On the Pricing of Auto-Callable Equity Structures in the Presence of Stochastic Volatility and Stochastic Interest Rates

Alexander Giese

Outline

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- 2. Requirements for the Pricing Model
- 3. Black-Scholes & Vasicek++ Model
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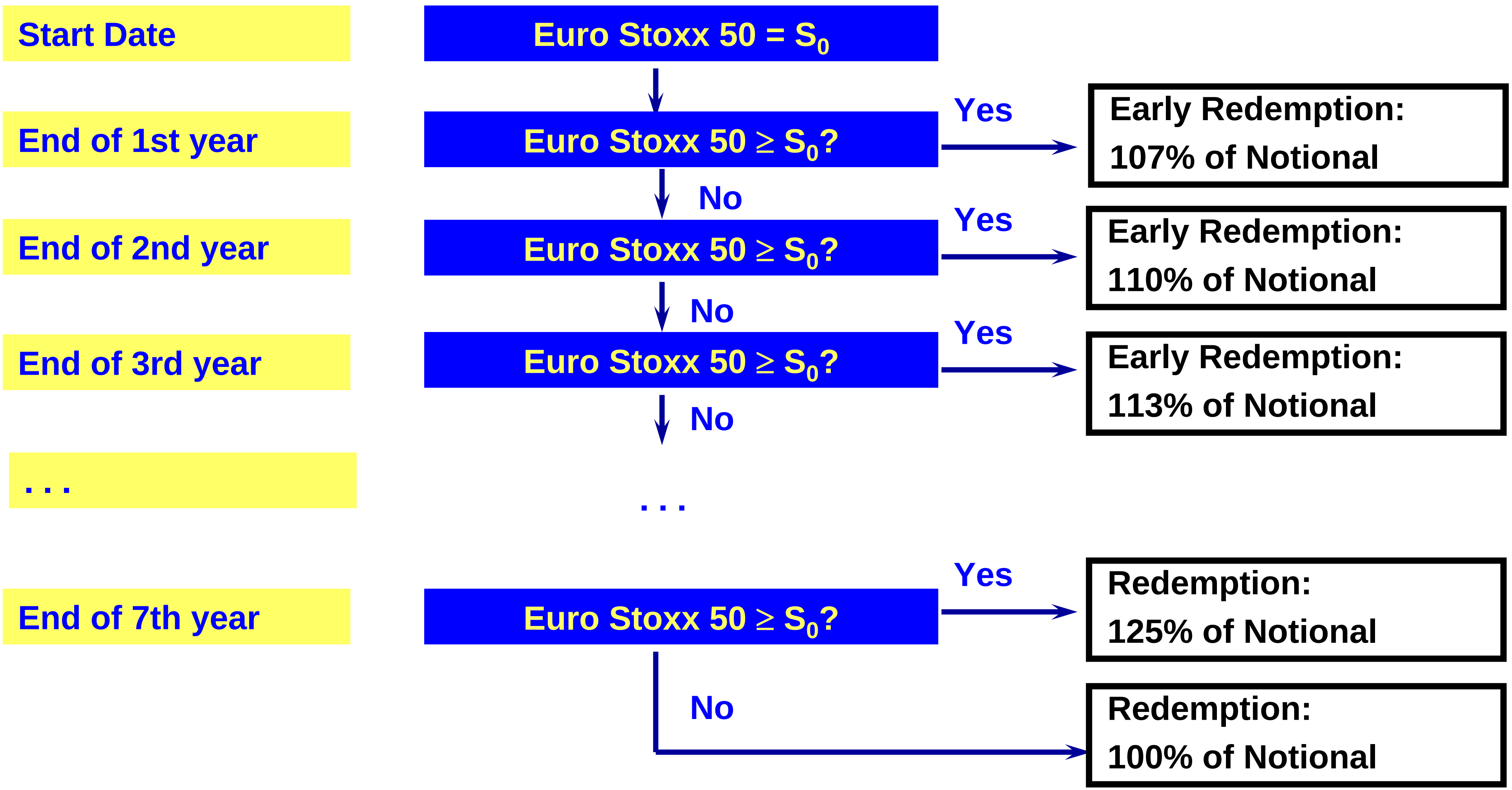
1. Introduction to Auto-Callable Structures

Characterization

An auto-callable equity structure is a structured product which depending on the path of the equity underlying is automatically called and redeemed early on pre-prescribed dates known as the auto-call dates.

Auto-callable structures are also called auto-trigger or express structures.

1. Introduction to Auto-Callable Structures



1. Introduction to Auto-Callable Structures

Some Figures:

- There are currently more than 235 auto-callable equity certificates ("express certificates") in the German certificate market.
- The open interest (units x market price) in these products is about €4.7 billion which corresponds to a 9% share of the overall German certificate market.
- HVB issued recently its 100th auto-callable equity certificate.

Source: Derivate Forum, Der Deutsche Markt für Derivative Produkte, Januar 2006.

