Implied Sampling: Properties and Pitfalls

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Introduction

- In modelling equity derivatives Dupire's local volatility model plays an important role. A market consisting of Vanilla options and nothing else almost naturally leads to this approach.
- However, for multi asset and period equity products this approach is numerically involved. Many practitioners now use more efficient approximations of this model derived from the equity implied distribution.
- In this presentation we discuss properties and pitfalls of this approach.

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Market Implied Distribution

- The distribution of S(t) can be derived from the Vanilla option market, as has been noted by Breeden and Litzenberger:

$$F_{S_t}(K) = \frac{\partial \mathrm{PV}(K,t)}{\partial K} = N(\text{-d\_2}) + \sqrt{t}\; K\; N'(\text{d\_2}) \frac{\partial \hat{\sigma}(K,t)}{\partial K}$$

  here PV(K,t) denotes the (not discounted) market price of a put with strike K and maturity t.

- This allows to obtain model independent prices for payoffs of the form f(S(t)).

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Black-Scholes Copula

- There is less market consensus on the distribution of (S(T),S(t)) for 0<t<T, namely the time copula of S.

- The Black-Scholes model assumes a Gaussian copula with correlation

$$\rho_{tT} = \frac{\hat{\sigma}_t}{\hat{\sigma}_T}\sqrt{\frac{t}{T}} \qquad \text{where} \qquad \hat{\sigma}_t^2 t = \int_0^t \sigma_u^2 \, du$$

  with a deterministic function $\sigma_u, u \geq 0.$

- A fast variant of Dupire's model consists in combining the Black-Scholes time copula with the market implied distribution. Such a process is now widely used by practitioners:

Standard Black-Scholes

- Introducing a Gaussian integral as factor $X_t = \int_0^t \sigma_u dW_u$

  the standard Black-Scholes model can be described (and simulated) as

$$S_t^{BS} = D_t^{BS}(X_t)$$

  where D_t^{BS} is an exponential function, namely

$$D_t^{BS}(x) = F_t \exp\left(x - \frac{\hat{\sigma}_t^2 t}{2} \right)$$

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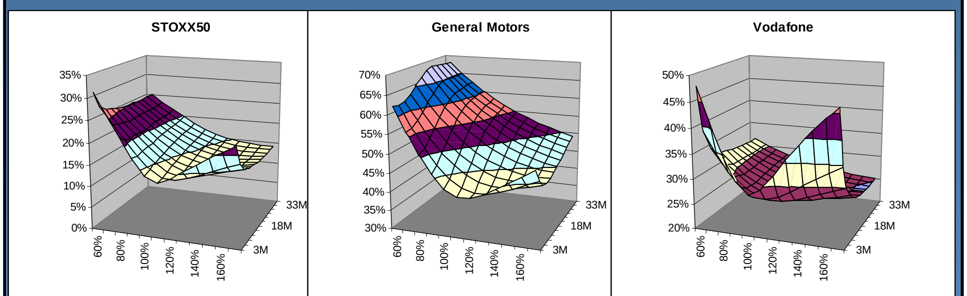
Implied Volatility Surfaces

- However, we have non-flat implied volatility surfaces:



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Implied Black-Scholes

- Thus, preserving the copula one replaces D_t^{BS} by

$$D_t^I(x) = F_{S_t}^{-1}\left(N\left(\frac{x}{\hat{\sigma}_t\sqrt{t}}\right)\right)$$

to obtain

$$S_t^I = D_t^I(X_t)$$

- Lacking a name for this process we shall refer to it as Implied Black-Scholes.

