

Interest Rate Convexity and the Volatility Smile

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1 Introduction – Interest Rate Convexity

Notation:

$$\begin{aligned} B(t, T) & \text{ zero bond at time } t \text{ with maturity } T \\ L(T_1, T_2) &= \frac{1}{B(T_1, T_2) \Delta} - 1 \quad \text{spot Libor for interval } [T_1, T_2] \\ L_0(T_1, T_2) &= \frac{B(0, T_1)}{B(0, T_2) \Delta} - 1 \quad \text{forward Libor for interval } [T_1, T_2] \end{aligned}$$

Libor payment in standard swap (set in advance, pay in arrears), price \mathbf{P} today

$$\mathbf{P}(L(T_1, T_2) \text{ paid at } T_2) = B(0, T_2)L_0(T_1, T_2).$$

Libor-in-arrears (set in arrears)

$$\mathbf{P}(L(T_1, T_2) \text{ paid at } T_1) \neq B(0, T_1)L_0(T_1, T_2)$$

$$\mathbf{P}(L(T_1, T_2) \text{ paid at } T_1) = B(0, T_1)L_0(T_1, T_2)(1 + \text{convexity})$$

Where is the convexity coming from:

$$\begin{aligned} & \mathbf{P}(L(T_1, T_2) \text{ @ } T_1) \\ &= \mathbf{P}(L(T_1, T_2)(1 + \Delta L(T_1, T_2)) \text{ @ } T_2) \\ &= \dots \\ &= B(0, T_1)L_0(T_1, T_2) \left(1 + \frac{\Delta L_0(T_1, T_2)(\exp(\sigma^2 T_1) - 1)}{(1 + \Delta L_0(T_1, T_2))} \right) \end{aligned}$$

Interest Rate Convexity and the Volatility Smile

Now here is the problem:

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EUR Caps/Floors - Implied Volatilities															
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	STK	ATM	1.50	1.75	2.00	2.25	2.5	3.0	3.5	4.0	5.0	6.0	7.0	8.0	10.0
1Y	2.31	15.99	20.4	17.6	15.7	16.0	18.0	18.6	19.2	19.7	22.1	24.1	25.3	26.6	29.0
2Y	2.58	20.94	26.5	23.3	20.2	19.7	20.7	21.4	22.0	22.3	23.0	24.3	25.6	26.8	28.8
3Y	2.87	21.35	27.7	25.1	22.6	21.6	21.5	21.3	21.3	21.6	22.7	24.0	25.3	26.5	30.1
4Y	3.04	20.73	27.5	25.1	23.0	21.9	21.5	20.8	20.4	20.4	21.3	22.4	23.6	24.7	27.0
5Y	3.20	19.90	27.1	24.9	23.0	21.9	21.3	20.2	19.6	19.3	19.8	20.8	21.9	22.9	24.6
6Y	3.34	19.23	27.0	24.9	23.0	21.9	21.1	19.9	19.0	18.6	18.7	19.6	20.5	21.4	23.1
7Y	3.47	18.64	26.9	24.8	23.0	21.9	21.0	19.6	18.6	18.0	17.9	18.5	19.4	20.2	21.8
8Y	3.58	18.08	26.6	24.6	22.8	21.8	20.9	19.3	18.2	17.5	17.2	17.7	18.4	19.3	20.7
9Y	3.68	17.63	26.5	24.5	22.7	21.7	20.7	19.1	18.0	17.2	16.7	17.1	17.8	18.5	20.0
10Y	3.76	17.27	26.2	24.4	22.6	21.6	20.6	19.0	17.8	16.9	16.3	16.6	17.3	18.0	19.3
12Y	3.88	16.80	26.0	24.3	22.6	21.6	20.6	18.8	17.5	16.6	15.8	16.0	16.6	17.3	18.6
15Y	4.01	16.01	25.4	23.8	22.3	21.3	20.2	18.4	17.0	16.0	15.1	15.2	15.7	16.4	17.7
20Y	4.13	15.06	24.7	23.2	21.7	20.8	19.7	17.8	16.4	15.3	14.2	14.3	14.8	15.5	16.8

Please note 1 and 2 yrs are vs 3 mth

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<VCAP2>

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OPTIONS Index <VCAP>
 ICAP Global Index <ICAP>

RIC Index Pages <ICAPREJ>
 Forthcoming changes <ICAPCHANGE>

Which volatility σ to take?? The at-the-money vol?