2. Requirements for the Pricing Model

Closed-form Solution in the Black-Scholes Model with time-dep Volatility:

$$\text{Price of Auto-Call Structure} = \sum_{i=1}^{8} V_i$$

where

$$V_1 = 107 \times df(T_1) \times P^*[S(T_1) \geq K]$$
$$= 107 \times df(T_1) \times N_1(d(K,T_1))$$
$$V_2 = 110 \times df(T_2) \times P^*[S(T_1) < K, S(T_2) \geq K]$$
$$= 110 \times df(T_2) \times N_2(-d(K,T_1), d(K,T_2), \Sigma_2)$$

...

$$V_7 = 125 \times df(T_7) \times N_7(-d(K,T_1), -d(K,T_2) \ldots, d(K,T_7), \Sigma_7)$$
$$V_8 = 100 \times df(T_7) \times N_7(-d(K,T_1), -d(K,T_2) \ldots, -d(K,T_7), \Sigma_7)$$

2. Requirements for the Pricing Model

Note that for the computation of the n-dimensional normal integrals we can apply the following reduction formula (Schroder (1989)):

$$N_n(b_1,\ldots,b_n;\Sigma)=\int_{-\infty}^{b_s} N_{s-1}(b_1^{'},\ldots,b_{s-1}^{'};\Sigma^{'})\times N_{n-s}(b_{s+1}^{''},\ldots,b_n^{''};\Sigma^{''})f(y)dy$$

where $2 \le s \le n\text{-}1$ and $f(.)$ denotes the univariate standard normal density if the correlation matrix $\Sigma = \{\rho_{\mathrm{i,j}}\}$ has the following structure

$$\rho_{\mathrm{i,j}} = \frac{\gamma_i}{\gamma_j} \text{ for } i < j \text{ with } |\gamma_i| < |\gamma_j|.$$

For example :

$$N_7(b_1,\ldots,b_7;\Sigma)=\int_{-\infty}^{b_4} N_3(b_1^{'},\ldots,b_3^{'};\Sigma^{'})\times N_3(b_4^{''},\ldots,b_7^{''};\Sigma^{''})f(y)dy$$

2. Requirements for the Pricing Model

Is the Black Scholes model with time-dep volatility

the right model for pricing auto-callable structures?

What are the requirements for the pricing model in

order to “correctly” price these structures?

2. Requirements for the Pricing Model

- Structures typically include multiple digital risks.

  ➔ 1. Pricing model should incorporate the implied volatility skew.

- The redemption time of the structures is a stopping time and depends on the path of the equity underlying.

  ➔ 2. Pricing model should incorporate stochastic interest rates and correlation between interest rates and underlying.

- Fast calibration to standard options in the equity and interest rate world (calls, puts, caps, swaptions) is required.

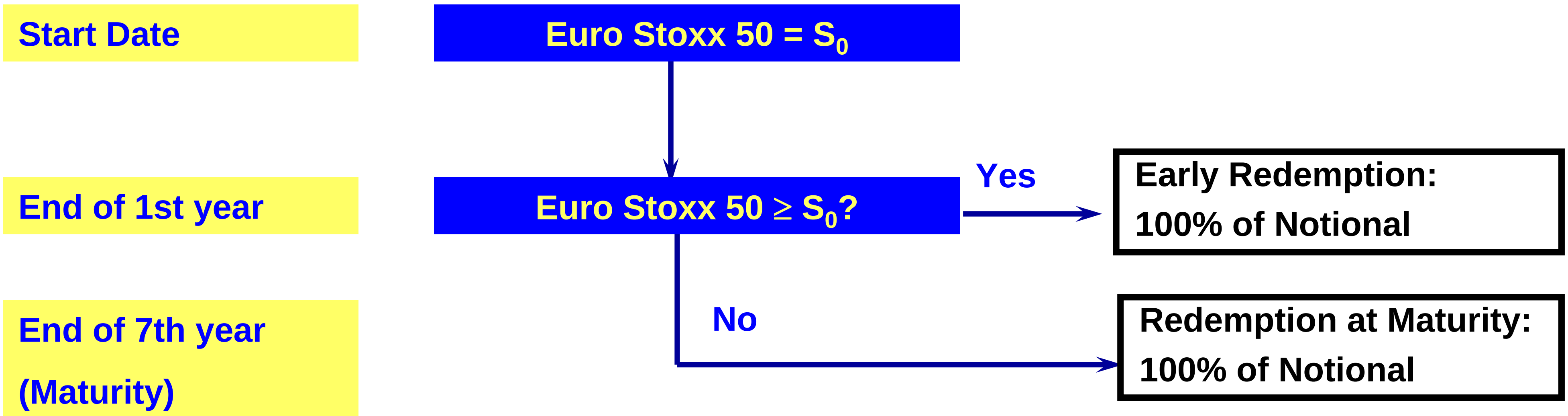
  ➔ 3. Pricing model should allow for the derivation of closed-form solutions for standard options.

2. Requirements for the Pricing Model

Why is it important to incorporate stochastic interest rates and correlation between interest rates and the equity underlying into our pricing model?