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Local Volatility of Implied Black-Scholes

- An application of Ito 's Lemma shows that

$$\frac{dS_t^I}{S_t^I} = \sigma_t^I(S_t^I)dW_t + \mu_t^I(S_t^I)dt$$

where $$\sigma_t^I(S) = \frac{\sigma_t}{\hat{\sigma}_t}\frac{N'\left(N^{-1}\left(\partial_K PV(S,t)\right)\right)}{S\partial_{KK}^2 PV(S,t)}$$

- Note that the time copula yields parallel shifts of this local volatility function.
- Also, one generally has $\mu_t^I(S) \neq r_t - d_t$

Implied Black-Scholes and Dupire

- Before all of this, Carr, Tari and Zariphopoulou (1999) discuss the problem of finding mapping functions $D_t(x)$ or, equivalently, implied volatility surfaces $\hat{\sigma}_t(K)$ such that Implied Black-Scholes coincides with Dupire, namely

$$\mu_t^I(S) = r_t - d_t$$

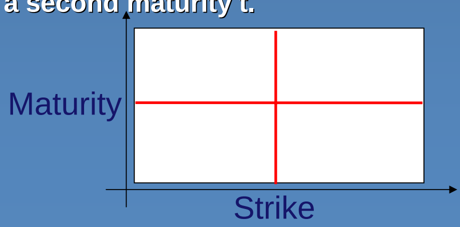
- One main result of their paper is a necessary condition in terms of a PDE on local volatility $\sigma_t^I(S)$

$$\frac{\left(\sigma^I\right)^2}{2}\partial_{SS}^2\sigma^I + (r-q)S\partial_S\sigma^I + \partial_t\sigma^I = \left(r - q + \frac{\sigma'}{\sigma}\right)\sigma^I$$

- Further results include solutions of that PDE.

Implied Black-Scholes and Dupire

- Fixing the distribution at time T and the copula volatility σ_t it follows from this necessary condition that the model is fully specified. Thus it is only possible to calibrate to options expiring at T and, additionally, a term structure of option prices: Matching the distribution at a second date t<T will typically violate the condition.
- An inspection of the special cases proposed by Carr et al. shows that good calibration of the skew for one maturity T can be achieved. However, there are no parameters left to calibrate a second maturity t.

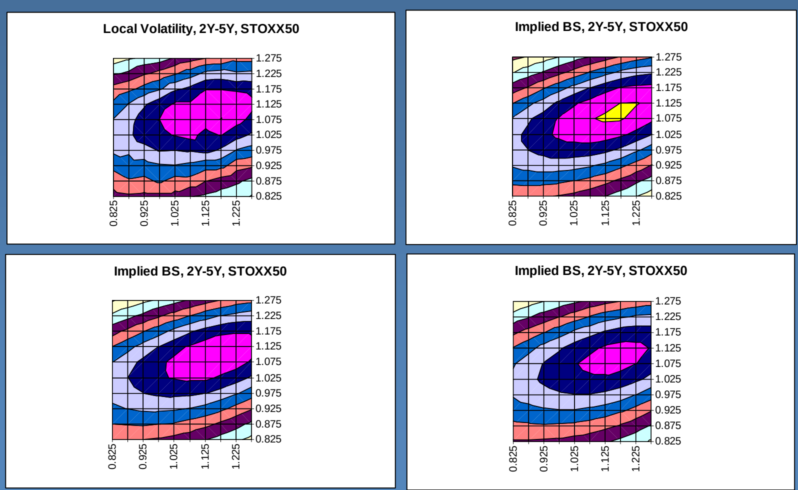


Implied Black-Scholes and Dupire

- Given an implied volatility surface $\hat{\sigma}_t(K)$ there is no reason why Implied Black-Scholes and Dupire should coincide:
- The Dupire process generally does not have a Gaussian time copula.
- The next slide displays the joint distribution of (S(2Y),S(5Y)) for STOXX50, assuming Local Volatility and Gaussian copulas with various levels of autocorrelation

$$\rho_{tT} = \frac{\hat{\sigma}_t}{\hat{\sigma}_T}\sqrt{\frac{t}{T}}$$

Implied Black-Scholes and Dupire



Time Copula: Defining the factor process

In order to specify the factor volatility process σ_t and thus autocorrelation $\rho_{tT} = \frac{\hat{\sigma}_t}{\hat{\sigma}_T}\sqrt{\frac{t}{T}}$ various approaches can be considered:

- Interpolate the term structure at spot: $\sigma_t = \sigma_t(S_0)$
- Interpolate the term structure at the forward: $\sigma_t = \sigma_t(F_t)$
- Interpolate the term structure at a level relevant for the payoff.
- Set the variance of X(t) equal to quoted variance swaps.