Other interest rate products involving convexity:

- In-arrears cap: $\max(L(T_1, T_2) - K, 0)$ paid at T_1
- Denote by $C(T_0, T_n)$ the fair swap rate at time T_0 for a swap with reference dates $T_0 < T_1 < \dots < T_n$,

$$C(T_0, T_n) = \frac{1 - B(T_0, T_n)}{\sum_{i=1}^n \Delta_i B(T_0, T_i)},$$

with Δ_i denoting the length of the interval $[T_{i-1}, T_i]$.

CMS swaplet: $C(T_0, T_n)$ paid at time T_1 .

- CMS caplet: $\max(C(T_0, T_n) - K, 0)$ paid at time T_1

2 Setup of the problem

Underlying with price $Y > 0$ at time T . Suppose liquid market for plain vanilla options. Somewhat more general, suppose that for all $K > 0$ the price $\mathbf{P}(K)$ of the "plain vanilla" derivative with payoff

$$g(Y) \max(Y - K, 0)$$

is known. Goal is to price an exotic contingent claim with payoff

$$f(Y)$$

at time T .

If

$$f(Y) = \int_0^\infty g(Y) \max(Y - K, 0) d\mu(K) \quad (1)$$

with some locally finite signed measure μ on $[0, \infty)$, then by no-arbitrage the fair price $\mathbf{P}(f(Y))$ is

$$\mathbf{P}(f(Y)) = \int_0^\infty \mathbf{P}(K) d\mu(K), \quad (2)$$

provided the integral on the right hand side is well-defined.

3 Target applications related to irregular interest rate derivatives

Example 1:

The standard caplet $\Delta \max(L(T_1, T_2) - K, 0)$ paying at time T_2 is equivalent to a product which pays the amount

$$\frac{\Delta}{1 + \Delta L(T_1, T_2)} \max(L(T_1, T_2) - K, 0)$$

at time T_1 . So the market gives us prices for the time T_1 payout

$$g(L(T_1, T_2)) \max(L(T_1, T_2) - K, 0) \text{ where } g(y) = \frac{\Delta}{1 + \Delta y}$$

for any strike K . Our goal is to price an "exotic" payoff

$$f(L(T_1, T_2))$$

at time T_1 , where $f(y)$ is, for example, $f(y) = \Delta y$ in case of a Libor-in-arrears, or, $f(y) = \Delta \max(y - K, 0)$ in the case of an in-arrears caplet.

Example 2:

The payoff of a *cash-settled payer swaption*,

$$\left(\sum_{i=1}^n \frac{\Delta_i}{(1 + C(T_0, T_n))^{T_i - T_0}} \right) \max(C(T_0, T_n) - K, 0), \quad (3)$$

is again of the form $g(C(T_0, T_n)) \max(C(T_0, T_n) - K, 0)$ where

$$g(y) = \text{DV01}(y) = \sum_{i=1}^n \frac{\Delta_i}{(1 + y)^{T_i - T_0}}. \quad (4)$$

The exotic payoffs $f(C(T_0, T_n))$ of interest are, for example, the payment of

$$f(C(T_0, T_n)) = C(T_0, T_n), \quad (5)$$

paid at time T_0 . This is like a cash flow in a CMS swap, but paid in advance. Another example of interest is a CMS caplet,

$$f(C(T_0, T_n)) = \max(C(T_0, T_n) - \bar{K}, 0) \quad (6)$$

to be paid at time T_0 .

In case the payoff is scheduled for time $p \geq T_0$ we can approximately think of a derivative with payoff

$$f(C(T_0, T_n)) = \frac{C(T_0, T_n)}{(1 + C(T_0, T_n))^{p-T_0}} \text{ for a CMS swap} \quad (7)$$

$$f(C(T_0, T_n)) = \frac{\max(C(T_0, T_n) - \bar{K}, 0)}{(1 + C(T_0, T_n))^{p-T_0}} \text{ for a CMS caplet.} \quad (8)$$

Example 3:

For a *physically-settled payer swaption* the payoff at time T_0 is

$$\sum_{i=1}^n \Delta_i B(T_0, T_i) \max(C(T_0, T_n) - K, 0). \quad (9)$$

The market prices swaptions under the so-called swap measure \mathbf{Q}_{Swap} which is the equivalent martingale measure referring to the numeraire $N_t = \sum_{i=1}^n \Delta_i B(t, T_i)$. The price of a swaption is

$$\mathbf{P}(K) = N_0 \mathbf{E}_{\mathbf{Q}_{\text{Swap}}} \max(C(T_0, T_n) - K, 0). \quad (10)$$

