

# Put No-Touch Duality

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## So You Want to Be a Quant, Eh?

- Put No-Touch Duality (PNTD) is a little known result that can be arrived at by answering a barrage of questions of the kind typically asked during a job interview.
- To derive PNTD, or at least to prepare for your next interview, please try to solve the quizzes posed on the next few slides.
- For all of the following quizzes, you should assume that the underlying asset is a stock whose price follows geometric Brownian motion forever. Although volatility is constant, you don't know what it is.
- You should further assume that the stock never pays any dividends and that you can borrow and lend at zero interest rate.

## Quiz #1: Perpetual One-Touch, Upper Barrier

- By definition, a one-touch pays one dollar at the first time that some underlying stock price touches a barrier.
- In practice, all one touches are finite-lived, but consider a perpetual one-touch that potentially lasts forever.
- Suppose that the stock price is presently 90 cents and that the one touch barrier is at a dollar.
- What is the exact arbitrage-free value of this perpetual one touch with an upper barrier?

## Quiz #2: Perpetual No-Touch, Upper Barrier

- By definition, a no-touch pays one dollar at its expiry provided that the underlying stock price never touches a barrier.
- If you know the value of a one-touch, then under zero rates, the no-touch is valued by subtracting the one-touch value from one.
- In practice, all no-touches have a finite life, but consider the value of a perpetual no-touch, defined as the limiting value of a finite-lived no-touch as its lifetime becomes infinite.
- Suppose again that the stock price is presently 90 cents and that the no-touch payoff is a dollar.
- What is the arbitrage-free value of this perpetual no-touch with an upper barrier?

## Quiz #3: Perpetual No-Touches with an Upper

- Let  $S$  be the stock price at the valuation date.
- Let  $N(S)$  be the perpetual no-touch value function.
- Suppose as in the last question that  $S < 1$ , where recall that the no-touch barrier is one dollar.
- What is  $N(S)$  in this case?

## Quiz #4: Perpetual No-Touch, Lower Barrier

- Now suppose that the stock price is presently \$1.20, but that the no-touch barrier is still one dollar.
- What is the arbitrage-free value of this perpetual no-touch with a lower barrier?

## Quiz #5: Perpetual No-Touch Valuation

- Recall that  $S$  denotes the stock price at the valuation date and that  $N(S)$  is the value of a perpetual no-touch with barrier of one dollar.
- Recall that when  $S < 1$ , the perpetual no-touch value is  $N(S) = 1 - S$ , while when  $S > 1$ , the perpetual no-touch value is  $N(S) = 0$ .
- What formula relates the Black Scholes value (under zero rates and dividends) of a perpetual no-touch to the initial stock price, when  $S > 0$ ?
- What is the vega of the perpetual no touch?
- What is its gamma at  $S = 1$ ?

# No-Touch Value as a Function of Starting Stock Price

- Here is a graph of the no-touch value as a function of the starting stock price

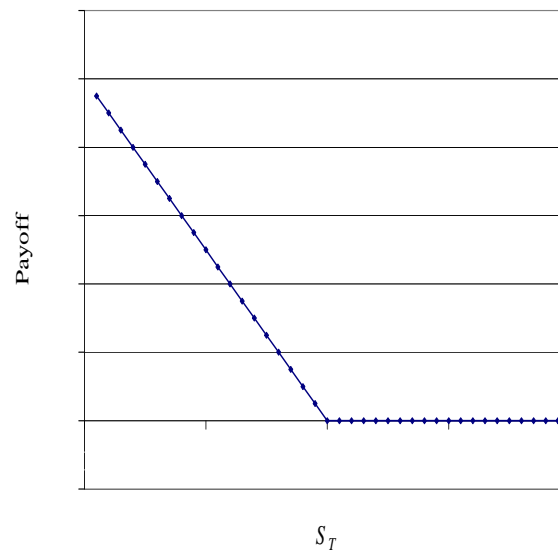


Figure 1: Put Payoff

- The value is the same as the payoff  $(1 - S)^+$  from a standard European put at 1.

