Efficient computations for the local volatility model and consistent pricing of multiunderlying derivatives

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MathFinance Workshop
April 14-15, 2005
Frankfurt am Main

- After 1987 BS model needed correction: constant volatility – incorrect assumption

- Many different models have been developed since then. Question: which model should we use in practice?? No firm answer…
- Depends on particular circumstances? (markets,…) Yes and No.
- Type of the model 🡐 The way we see the underlying process

- Statistical inference… for financial data rarely gives high and stable confidence levels (nonstationarity!)

- Multiple markets…
- Global risk management…

➔ We need some robust and ***universal*** approach!

Good practical model

- Complete parametric form
- Clear meaning of parameters: should be easy to calibrate ➔ few variable parameters
- Change in each parameter – separate risk
- Efficient calculations

Local Volatility Models

- Local volatility model by Derman & Kani (1994). Diffusion process

$$d\ \ln X_t = r_t\, dt + \sigma(X_t, t)\, dW_t$$

- Constant volatility BS ➔ Constant volatility surface LV

$$\sigma_{const} \quad \Rightarrow \quad \sigma_{const}(X_t, t)$$

Local Volatility Models

- Most used in practice – most criticized in theory
- Two objectives: static and dynamic fit of the market
- Strong advantage – good static fit: easy and perfect
- Big problem – bad dynamic fit: underlying moves away – theos go wrong

Alternative Models

- Stochastic volatility (SV) models: two sources of uncertainty

$$d \ln X_t = r_t \, dt + \sigma_t \, dW_t$$

$$d\sigma_t^2 = \quad \ldots \quad d\tilde{W}_t$$

$$< dW_t, d\tilde{W}_t > = \rho$$

Alternative Models

- Define "equivalent" Markov process: combine all local distributions

$$P_X(x,t)$$

(mixtures of normals) into the Markov process

$$d \ln \hat{X}_t \cong r_t \, dt + \hat{\sigma}(\hat{X}_t, t) \, d\hat{Z}_t$$

All 1-dim local distributions are the same.

➔ Prices of european options are equal.

- Back to one source of uncertainty
- Similar asymptotic properties
- Exotic options prices?
- Significant difference between local volatilities $\hat{\sigma}$ and σ . First is not constant, it depends on the initial value of the process, but! dependence is smooth and deterministic
- Modifying: LV ➔ DLV

$$\sigma_{const}(X_t, t) \quad \Rightarrow \quad \sigma_{X_0}(X_t, t)$$

Parametric DLV model

- Static parameters: $p_1 \cdots p_n$
- Dynamic parameters: slopes $\pi_1 \ldots \pi_n$

$$\pi_j = \frac{dp_j}{d \ln X_0} \quad \Rightarrow \quad p_j^{new} \cong p_j^{old} + \pi_j \, d \ln X_0$$

- Smooth and stable in some neighborhood
- Too many parameters?
- Parameter stability

Building DLV tree

- Special case:

$$\sigma(X_t, t) = \sigma(X) \times \sigma(t)$$

- easier for listed options, wrong for exotics
- term structure of parameters:

  (a) const (+0)

  (b) power law (+1)

  (c) exponential convergence (+2)

  automatic fit, parameter stability

Multiasset options

- N assets:

$$X_1, \ldots X_N \quad \sigma_1(X_1), \ldots \sigma_N(X_N) \quad (\rho_{i\,j})$$

- Multivariate function:

$$F(X_1, \ldots X_N)$$

- Examples:

$$F = \sum_i w_i \; X_i \quad w_i <> 0$$

$$F = \min / \max\,(w_1 X_1, \ldots w_N X_N)$$

Multiasset options

1-dim tree for F!

- must be consistent with all given information
- must be asymptotically precise
- General algorithm:

(0) choose small time step dt

(1) calculate

$$\sigma^0(F(X_1^0, \ldots X_N^0))$$

Multiasset options

(2) build initial K-node with neutral mean and given volatility

(3) for point C for every underlying calculate

$$X_i^C = E(X_i \mid F = C)$$

$$i = 1, \ldots N$$

Multiasset options

(4) for all underlyings get new values

$$\sigma_i^C\,(X_i^C) \quad i = 1, \dots N$$

(5) out of those and correlations calculate

$$\sigma_F^C(F(X_1^C, \dots X_N^C))$$

(6) build next point D in the tree and assign probabilities with neutral expectation

(7) repeat from (3) with a new point C’

Multiasset options (baskets)

- Basket options:

  underlyings are lognormals

  for small time-space neighborhood we use

  normal approximation to calculate local volatility of the basket at each point

  → analytical solution, no MC needed
- The way correlations are specified…

Multiasset options (min/max)

- If correlations are nonzero (which is normally the case), no analytical solution exists to calculate local volatilities σ_F^C
- need to use MC for this step
- but if we use normal approximations for lognormals then quite straightforward analytical solution exists for calculation of conditional expectations in step (3)

Multiasset options (general case)

- In general case for arbitrary F we use MC for estimation of local volatilities σ_F^C
- **in parallel** with this we can estimate conditional means needed in step (3) for each underlying through building regressions of these means on the values of F; usually quadratic regression is enough
- some smoothness properties of F needed

Conclusions

- …