An exotic derivative paying $f(C(T_0, T_n))$ at some time $p \geq T_0$ is equivalent to a product which pays $B(T_0, p) f(C(T_0, T_n))$ at time T_0 . Under the assumption

of a linear swap rate model we have

$$\frac{B(T_0, p)}{N_{T_0}} = \alpha + \beta_p C(T_0, T_n),$$

and the price of $f(C(T_0, T_n))$ to be paid at time p is given by

$$N_0 \mathbf{E}_{\mathbf{Q}_{\text{Swap}}} \frac{B(T_0, p) f(C(T_0, T_n))}{N_{T_0}} = N_0 \mathbf{E}_{\mathbf{Q}_{\text{Swap}}} (\alpha + \beta_p C(T_0, T_n)) f(C(T_0, T_n)). \quad (11)$$

Here

$$\alpha = \frac{1}{\sum_{i=1}^n \Delta_i} \quad (12)$$

$$\beta_p = \frac{\frac{B(0,p)}{\sum_{i=1}^n \Delta_i B(0,T_i)} - \alpha}{C_0(T_0, T_n)} \quad (13)$$

$$C_0(T_0, T_n) = \frac{B(0, T_0) - B(0, T_n)}{\sum_{i=1}^n \Delta_i B(0, T_i)}. \quad (14)$$

In our replication approach we are therefore looking for a representation (1) of the form

$$(\alpha + \beta_p Y) f(Y) = \int_0^\infty \max(Y - K, 0) d\mu(K). \quad (15)$$

4 The result

Proposition 1. *The exotic payoff $f(Y)$ allows for a replication (1) with some locally finite signed measure μ on $[0, \infty)$ if and only if the function $\frac{f}{g}$ is a difference of convex functions on $[0, \infty)$. The measure μ is then generated by the following right continuous generalized distribution function (function of locally bounded variation)*

$$d\mu(y) = dD_+ \left(\frac{f(y)}{g(y)} \right),$$

with D_+ denoting the right hand derivative and defining $D_+ \left(\frac{f}{g} \right) (0-) = 0$.

Proof. We identify the signed measure μ with its generalized right continuous distribution function, $\mu(0-) = 0$. Integrating by parts the representation (1) is equivalent to

$$\begin{aligned}\frac{f(y)}{g(y)} &= \int_0^\infty \max(y - K, 0) d\mu(K) \\ &= \int_{[0, y]} (y - K) d\mu(K) \\ &= y \mu(y) - \int_0^y \mu(K) dK \\ &= \int_0^y \mu(K) dK.\end{aligned}$$

The assertion now follows. □

5 Application to the examples

Let us apply the above result to our examples. As we shall see in these examples the measure μ normally possesses a jump.

Example 1:

For the in-arrears caplet $f(L(T_1, T_2)) = \Delta \max(L(T_1, T_2) - \bar{K}, 0)$ paid at time T_1 the corresponding distribution function μ is

$$\begin{aligned} \frac{f(y)}{g(y)} &= \max(y - \bar{K}, 0)(1 + \Delta y) \\ \mu(y) &= 0, \quad y < \bar{K}, \\ \mu(y) &= (1 - \Delta \bar{K}) + 2\Delta y, \quad y \geq \bar{K}, \end{aligned}$$

and the resulting pricing equation is

$$\mathbf{P}(\Delta \max(L(T_1, T_2) - \bar{K}, 0) @ T_1) = \mathbf{P}(\bar{K})(1 + \Delta \bar{K}) + 2\Delta \int_{\bar{K}}^{\infty} \mathbf{P}(K) dK. \quad (16)$$

Example 2:

Write

$$\text{DV01}_p(y) = \sum_{i=1}^n \frac{\Delta_i}{(1+y)^{T_i-p}}. \quad (17)$$

For the CMS caplet (8) which pays out at time $p \geq T_0$ the replication by plain-vanilla swaptions is derived as

$$\begin{aligned} \frac{f(y)}{g(y)} &= \frac{\max(y - \bar{K}, 0)}{\text{DV01}_p(y)} \\ \mu(y) &= 0, \quad y < \bar{K}, \\ \mu(y) &= \frac{\text{DV01}_p(y) - (y - \bar{K})\text{DV01}_p'(y)}{\text{DV01}_p^2(y)}, \quad y \geq \bar{K}, \end{aligned}$$

and we end up with the pricing formula

$$\mathbf{P}(\max(C(T_0, T_n) - \bar{K}, 0)) = \frac{\mathbf{P}(\bar{K})}{\text{DV01}_p(\bar{K})} + \int_{\bar{K}}^{\infty} h(K) \mathbf{P}(K) dK, \quad (18)$$

where

$$h(K) = \left\{ (K - \bar{K}) \left[2 \frac{(\text{DV01}_p'(K))^2}{\text{DV01}_p^3(K)} - \frac{\text{DV01}_p''(K)}{\text{DV01}_p^2(K)} \right] - 2 \frac{\text{DV01}_p'(K)}{\text{DV01}_p^2(K)} \right\}.$$