2. Requirements for the Pricing Model

In order to simplify matters, consider the following elementary auto-callable structure:



Assume that we delta hedge the auto-callable structure using the equity underlying, a 1 year zero bond and a 7 year zero bond.

2. Requirements for the Pricing Model

- **1. Case: Equity underlying increases**

  ➔ early redemption becomes more likely

  ➔ need to buy more of the 1y zero bond and to sell parts of the 7y zero bond position

If the correlation between equity and interest rates is <u>positive</u>, then interest rates will also increase on average and the bonds will decrease. However, value of the 7y zero bond, which we have to sell, will decrease more than the value of the 1y zero bond, which we have to buy. Thus, we would make on average a <u>net loss</u> on the interest rate delta rebalancing of the zero bond positions.

On the other hand, if correlation between equity and interest rates is <u>negative</u>, then interest rates will decrease on average and bond will increase. Therefore, we would make on average a <u>net profit</u> on the interest rate delta rebalancing of the zero bond positions since the value of the 7y zero bond will increase more than the value of the 1y zero bond.

2. Requirements for the Pricing Model

- **2. Case: Equity underlying decreases**

  ➔ early redemption becomes less likely

  ➔ need to sell parts of the 1y zero bond position and to buy more of the 7y zero bond

If the correlation between equity and interest rates is positive, then interest rates will also decrease on average. Thus, we would make again a net loss on the interest rate delta rebalancing of the zero bond positions since the value of the 7y zero bond, which we have to buy, will increase more than the value of the 1y zero bond. If the correlation between equity and interest rates is negative, then interest rate will increase on average and we would make on average a net profit on the interest rate delta rebalancing of the zero bond positions.

Consequently, we expect a higher price of the auto-callable structure if a positive correlation between equity and interest rates is assumed and a lower price if a negative correlation is assumed.

3. Black-Scholes & Vasicek++ Model

- **Definition of the Model (Merton (1973), Brigo and Mercurio (2001)) :**

  - **Equity part:** Black-Scholes with time-dependent deterministic volatility function $\sigma_S(t)$

  - **Interest rate part:** instantaneous, mean reverting interest rate process (Vasicek) with deterministic shift function $\varphi(t)$

  - **Correlation:** Brownian motions are correlated

$$
\begin{aligned}
&dS(t) = (r(t) - d)\, S(t)dt + \sigma_S(t)S(t)dW^S(t) \\
&r(t) = R(t) + \varphi(t) \\
&dR(t) = \kappa_R\big(\theta_R - R(t)\big)dt + \sigma_R dW^R(t), \\
&d\left[W^S, W^R\right]_t = \rho_{S,R}dt.
\end{aligned}
$$

3. Black-Scholes & Vasicek++ Model

- **Closed-from Solution for Standard Equity Options (Merton (1973)):**

Price of a call option with strike K and maturity T:

$$C(K,T) = df(T)\big(F(T)N(d_1) - KN(d_2)\big)$$

$$d_{1,2} = \left(\ln\left(\frac{F(T)}{K} \right) \pm \frac{1}{2} v^2(T) \right) / v(T)$$

$$v^2(T) = \int_0^T \sigma_S^2(u) + 2\rho_{S,R}\sigma_S(u)\sigma_R \eta(\mathrm{u},\mathrm{T}) + \sigma_R^2 \eta^2(\mathrm{u},\mathrm{T})\mathrm{du}$$

$$\eta(\mathrm{u},\mathrm{T}) = \frac{1 - \mathrm{e}^{-\kappa_R(\mathrm{T}-\mathrm{u})}}{\kappa_R}$$

Note that in comparison to the standard Black-Scholes formula only the volatility changes. The “total” volatility now depends on the equity and interest rate volatility as well as on the correlation between equity and interest rates.

3. Black-Scholes & Vasicek++ Model

- **Closed-from Solution for Zero Bond Options (Brigo and Mercurio (2001)):**

Price of a call option with strike K and maturity T_1 on a zero bond with maturity T_2 :

$$V_{Call}(0,T_1,T_2,K) = e^{-\int_0^{T_2}\varphi(t)dt}\left[A(0,T_2)e^{-B(0,T_2)R(0)}N(h_1) - Ke^{\int_{T_1}^{T_2}\varphi(t)dt}A(0,T_1)e^{-B(0,T_1)R(0)}N(h_2)\right]$$

$$h_{1/2} = \frac{1}{\overline{v}}\ln\left[\frac{A(0,T_2)e^{-B(0,T_2)R(0)}}{Ke^{\int_{T_1}^{T_2}\varphi(t)dt}A(0,T_1)e^{-B(0,T_1)R(0)}}\right] \pm \frac{\overline{v}}{2};\ \overline{v} = \sigma_R B(T_1,T_2)\sqrt{\frac{1-\mathrm{e}^{-2\kappa_R T_1}}{2\kappa_R}}$$

$$A(t,T) = \exp\left[\frac{\left(B(t,T)-T+t\right)\left(\kappa_R^2\theta_R - 0.5\sigma_R^2\right)}{\kappa_R^2} - \frac{\sigma_R^2 B^2(t,T)}{4\kappa_R}\right];\ \mathrm{B}(t,T) = \frac{1-\mathrm{e}^{-\kappa_R(T-t)}}{\kappa_R}$$

Through the formula above we can also price caps and floors as well as swaptions.

