[u'(V_T(\widehat{\theta}))\right] < \infty$. Define $Q \sim P$ by

$$\frac{dQ}{dP} = \frac{u'(V_T(\widehat{\theta}))}{Eu'(V_T(\widehat{\theta}))}.$$

For every $\vartheta \in \mathbb{R}$ the "canonical" Q-drift rate of $S(\vartheta,\cdot)$ vanishes on the (predictable) set $\{(\omega,t) \in \Omega \times [0,T] \mid \widehat{\theta}_t(\omega) = \vartheta\}$.
"Increase investment as long as the Q-drift rate is positive"

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