Example 3:

For a CMS caplet $\max(C(T_0, T_n) - \bar{K}, 0)$ paid at time $p \geq T_0$ the distribution μ is

$$\begin{aligned}\mu(y) &= 0, y < \bar{K} \\ \mu(y) &= (\alpha - \bar{K}\beta_p) + 2\beta_p y, y \geq \bar{K},\end{aligned}$$

and we end up with the pricing formula

$$\mathbf{P}(\max(C(T_0, T_n) - \bar{K}, 0) \text{ paid at time } p) = \mathbf{P}(\bar{K})(\alpha + \beta_p \bar{K}) + \int_{\bar{K}}^{\infty} \mathbf{P}(K) 2\beta_p dK. \quad (19)$$

6 Numerical examples

We illustrate the smile effect on the convexity by some numerical examples. The examples are based on the EUR interest rate curve as of October 18, 2004.

6.1 Libor-in-arrears and in-arrears caps

Examples are based on the caplet volatility surface as of October 18, 2004 which is the primary input for getting the price $\mathbf{P}(K)$ for an individual caplet with strike K .

Interest Rate Convexity and the Volatility Smile

strike $\bar{K} = 0\%$			strike $\bar{K} = 2\%$		
Replication	adj Fwd	adj Fwd & Vol	Replication	adj Fwd	adj Fwd & Vol
34,253%	34,224%	34,224%	18,883%	18,858%	18,870%
1,396%	1,396%	1,396%	0,412%	0,412%	0,412%
1,523%	1,523%	1,523%	0,567%	0,567%	0,567%
1,669%	1,669%	1,669%	0,724%	0,723%	0,723%
1,632%	1,632%	1,632%	0,718%	0,717%	0,717%
1,828%	1,826%	1,826%	0,910%	0,909%	0,909%
1,764%	1,763%	1,763%	0,886%	0,884%	0,885%
1,875%	1,873%	1,873%	1,010%	1,009%	1,009%
1,836%	1,834%	1,834%	0,994%	0,993%	0,993%
1,975%	1,973%	1,973%	1,142%	1,140%	1,141%
1,920%	1,918%	1,918%	1,113%	1,112%	1,113%
1,954%	1,952%	1,952%	1,161%	1,160%	1,160%
1,897%	1,896%	1,896%	1,130%	1,129%	1,130%
1,959%	1,957%	1,957%	1,205%	1,203%	1,204%
1,911%	1,908%	1,908%	1,179%	1,177%	1,178%
1,929%	1,926%	1,926%	1,206%	1,204%	1,205%
1,850%	1,848%	1,848%	1,157%	1,156%	1,157%
1,818%	1,816%	1,816%	1,144%	1,142%	1,143%
1,782%	1,780%	1,780%	1,122%	1,120%	1,121%
1,735%	1,733%	1,733%	1,103%	1,101%	1,103%

Interest Rate Convexity and the Volatility Smile

strike $\bar{K} = 8\%$			strike $\bar{K} = 10\%$		
Replication	adj Fwd	adj Fwd & Vol	Replication	adj Fwd	adj Fwd & Vol
1,072%	1,055%	1,064%	0,420%	0,410%	0,416%
0,000%	0,000%	0,000%	0,000%	0,000%	0,000%
0,001%	0,001%	0,001%	0,000%	0,000%	0,000%
0,007%	0,007%	0,007%	0,001%	0,001%	0,001%
0,012%	0,011%	0,011%	0,003%	0,003%	0,003%
0,027%	0,026%	0,026%	0,008%	0,008%	0,008%
0,027%	0,026%	0,027%	0,009%	0,008%	0,008%
0,041%	0,040%	0,040%	0,014%	0,014%	0,014%
0,038%	0,037%	0,037%	0,013%	0,012%	0,012%
0,061%	0,060%	0,061%	0,022%	0,022%	0,022%
0,060%	0,059%	0,059%	0,022%	0,021%	0,022%
0,069%	0,068%	0,068%	0,026%	0,025%	0,025%
0,065%	0,064%	0,065%	0,024%	0,023%	0,024%
0,085%	0,084%	0,084%	0,033%	0,033%	0,033%
0,085%	0,084%	0,085%	0,034%	0,033%	0,034%
0,098%	0,096%	0,097%	0,040%	0,040%	0,040%
0,097%	0,096%	0,096%	0,041%	0,040%	0,040%
0,100%	0,099%	0,100%	0,043%	0,042%	0,042%
0,100%	0,098%	0,099%	0,043%	0,042%	0,043%
0,101%	0,099%	0,100%	0,044%	0,043%	0,043%