For details see Brigo and Mercurio (2001).

3. Black-Scholes & Vasicek++ Model

Calibration of the Model:

Benchmark instruments: equity calls and puts, interest rate caps and floors, swaptions

Should we first calibrate to equity options and then try to best fit the interest rate benchmark instruments, or vice versa, or should we simultaneously calibrate to all benchmark products? We propose the following calibration procedure:

1. Calibrate the interest rate process to caps, floors, swaptions and the yield curve

   ➔ κ_R, θ_R, σ_R and shift function φ determined

2. Fix the correlation between interest rates and equity $\rho_{S,R}$ based on time series analysis or OTC market quotes

3. Calibrate the equity process to ATM standard call options holding the interest rate parameters constant.

   ➔ time-dependent deterministic volatility function σ_S determined

3. Black-Scholes & Vasicek++ Model

- **Pros:**

  1. **The model incorporates stochastic interest rates.**

  2. **Direct correlation between stochastic interest rates and the equity underlying.**

  3. **Closed-form solutions for standard call and put options as well as for caps and swaptions → fast model calibration possible.**

  4. **The Vasicek++ interest rate process produces a reasonably good fit to market prices of caps and swaptions.**

  5. **The model is able to provide a perfect fit to all ATM call options.**

3. Black-Scholes & Vasicek++ Model

- **Cons:**

  The model produces no implied volatility skew on the equity side.

- **Conclusion:**

  Since the implied volatility skew is an important factor for the price of any structure with digital risk, we need to look for another model.

4. Heston & Vasicek++ and Heston & CIR++ Model

- **Definition of the Heston & Vasicek++ Model:**

- Equity part: mean reverting stochastic volatility process (Heston model)
- Interest rate part: instantaneous, mean reverting interest rate process (Vasicek) with deterministic shift function φ(t)

$$dS(t) = (r(t) - d)\, S(t)dt + \sqrt{v(t)}\, S(t)\, dW^S(t) + \sigma_{S,R} S(t) dW^R(t),$$

$$dv(t) = \kappa_v \left(\theta_v - v(t)\right)dt + \sigma_v \sqrt{v(t)}\, dW^v(t),$$

$$d\left[W^S, W^v\right]_t = \rho_{S,v} dt,$$

$$r(t) = R(t) + \varphi(t),$$

$$dR(t) = \kappa_R \left(\theta_R - R(t)\right)dt + \sigma_R dW^R(t),$$

$$d\left[W^S, W^R\right]_t = 0,$$

$$d\left[W^v, W^R\right]_t = 0.$$

4. Heston & Vasicek++ and Heston & CIR++ Model

- **Definition of the Heston & CIR++ Model:**

- Equity part: mean reverting stochastic volatility process (Heston model)
- Interest rate part: instantaneous, mean reverting interest rate process (CIR) with deterministic shift function φ(t)

$$dS(t) = (r(t) - d)\, S(t)dt + \sqrt{v(t)}\, S(t)\, dW^{S}(t) + \sigma_{S,R}\sqrt{R(t)}\, S(t)\, dW^{R}(t),$$

$$dv(t) = \kappa_v\left(\theta_v - v(t)\right)dt + \sigma_v\sqrt{v(t)}\, dW^{v}(t),$$

$$d\left[W^{S}, W^{v}\right]_t = \rho_{S,v}\, dt,$$

$$r(t) = R(t) + \varphi(t),$$

$$dR(t) = \kappa_R\left(\theta_R - R(t)\right)dt + \sigma_R\sqrt{R(t)}\, dW^{R}(t),$$

$$d\left[W^{S}, W^{R}\right]_t = 0,$$

$$d\left[W^{v}, W^{R}\right]_t = 0.$$

4. Heston & Vasicek++ and Heston & CIR++ Model

- **Closed-form Solution for Standard Equity Options (G. (2004)):**

Price of standard call with strike K and maturity T:

$$C\left(K,T,S(0),v(0),R(0)\right)=S(0)e^{-dT}P_1-Kdf(T)P_2, \text{ where}$$

$$P_j\left(K,T,S(0),v(0),R(0)\right)=\frac{1}{2}+\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\phi\ln K}f_j}{i\phi}\right]d\phi \quad \text{for } j=1,2.$$

with

$$f_1=e^{C_1(T,\phi)+D_1(T,\phi)v(0)+E_1(T,\phi)R(0)+i\phi\ln S(0)}$$

$$f_2=e^{C_2(T,\phi)+D_2(T,\phi)v(0)+E_2(T,\phi)R(0)+i\phi\ln S(0)-\ln df(T)}$$

4. Heston & Vasicek++ and Heston & CIR++ Model

Link between coupling factor $\sigma_{S,R}$ and correlation

- **Produced (stochastic) instantaneous correlation:**

$$\rho_{S,R}(t) = \frac{d[S,r]_t}{\sqrt{d[S]_t}\sqrt{d[r]_t}} = \frac{\sigma_{S,R}\sqrt{R(t)}}{\sqrt{v(t) + \sigma_{S,R}^2 R(t)}} \qquad (\text{Heston } \& \text{ CIR}++)$$

$$\rho_{S,R}(t) = \frac{d[S,r]_t}{\sqrt{d[S]_t}\sqrt{d[r]_t}} = \frac{\sigma_{S,R}}{\sqrt{v(t) + \sigma_{S,R}^2}} \qquad (\text{Heston } \& \text{ Vasicek}++)$$