## Quiz #6: Forward-Start Perpetual No-Touch

- Some barrier options are forward-start, which means that the barrier is not monitored until some future date.
- Consider a forward-start perpetual no-touch with a barrier of one dollar which is monitored in one year.
- Suppose you know that a European put with strike and maturity both one is worth 10 cents at 10 cents.
- What is the arbitrage-free value of the forward-start perpetual no-touch?

## Put No-Touch Duality

- Let  $S$  denote the spot price of a limited liability asset with continuous price whose total variance grows without bound:

$$\lim_{t \uparrow \infty} \langle \ln S \rangle_t = \infty.$$

- Let  $M_{[T, \infty)} \equiv \max_{t \in [T, \infty)} S_t$  denote the maximum spot price over the time period
- Let  $P_0(K, T)$  denote the initial price of a European put paying  $(K - S_T)^+$
- Then under zero interest rates and dividends, Put No Touch Duality is that:

$$\frac{P_0(K, T)}{K} = \mathbb{Q}\{M_{[T, \infty)} < K\}.$$

- In words, when normalized by its strike price, the European put price is just the probability that the underlying stock price avoids the strike price after maturity.

## So What?

- We give two applications of Put No Touch Duality (PNTD):
  1. Interpreting implied volatility in stochastic volatility models.
  2. Valuing finite-lived no-touches in stochastic volatility models.
- Both applications are possible without PNTD if one assumes that instantaneous variance evolves independently of the standard Brownian motion driving stock returns.
- The advantage of PNTD is that one can allow nonzero correlation.

## Last Passage Times

- Recall that by PNTD, the price of  $\frac{1}{K}$  puts struck at  $K$  and maturing at  $T$  is  $\frac{1}{K} \mathbb{P}^Q(S_T \leq K)$  under risk-neutral probability that  $S$  never hits  $K$  after  $T$ .
- The probability can be expressed in terms of the last passage time  $L$  of the price  $S$  to  $K$ :

$$L_t \equiv \sup_{t \in [0, \infty)} S_t = K.$$

If  $S$  never hits  $K$ , then we set  $L = 0$ . We note that the random time  $L$  is a stopping time.

## Quadratic Variation at the Last Passage Time

- Let  $SV P_0(K, T)$  denote the initial value of a European put in a stochastic volatility model.
- If the underlying stock price  $S$  never hits the put strike  $K$  after the put matures, then either  $S$  hits  $K$  in  $[0, T]$  or else  $S$  never hits  $K$ .
- The scaled put value gives the risk-neutral probability that either of these two events occurs:

$$\frac{SV P_0(K, T)}{K} = \mathbb{Q}\{L < T\}.$$

- Since quadratic variation is an increasing process, we also have:

$$\frac{SV P_0(K, T)}{K} = \mathbb{Q}\{\langle \ln S \rangle_L < \langle \ln S \rangle_T\}.$$

- This shows exactly how put prices forecast subsequent realized variance as explained upon further on the next slide.

## Put Price and Subsequent Realized Variance

- Recall that the scaled initial SV model value of a European put gives the risk-neutral probability that the quadratic variation of returns at the last passage time  $L$  of stock price  $S$  to the put strike  $K$  is less than the quadratic variation of returns at the put's maturity  $R$ :

$$\frac{SVP_0(K, T)}{K} = \mathbb{Q}\{\langle \ln S \rangle_L < \langle \ln S \rangle_T\}.$$

- Note that  $\langle \ln S \rangle_T$  depends only on  $T$  and is independent of  $K$ .
- Likewise,  $\langle \ln S \rangle_L$  depends only on  $K$  and is independent of  $T$ .
- The put value on the LHS depends on both  $K$  and  $T$ , so the RHS represents a convenient way to understand the impact on option prices of varying  $K$ , while  $T$  fixed, or alternatively of varying  $T$  while holding  $K$  fixed.