Overall the results are quite close and the two naive approaches (adjusted forward rates without and with additional adjusted volatilities) yield results whose differences to the correct one from replication are negligible in practice. In general the results from adjusted forwards and adjusted volatilities are somewhat closer to the correct one.

These are good news for practitioners indicating that naive approaches are fairly sufficient for in-arrears caps and swaps.

6.2 Constant maturity swaps and CMS caps

The swaption market quotes at-the-money implied volatilities for a variety of swaption maturities and tenors for the underlying swap. The swaption smile is rarely quoted on publicly available data sources. Our analysis below is based on a swaption smile surface which contains implied volatilities for all combinations of the following option maturities, swap tenors and strikes

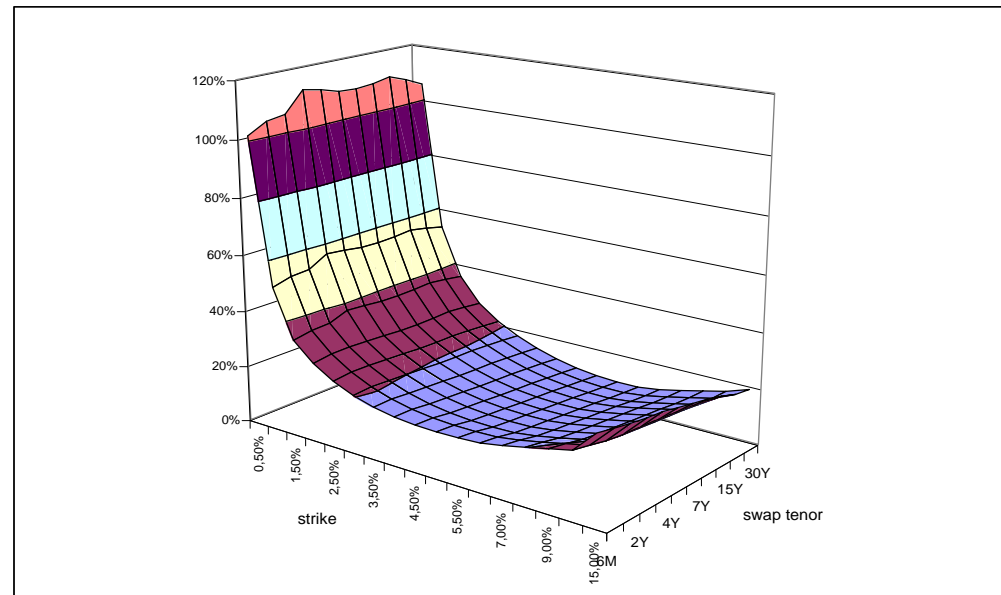
Interest Rate Convexity and the Volatility Smile

Maturities	Tenors	Strikes
1M	6M	0,50%
2M	1Y	1,00%
3M	2Y	1,50%
6M	3Y	2,00%
1Y	4Y	2,50%
2Y	5Y	3,00%
3Y	7Y	3,50%
4Y	10Y	4,00%
5Y	15Y	4,50%
7Y	20Y	5,00%
10Y	30Y	5,50%
15Y		6,00%
20Y		7,00%
30Y		8,00%
		9,00%
		10,00%
		15,00%

Since this volatility surface consists of 2618 data points we are not able to show it here completely.

Interest Rate Convexity and the Volatility Smile

As an example here is the implied swaption volatility surface for 5 years maturity options with different swap tenors and strike rates.



Here are the results for a 10 years CMS cap with semi-annual periods and a 30/360 payment convention for different strikes. The tenor of the CMS rate is 10Y. The case of strike $\bar{K} = 0\%$ corresponds to the floating leg of a CMS swap.