- **Isolating the coupling factor yields:**

$$\sigma_{S,R} = \frac{\rho_{S,R}(t)\sqrt{v(t)}}{\sqrt{R(t)}\sqrt{1-\rho_{S,R}^2(t)}} \qquad (\text{Heston } \& \text{ CIR}++)$$

$$\sigma_{S,R} = \frac{\rho_{S,R}(t)\sqrt{v(t)}}{\sqrt{1-\rho_{S,R}^2(t)}} \qquad (\text{Heston } \& \text{ Vasicek}++)$$

4. Heston & Vasicek++ and Heston & CIR++ Model

Link between coupling factor $\sigma_{S,R}$ and correlation

If we want to know which value for $\sigma_{S,R}$ is approximately needed in order to produce an average instant. correlation of $\bar{\rho}_{S,R}$, we can use the following crude approximation:

$$\sigma_{S,R} \approx \frac{\bar{\rho}_{S,R}\, E\left[\sqrt{\frac{1}{T}\int_0^T v(t)dt}\right]}{E\left[\sqrt{\frac{1}{T}\int_0^T R(t)dt}\right]\sqrt{1-\bar{\rho}_{S,R}^2}} \qquad \text{(Heston \& CIR++)}$$

$$\sigma_{S,R} \approx \frac{\bar{\rho}_{S,R}\, E\left[\sqrt{\frac{1}{T}\int_0^T v(t)dt}\right]}{\sqrt{1-\bar{\rho}_{S,R}^2}} \qquad \text{(Heston \& Vasicek++)}$$

for some time horizon T.

4. Heston & Vasicek++ and Heston & CIR++ Model

- **With**

$$E\left[\sqrt{\frac{1}{T}\int_0^T v(t)dt}\right] \approx \sqrt{E\left[\frac{1}{T}\int_0^T v(t)dt\right] - \mathrm{Var}\left[\frac{1}{T}\int_0^T v(t)dt\right] \Big/ 8\,E\left[\frac{1}{T}\int_0^T v(t)dt\right]^{\frac{3}{2}}}$$

$$E\left[\frac{1}{T}\int_0^T v(t)dt\right] = \frac{1-e^{-\kappa_v T}}{\kappa_v T}\left(v(0)-\theta_v\right)+\theta_v$$

$$\mathrm{Var}\left[\frac{1}{T}\int_0^T v(t)dt\right] = \frac{\sigma_v^2 e^{-2\kappa T}}{2\kappa^3 T^2}\begin{pmatrix} 2\left(-1+e^{2\kappa_v T}-2e^{\kappa T}\kappa_v T\right)\left(v(0)-\theta_v\right) \\ +\left(-1+4e^{\kappa_v T}-3e^{2\kappa_v T}+2e^{\kappa_v T}\kappa_v T\right)\theta_v \end{pmatrix}$$

Alternatively,

$$E\left[\sqrt{\frac{1}{T}\int_0^T v(t)dt}\right] = \frac{1}{2\pi}\int_0^\infty \left(1-f(\lambda)\right)/\,\lambda^{3/2}d\lambda \text{ where } f(\lambda) = E\left[e^{-\lambda\frac{1}{T}\int_0^T v(t)dt}\right] = Ae^{-\lambda v(0)B}$$

4. Heston & Vasicek++ and Heston & CIR++ Model

- **Example (Heston & CIR++):**

$$v(0) = 0.0625\ ,\ \kappa_v = 0.4,\ \theta_v = 0.09,\ \sigma_v = 0.5,\ \rho_{S,v} = -0.7$$

$$R(0) = 0.02,\ \kappa_R = 0.1,\ \theta_R = 0.05,\ \sigma_R = 0.06,\ T = 4 \text{ years}$$

We would like to have an average instant. correlation of $\rho_{S,R} = 0.35.$

Our approximation suggests to use $\sigma_{S,R} = 0.54.$

Average instantaneous correlation after 50.000 Monte Carlo runs: $\hat{\rho}_{S,R} = 0.3473.$

4. Heston & Vasicek++ and Heston & CIR++ Model

- **Calibration of the models**

1. **Calibrate the CIR++/ Vasicek++ interest rate process to the yield curve, caps and swaptions.**

2. **Estimate the correlation between interest rates and underlying using historical time series / OTC market quotes and estimate corresponding coupling factor $\sigma_{S,R}$.**