## Distribution Function of Last Passage Time

- Let  $F^{sv}(q) : \mathbb{R}^+ \mapsto [0, 1]$  denote the conditional distribution function of  $L$  under

$$F^{sv}(q) \equiv \mathbb{P}\{\langle \ln S \rangle_L < q \mid \langle \ln S \rangle_T = q\}.$$

- Note that  $SVF$  has domain  $\mathbb{R}^+$  and range  $[0, 1]$ . By the law of iterated expectation

$$\frac{SVP_0(K, T)}{K} = E^{\mathbb{Q}}\{F^{sv}(\langle \ln S \rangle_T) \mid \mathcal{F}_0\}.$$

- Let  $F^{bs}(q) : \mathbb{R}^+ \mapsto [0, 1]$  be the dist'n function of  $L$  under constant volatility

$$F^{bs}(\sigma^2 T) \equiv \mathbb{Q}\{\langle \ln S \rangle_L < \sigma^2 T\}.$$

- Since the Black Scholes model satisfies our assumptions:  $\frac{BSP_0}{K} = F^{bs}(\sigma^2 T)$ .

- If  $\sigma^2 T = I_0$ , then:  $F^{bs}(I_0) = E\{F^{sv}(\langle X \rangle_T) \mid \mathcal{F}_0\}$ .

- Since  $F^{bs}$  is a distribution function,  $(F^{bs})^{-1}$  exists, and hence:

$$I_0 = (F^{bs})^{-1}(E^{\mathbb{Q}}\{F^{sv}(\langle \ln S \rangle_T) \mid \mathcal{F}_0\}).$$

- Thus, initial implied total variance  $I_0$  is linked to terminal quadratic variation  $\langle X \rangle_T$ .

## Varying Strike and Maturity

- Recall the expression linking initial implied total variance  $I_0$  to terminal quadratic variation  $\langle \ln S \rangle_T$ :

$$I_0 = (F^{bs})^{-1}(E^{\mathbb{Q}}\{F^{sv}(\langle \ln S \rangle_T) | \mathcal{F}_0\}).$$

- Given a particular model of stochastic volatility, one can analyze how the distribution of  $\langle X \rangle_T$  and the form of  $SVF$  affect the initial implied total variance  $I_0$ .
- Note that if we vary  $T$  while holding  $S_0$  and  $K$  fixed, then the impact on implied volatility occurs only through the dependence of the distribution of  $\langle X \rangle_T$  on  $T$ . In other words,  $BSF^{-1}$  and  $SVF$  are both independent of  $T$  and depend only on  $S_0$ .
- Conversely, if we vary  $K$  while holding  $S_0$  and  $T$  fixed, then the impact on implied volatility occurs only through the dependence of  $BSF^{-1}$  and  $SVF$  on  $K$ . In other words, the distribution of  $\langle X \rangle_T$  is independent of  $K$  and depends only on  $T$ .

## Valuing Finite Lived No-Touches as Puts

- Recall that under zero interest rates and dividends, Put No-Touch Duality is

$$\frac{P_0(K, T)}{K} = \mathbb{Q}\{M_{[T, \infty)} < K\}.$$

- In words, when normalized by its strike price, a European put has the same value as a forward-start perpetual no touch.
- We now show how time inversion can be used to reduce the problem of valuing a finite-lived no-touch into the simpler problem of valuing a European put in a complete market economy.
- For simplicity, we begin with the Black Scholes model with constant interest rate  $r \in \mathbb{R}$  and dividend yield  $q \in \mathbb{R}$ .

## No-Touch Valuation in Black Scholes

- Consider the familiar problem of finding the Black Scholes value of a no-touch option with the spot price path  $\{S_t, t \in [0, T]\}$  with lower barrier  $L < S_0$  and maturity  $T$ .
- It will be convenient to measure the underlying asset price in barriers and time in maturities:

$$U_m \equiv \frac{S_t}{L}, \quad m \equiv \frac{t}{T}.$$

- The process  $U$  has initial level  $U_0 \equiv \frac{S_0}{L} > 1$  and subsequently solves the SDE

$$\frac{dU_m}{U_m} = (r - q)\sigma^2 T dm + \sigma\sqrt{T} dZ_m, \quad m > 0,$$

where  $\{Z_m : m > 0\}$  is a  $\mathbb{Q}$  standard Brownian motion in the  $m$  clock.