Interest Rate Convexity and the Volatility Smile

strike $K = 0\%$			
Replication (19)	Replication (18)	adj Fwd	adj Fwd & Vol
40,882%	41,211%	40,351%	40,351%
2,100%	2,121%	2,095%	2,095%
2,149%	2,169%	2,147%	2,147%
2,190%	2,209%	2,186%	2,186%
2,217%	2,236%	2,210%	2,210%
2,257%	2,273%	2,248%	2,248%
2,231%	2,248%	2,219%	2,219%
2,234%	2,248%	2,217%	2,217%
2,245%	2,260%	2,225%	2,225%
2,240%	2,253%	2,216%	2,216%
2,230%	2,247%	2,203%	2,203%
2,207%	2,223%	2,175%	2,175%
2,181%	2,197%	2,145%	2,145%
2,155%	2,170%	2,115%	2,115%
2,125%	2,142%	2,082%	2,082%
2,114%	2,132%	2,067%	2,067%
2,058%	2,076%	2,009%	2,009%
2,013%	2,032%	1,962%	1,962%
2,001%	2,020%	1,948%	1,948%
1,936%	1,956%	1,883%	1,883%

Interest Rate Convexity and the Volatility Smile

strike $K = 5\%$			
Replication (19)	Replication (18)	adj Fwd	adj Fwd & Vol
5,104%	5,224%	4,766%	4,788%
0,010%	0,010%	0,009%	0,009%
0,039%	0,040%	0,038%	0,038%
0,078%	0,080%	0,077%	0,077%
0,117%	0,119%	0,114%	0,114%
0,155%	0,157%	0,150%	0,151%
0,188%	0,191%	0,182%	0,183%
0,221%	0,224%	0,213%	0,214%
0,254%	0,258%	0,244%	0,245%
0,285%	0,290%	0,272%	0,273%
0,314%	0,321%	0,298%	0,299%
0,334%	0,341%	0,314%	0,315%
0,351%	0,359%	0,328%	0,329%
0,366%	0,375%	0,340%	0,342%
0,380%	0,389%	0,350%	0,352%
0,393%	0,403%	0,361%	0,362%
0,397%	0,408%	0,363%	0,365%
0,401%	0,413%	0,366%	0,368%
0,411%	0,424%	0,374%	0,376%
0,411%	0,423%	0,373%	0,375%

Interest Rate Convexity and the Volatility Smile

strike $K = 8\%$			
Replication (19)	Replication (18)	adj Fwd	adj Fwd & Vol
0,978%	1,054%	0,657%	0,678%
0,000%	0,000%	0,000%	0,000%
0,000%	0,000%	0,000%	0,000%
0,003%	0,004%	0,003%	0,003%
0,010%	0,010%	0,008%	0,009%
0,016%	0,017%	0,013%	0,014%
0,021%	0,022%	0,017%	0,018%
0,028%	0,030%	0,022%	0,023%
0,035%	0,037%	0,026%	0,027%
0,043%	0,046%	0,032%	0,033%
0,052%	0,056%	0,037%	0,038%
0,060%	0,065%	0,042%	0,043%
0,068%	0,073%	0,046%	0,047%
0,075%	0,080%	0,049%	0,051%
0,080%	0,087%	0,052%	0,053%
0,088%	0,095%	0,056%	0,058%
0,093%	0,101%	0,059%	0,061%
0,097%	0,106%	0,062%	0,064%
0,102%	0,111%	0,065%	0,067%
0,105%	0,114%	0,067%	0,069%

For all strikes K we see a significant difference between the prices obtained from replication compared with the prices from naive adjustment approaches. In case of a CMS swap (i.e., $K = 0\%$) the effect from the swaption smile leads to difference of about 7 basis point in case of replication (19) or even 11 basis point for a replication based on (18) in the fair fixed rate relative to the naive valuation. This is clearly not ignorable

For at-the-money CMS caps the effect from the swaption smile is even more pronounced. The naive approaches, although taking into account the smile by taking the caplet volatility from the swaption smile, show a significant mispricing relative to the correct valuations based on the replication (18), (19).

References

- [1] BOENKOST, W. AND SCHMIDT, W.: Notes on convexity and quanto adjustments for interest rates and related options, Working Paper No. 47, Hochschule für Bankwirtschaft, 2003
<http://www.hfb.de/Dateien/Arbeits47e.pdf>

- [2] HAGAN, P.S.: Convexity Conundrums: Pricing CMS Swaps, Caps and Floors, *Wilmott Magazine*, 2004

- [3] HUNT, P.J., KENNEDY, J.E.: *Financial Derivatives in Theory and Practice*, Wiley, 2000