- Which one is best?

Time Copula: Defining the factor process

- From a practical prospective arbitrage emerges as a violation of call-put parity for forward starting options, namely $E\left[\frac{S_T}{S_t}\right] \neq \frac{F_T}{F_t}$
- For STOXX 50 we obtain $\log\left(E\left[\frac{S_T}{S_t}\right]\frac{F_t}{F_T}\right)/(T-t)$ as

	1M	2M	3M	4M	5M	6M	7M	8M	9M	10M	11M	12M
1M		-0.03%	-0.02%	0.00%	0.01%	0.03%	0.04%	0.05%	0.06%	0.07%	0.07%	0.08%
2M			0.08%	0.12%	0.17%	0.21%	0.24%	0.27%	0.30%	0.32%	0.34%	0.36%
3M				0.10%	0.19%	0.27%	0.33%	0.38%	0.43%	0.47%	0.51%	0.55%
4M					0.17%	0.31%	0.40%	0.49%	0.57%	0.63%	0.70%	0.75%
5M						0.20%	0.34%	0.47%	0.59%	0.68%	0.76%	0.84%
6M							0.20%	0.38%	0.54%	0.67%	0.79%	0.90%
7M								0.25%	0.48%	0.66%	0.83%	0.97%
8M									0.29%	0.52%	0.73%	0.92%
9M										0.30%	0.58%	0.83%
10M											0.34%	0.63%
11M												0.36%
12M												

- Here we interpolated volatilities of the driver at spot.

Time Copula: Defining the factor process

- The same for Vodafone

	1M	2M	3M	4M	5M	6M	7M	8M	9M	10M	11M	12M
1M		0.08%	0.78%	1.07%	1.28%	1.40%	1.53%	1.62%	1.65%	1.73%	1.78%	1.81%
2M			1.71%	2.46%	3.02%	3.35%	3.64%	3.83%	3.92%	4.10%	4.22%	4.30%
3M				1.06%	1.86%	2.35%	2.75%	3.01%	3.13%	3.38%	3.55%	3.66%
4M					1.21%	1.94%	2.53%	2.90%	3.08%	3.43%	3.67%	3.84%
5M						0.91%	1.63%	2.08%	2.30%	2.73%	3.02%	3.22%
6M							0.86%	1.40%	1.66%	2.18%	2.53%	2.77%
7M								0.66%	1.00%	1.63%	2.06%	2.35%
8M									0.39%	1.11%	1.60%	1.94%
9M										0.86%	1.45%	1.86%
10M											0.67%	1.14%
11M												0.55%
12M												

- and General Motors

	1M	2M	3M	4M	5M	6M	7M	8M	9M	10M	11M	12M
1M		-0.60%	-1.34%	-1.42%	-1.43%	-1.47%	-1.38%	-1.26%	-1.14%	-1.04%	-0.94%	-0.83%
2M			-2.06%	-2.15%	-2.05%	-2.03%	-1.75%	-1.45%	-1.13%	-0.90%	-0.65%	-0.40%
3M				0.20%	1.03%	1.58%	2.42%	3.22%	3.98%	4.57%	5.14%	5.68%
4M					2.21%	3.82%	5.69%	7.41%	8.94%	10.13%	11.25%	12.27%
5M						3.15%	6.43%	9.40%	11.96%	13.95%	15.80%	17.42%
6M							5.35%	10.20%	14.31%	17.54%	20.47%	23.00%
7M								7.74%	14.30%	19.51%	24.21%	28.20%
8M									9.32%	16.86%	23.64%	29.37%
9M										10.94%	20.87%	29.26%
10M											13.24%	24.51%
11M												15.06%
12M												