3. **Calibrate the stochastic variance process to the implied volatility surface of the equity underlying holding all interest rate parameters and the coupling factor constant.**

4. Heston & Vasicek++ and Heston & CIR++ Model

- **Pros:**
  - **Both models incorporate stochastic interest rates and correlation between stochastic interest rates and equity underlying.**
  - **Closed-form solutions for standard call and put options as well as for caps and swaptions → fast model calibration possible.**
  - **The CIR++ and the Vasicek++ interest rate processes produce a reasonably good fit to market prices of caps and swaptions.**
  - **Both models are able to provide a good fit to the whole implied volatility surface of the equity underlying.**

4. Heston & Vasicek++ and Heston & CIR++ Model

- **Cons:**

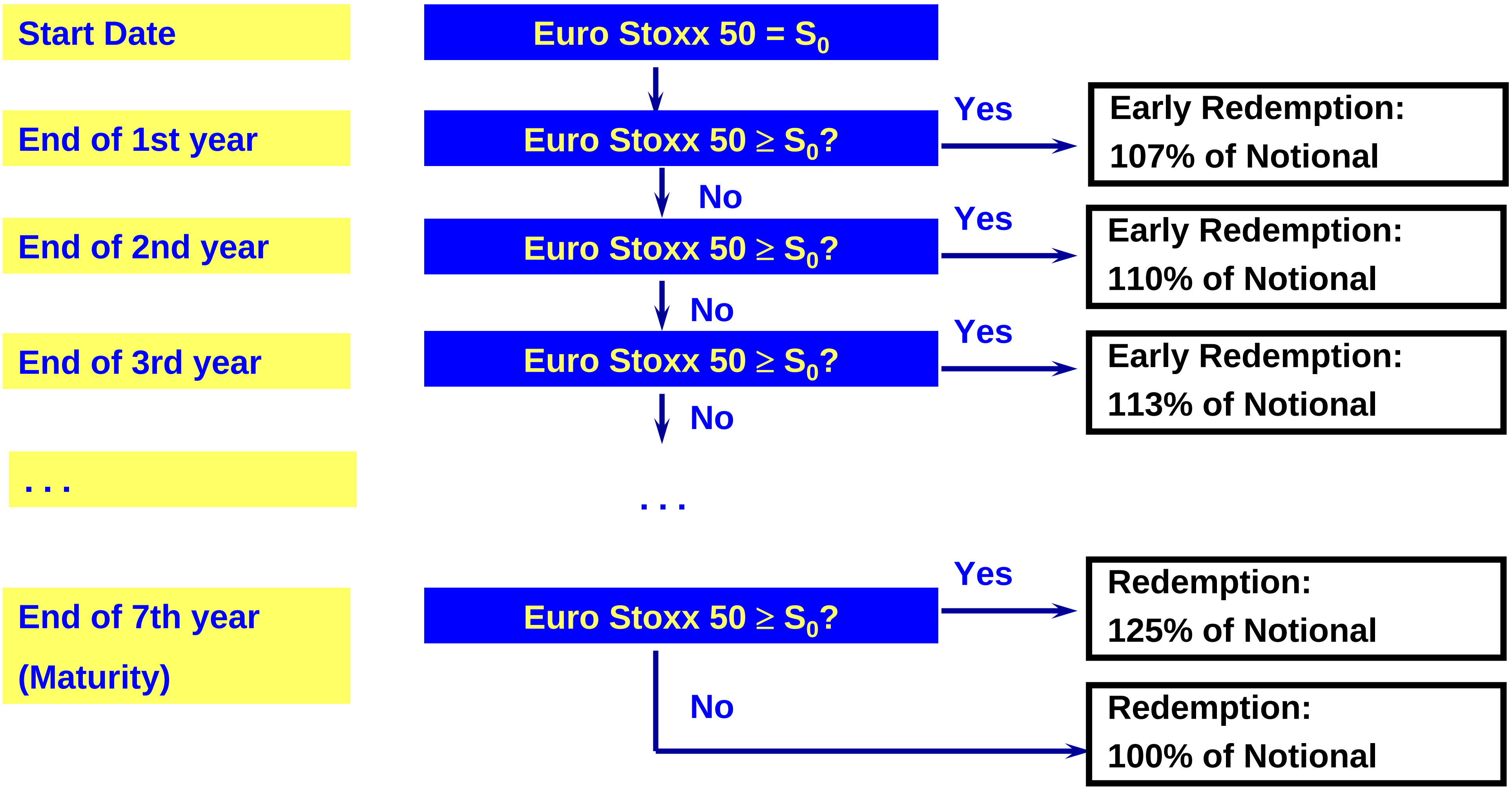
  - **Correlation between interest rates and underlying is modeled only <u>indirectly</u> and extreme positive/negative correlations cannot be produced.**

  - **The models have problems to obtain a good fit to the implied volatility surface of the equity underlying for high positive correlations.**

  - **The interest rate part is modeled only by a short rate processes.**

5. Pricing Example

We consider again the auto-callable equity structure from the introduction:



5. Pricing Example

After the calibration of all 3 models to market data (Feb 2006) and for different values of the correlation between equity and interest rate, we obtain the following prices and price differences:

Model / Correlation	**-40%**	**-30%**	**-20%**	**-10%**	**0%**	**10%**	**20%**	**30%**	**40%**
BlackScholes & Vasicek++	96.37 *-0.47*	96.49 *-0.35*	96.61 *-0.23*	96.74 *-0.10*	**96.84** *0.0*	96.97 *0.13*	97.10 *0.26*	97.25 *0.41*	97.38 *0.54*
Heston & Vasicek++	97.90 *-0.42*	98.00 *-0.32*	98.09 *-0.23*	98.18 *-0.14*	**98.32** *0.0*	98.48 *0.16*	98.68 *0.36*	98.92 *0.60*	99.08 *0.76*
Heston & CIR++	97.85 *-0.47*	97.96 *-0.36*	98.07 *-0.25*	98.22 *-0.10*	**98.32** *0.0*	98.45 *0.13*	98.62 *0.30*	98.90 *0.58*	99.04 *0.72*

5. Pricing Example

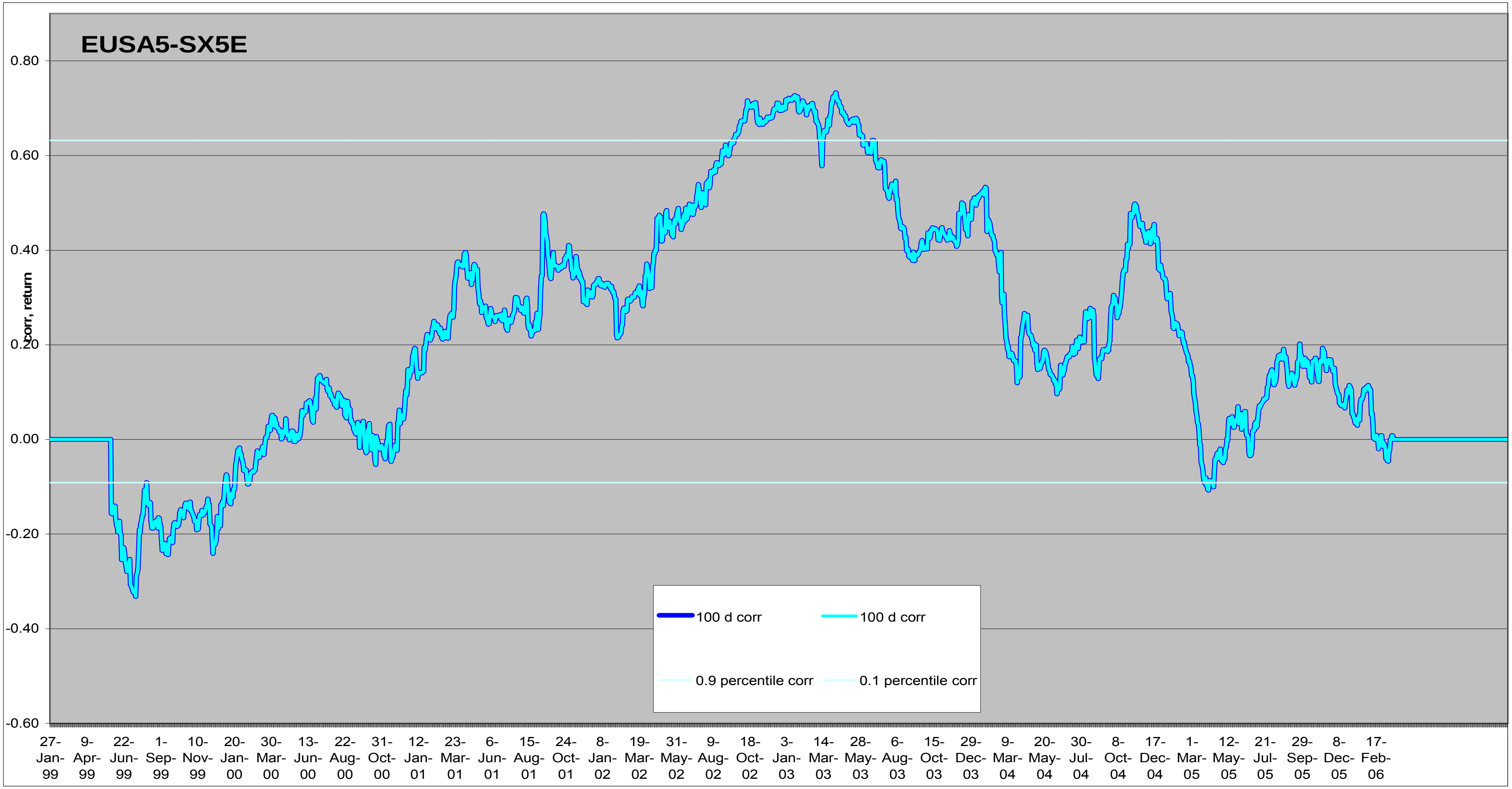
Conclusion:

- Ignoring the implied volatility skew or the correlation between equity and interest rates leads to significant mispricing.