- The solution to this SDE is:

$$U_m = U_0 e^{\sigma\sqrt{T}Z_m - p\sigma^2 T m}, \quad m > 0,$$

where:

$$p \equiv \frac{1}{2} - \frac{r - q}{\sigma^2}.$$

## No-Touch Valuation in Black Scholes (Con'd)

- The underlying spot price  $S$  stays above  $L$  dollars for the first  $T$  years if and only if the stochastic process  $U$  stays above one barrier for the first maturity:

$$1_{(\min_{t \in [0, T]} S_t > L)} = 1_{(\min_{m \in [0, 1]} U_m > 1)}.$$

- Hence, the no-touch paying one dollar at  $t = T$  years if  $S$  stays above  $L$  before  $t = T$  is equivalently paying one dollar at  $m = 1$  maturity if  $U$  stays above one before  $m = 1$ . As a result, a no-touch on  $S$  with lower barrier  $L < S_0$  and maturing at  $t = T$  has the same forward price as a no-touch on  $U$  with lower barrier  $1 < U_0$  and maturing at  $m = 1$ :

$$N_0(L, T)e^{rT} = \mathbb{Q}\{\min_{m \in [0, 1]} U_m > 1\}.$$

## Time Inversion

- With calendar time measured in maturities, suppose that we consider a time in

$$\omega = \frac{1}{m}, \quad m > 0.$$

- We henceforth refer to  $\omega$  as the inverted clock. Consider the following process defined by raising the GBM  $U$  to the power  $\pi\omega$  for some constant  $\pi$  and run on the inverted clock:

$$M_\omega = U_m^{\pi\omega}, \quad \omega > 0.$$

- If the underlying price process  $U$  touches the barrier one before time  $m = 1$ , the new process  $M$  touches the barrier one *after* time  $\omega = 1$ . Hence, the price of a claim paying \$1 if the underlying  $U$  avoids one before time  $m = 1$  has the same price as a claim paying \$1 if the process  $M$  avoids the barrier one after time

$$N_0(L, T)e^{rT} \equiv \mathbb{Q}\left\{\min_{m \in [0, 1]} U_m > 1\right\} = \mathbb{Q}\left\{\max_{\omega \in [1, \infty]} M_\omega < 1\right\}.$$

## Martingality on the Inverse Clock

- By defining the power  $\pi \equiv \frac{-2\ln(S_0/L)}{\sigma^2 T}$ , it can be shown that the process  $M$  obtained by raising the GBM  $U$  to the power  $\pi\omega$  is a  $\mathbb{Q}$  martingale when running in the inverted clock  $\omega \equiv \frac{1}{m}$ :

$$M_\omega = U_m^{\frac{-2\ln(S_0/L)}{\sigma^2 T} \omega}, \quad \omega > 0.$$

- It can also be shown that the lognormal volatility of  $M$  is constant at  $\sigma_m \equiv \frac{-2}{\sigma}$  and its initial value is  $M_0 = \left(\frac{S_0}{L}\right)^{2p}$ , where  $p \equiv \frac{1}{2} - \frac{r-q}{\sigma^2}$ .
- Since  $M$  is a GBM martingale running in the  $\omega$  clock, the claim paying \$1 at maturity avoids one after  $\omega = 1$  has the same value as the corresponding European Put No-Touch Duality, and hence:

$$N_0(L, T)e^{rT} = \mathbb{Q} \left\{ \max_{\omega \in [1, \infty]} M_\omega < 1 \mid M_0 = \left(\frac{S_0}{L}\right)^{2p} \right\} = E^{\mathbb{Q}} \left[ (1 - M_1)^+ \mid M_0 \right]$$