Time Copula: Defining the factor process

- Enforcing the condition $E\left[\frac{S_T}{S_t}\right]=\frac{F_T}{F_t}$ leads to a matrix of correlations

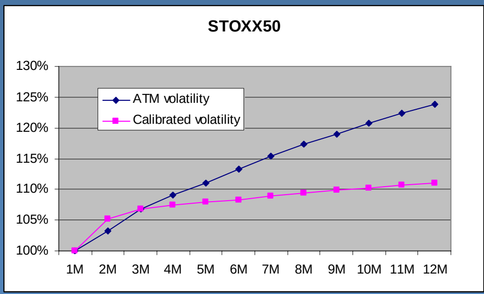
$$\rho_{tT} = f_{tT}\sqrt{\frac{t}{T}}$$

- Note that such a matrix does not necessarily exist.
- For STOXX50 we calibrated the factors f(t,T):

	1M	2M	3M	4M	5M	6M	7M	8M	9M	10M	11M	12M
1M		95.11%	92.79%	91.62%	90.59%	89.57%	88.58%	87.60%	86.73%	85.91%	85.13%	84.47%
2M			98.41%	97.53%	96.72%	95.85%	94.93%	93.99%	93.16%	92.36%	91.59%	90.93%
3M				99.38%	98.80%	98.08%	97.25%	96.39%	95.62%	94.86%	94.13%	93.51%
4M					99.64%	99.09%	98.38%	97.60%	96.89%	96.20%	95.52%	94.93%
5M						99.60%	99.00%	98.33%	97.69%	97.05%	96.42%	95.87%
6M							99.51%	98.93%	98.36%	97.79%	97.21%	96.70%
7M								99.51%	99.02%	98.51%	97.98%	97.52%
8M									99.58%	99.14%	98.68%	98.25%
9M										99.62%	99.21%	98.83%
10M											99.65%	99.32%
11M												99.72%
12M												

Time Copula: Defining the factor process

- The first off diagonal values determine a term structure of volatilities $\hat{\sigma}_t$ for a driver process, which we compare with atm volatilities with the starting point set to 100%:



Time Copula: Defining the factor process

- The result is a reduced drift error for STOXX50

	1M	2M	3M	4M	5M	6M	7M	8M	9M	10M	11M	12M
1M		0.00%	-0.02%	-0.03%	-0.04%	-0.06%	-0.07%	-0.08%	-0.09%	-0.10%	-0.11%	-0.12%
2M			0.00%	-0.01%	-0.03%	-0.06%	-0.08%	-0.10%	-0.12%	-0.14%	-0.17%	-0.19%
3M				0.00%	-0.02%	-0.04%	-0.07%	-0.09%	-0.12%	-0.15%	-0.18%	-0.21%
4M					0.00%	-0.02%	-0.04%	-0.07%	-0.10%	-0.14%	-0.18%	-0.21%
5M						0.00%	-0.01%	-0.04%	-0.07%	-0.11%	-0.15%	-0.19%
6M							0.00%	-0.02%	-0.04%	-0.08%	-0.12%	-0.16%
7M								0.00%	-0.02%	-0.04%	-0.08%	-0.12%
8M									0.00%	-0.02%	-0.04%	-0.08%
9M										0.00%	-0.02%	-0.05%
10M											0.00%	-0.02%
11M												0.00%
12M												

in comparison with the original obtained with atm volatilities:

	1M	2M	3M	4M	5M	6M	7M	8M	9M	10M	11M	12M
1M		-0.03%	-0.02%	0.00%	0.01%	0.03%	0.04%	0.05%	0.06%	0.07%	0.07%	0.08%
2M			0.08%	0.12%	0.17%	0.21%	0.24%	0.27%	0.30%	0.32%	0.34%	0.36%
3M				0.10%	0.19%	0.27%	0.33%	0.38%	0.43%	0.47%	0.51%	0.55%
4M					0.17%	0.31%	0.40%	0.49%	0.57%	0.63%	0.70%	0.75%
5M						0.20%	0.34%	0.47%	0.59%	0.68%	0.76%	0.84%
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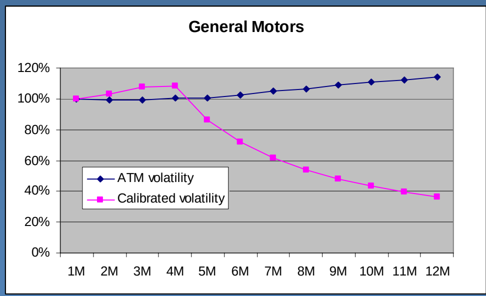
8M									0.29%	0.52%	0.73%	0.92%
9M										0.30%	0.58%	0.83%
10M											0.34%	0.63%
11M												0.36%
12M												

Time Copula: Defining the factor process

- This worked well for STOXX50. However, for GM we have



which implies negative forward variances.

Local Drift

- A more general (sufficient) condition for absence of arbitrage is given by

$$E\left[\frac{S_T}{S_t}1_{\{S_t>B\}}\right]=\frac{F_T}{F_t}P[S_t>B]$$

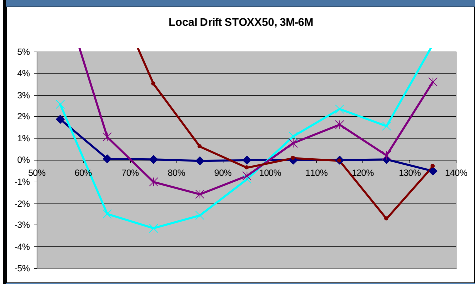
- We now display the local drift in terms of

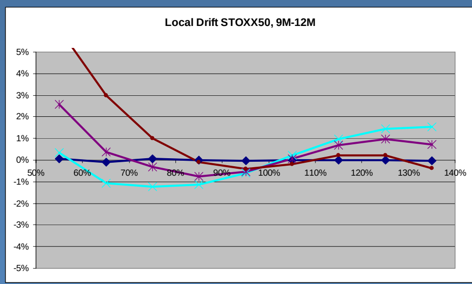
$$\log\left(E\left[\frac{S_{t+dt}}{S_t}1_{\{S_t\in dB\}}\right]\frac{F_t}{F_{t+dt}}/P[S_t\in dB]\right)/dt$$

where dt = 3M and dB = 10%

Local Drift

- The blue line refers to Local Volatility, the other lines refer to different interpolation levels of the volatility of the driver:





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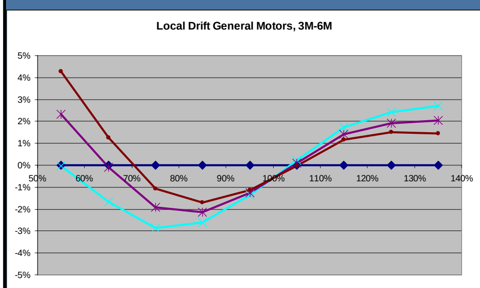
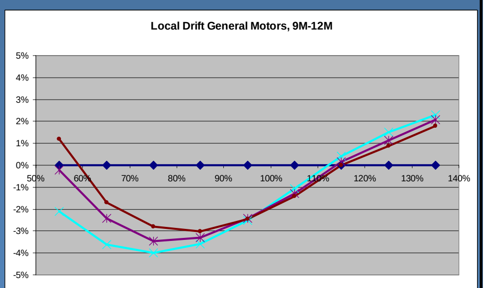
Local Drift

- The same for General Motors:





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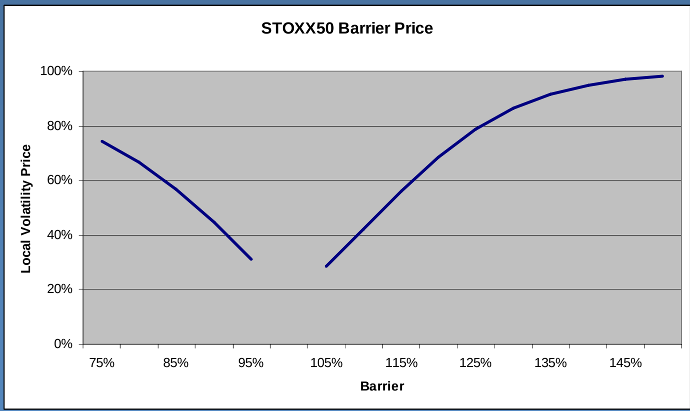
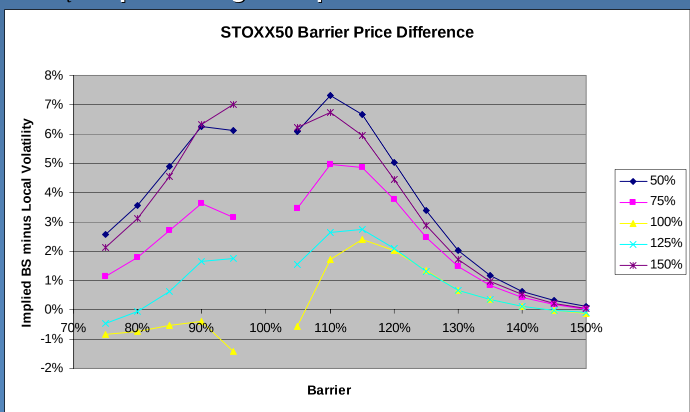
Barrier Prices

- Prices for Knock Out Barriers (paying 1 if down- or up-barrier is not hit, with maturity 2Y and sampling every 2M), no discounting:



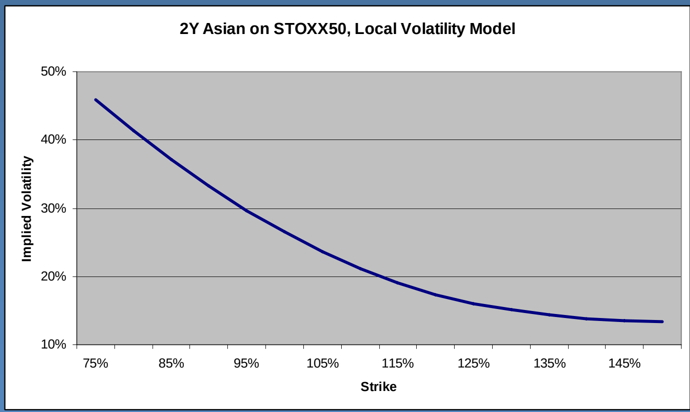
Barrier Prices

- And price difference for Knock Out Barriers (paying 1 if down- or up-barrier is not hit, with maturity 2Y and sampling every 2M). Each line corresponds to an interpolation level of σ_t in percentage of spot.



Asian Option

- Implied volatility for Asian Option (2Y Asian with sampling every 2M) using Local Volatility model:



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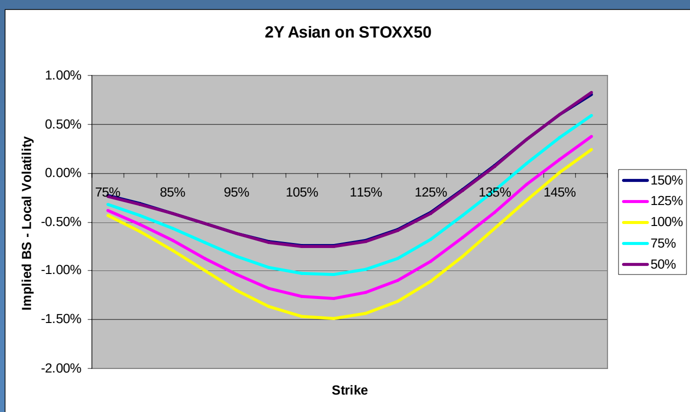
Asian Option

- And implied volatility difference to Implied Black Scholes model, interpolated at different percentages of spot:



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