- Heston & CIR++ and Heston & Vasicek++ produce similar results and their correlation exposure is comparable to the correlation exposure of the Black-Scholes & Vasicek++ model for moderate correlation values.

5. Pricing Example

Remaining Question: Which correlation value should be used?



5. Pricing Example

	1Y historical Correlation (1990-2002)	
	EuroStoxx 50 and 5y EUR Swap rate	S&P 500 and 5y USD Swap rate
Maximum	66%	70%
Average	-10%	-7%
Minimum	-62%	-58%

- Historical correlation between interest rates and equity varies considerably over time.
- Inflation expectations, monetary policy and business cycle have impact on the equity-interest rate correlation.

5. Pricing Example

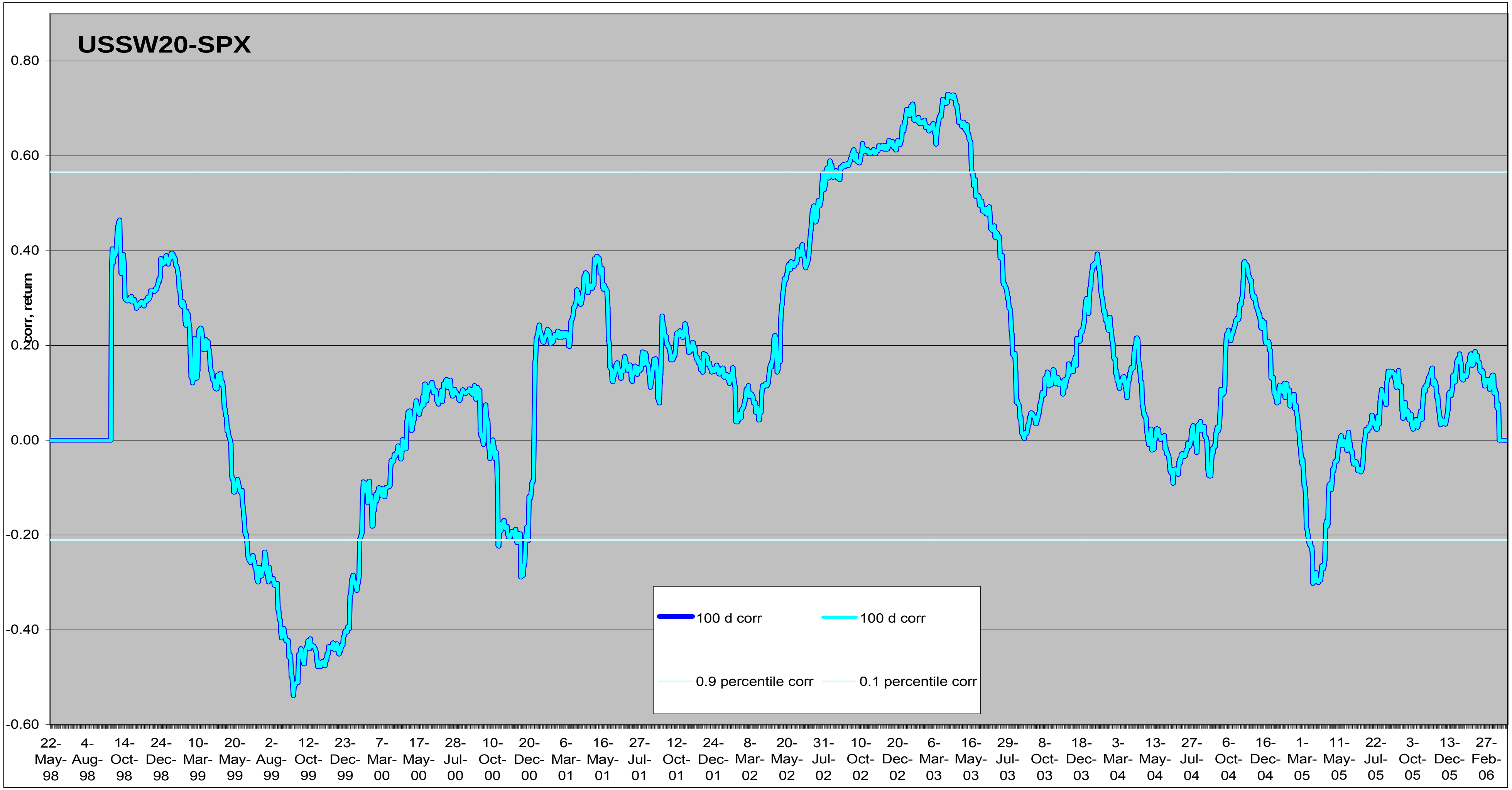
OTC Quote (February 16, 2006):

1 year correlation swap* between 20 year USD swap rate and S&P 500:

-10 / +45

*** based on daily realized log returns**

5. Pricing Example



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