## Checking the Result

- The forward value of a put on the  $M$  process, with initial value  $M_0$ , strike maturity  $T_m$  and volatility  $\sigma_m$  is given by the following Black put formula:

$$\hat{P}_0(K_m, T_m) \equiv N\left(\frac{\ln(K_m/M_0)}{\sigma_m\sqrt{T_m}} + \frac{\sigma_m\sqrt{T_m}}{2}\right) - M_0 N\left(\frac{\ln(K_m/M_0)}{\sigma_m\sqrt{T_m}} - \frac{\sigma_m\sqrt{T_m}}{2}\right)$$

normal distribution function.

- Setting  $\sigma_m \equiv \frac{-2\ln(S_0/L)}{\sigma\sqrt{T}}$ ,  $M_0 = \left(\frac{S_0}{L}\right)^{2p}$ ,  $K_m = T_m = 1$  implies:

$$\hat{P}_0(1, 1) = N\left(\frac{\ln(S_0/L)}{\sigma\sqrt{T}} - p\sigma\sqrt{T}\right) - \left(\frac{S_0}{L}\right)^{2p} N\left(\frac{\ln(L/S_0)}{\sigma\sqrt{T}} - p\sigma\sqrt{T}\right)$$

- Hence, we have:

$$N_0(L, T) = e^{-rT} \hat{P}_0(1, 1) = e^{-rT} \left[ N\left(\frac{\ln(S_0/L)}{\sigma\sqrt{T}} - p\sigma\sqrt{T}\right) - \left(\frac{S_0}{L}\right)^{2p} N\left(\frac{\ln(L/S_0)}{\sigma\sqrt{T}} - p\sigma\sqrt{T}\right) \right]$$

where recall that  $p \equiv \frac{1}{2} - \frac{r-q}{\sigma^2}$ . This result agrees with the well-known formula for the value of a no-touch in the Black Scholes model.

## No-Touch valuation Under SV

- Encouraged by our success in duplicating the BS formula for a no touch, we seek to value a finite-lived no touch in an SV model with nonzero correlation.
- If one assumes zero risk-neutral drift in the underlying, then one can show that the valuation of a finite-lived no touch again reduces to the valuation of a European put on a  $\mathbb{Q}$  martingale when running on the inverted clock  $\omega = \frac{T}{t}$ . The underlying martingale  $M$  is defined by:

$$M_\omega \equiv \begin{cases} \left(\frac{F_t}{L}\right)^{-\frac{2\ln(F_0/L)}{\langle \ln F \rangle_t}}, & \text{if } t \in (0, \infty) \\ \frac{F_0}{L}, & \text{if } t = \infty. \end{cases}$$

- When viewed in calendar time, the problem still reduces to put valuation, with the log of the underlying,  $\ln M$ , is actually proportional to the ratio  $\frac{\ln F_T}{\langle \ln F \rangle_T}$ .
- Thus, closed form solution for no touches is possible whenever the joint distribution or characteristic function of  $\ln F_T$  and  $\langle \ln F \rangle_T$  is known.
- The Romano Touzi mixing method can also be applied.

## Summary and Extensions

- Recall that under zero interest rates and dividends, Put No-Touch Duality is

$$\frac{P_0(K, T)}{K} = \mathbb{Q}\{M_{[T, \infty)} < K\}.$$

- When the riskfree rate  $r \neq 0$ , the barrier  $K$  for  $M$  becomes  $Ke^{rt}$  for  $t > T$ .
- Put no touch duality is robust to down jumps; the possibility of up jumps at maturity leads to a bound.
- Using Put-Call Equivalence, there is a Call No-Touch duality as well.
- Using Vanilla No-Touch Duality and time inversion, one can value all finite maturity barrier options in BS.
- By assuming zero risk-neutral drift and employing stochastic time change, the conclusion holds for SV models.
- Bounds arise whenever the underlying asset price can jump across the barrier.