Some valuation models for CDOs and Basket Credit Derivatives[1]

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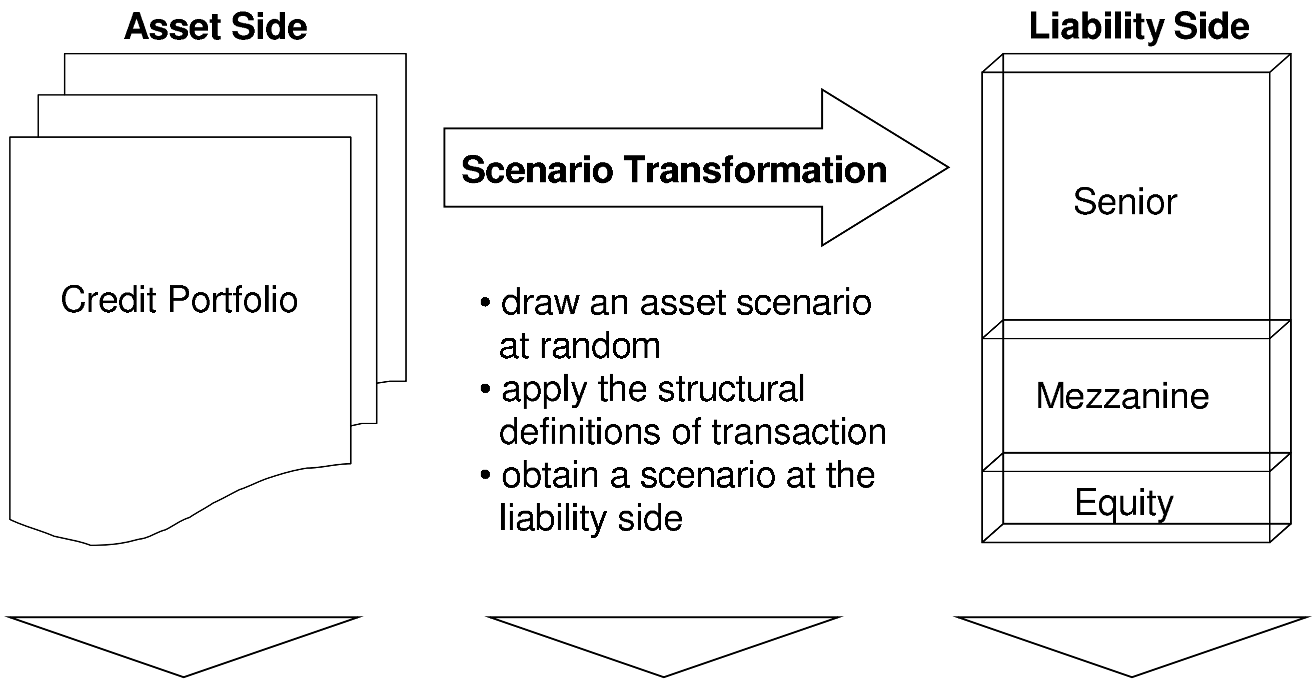
Agenda

- Basic structure of CDOs

- Comonotonicity in Default Timing



Asset Side

Credit Portfolio

Scenario Transformation

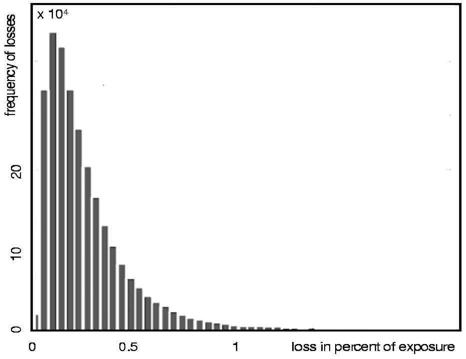
- draw an asset scenario at random
- apply the structural definitions of transaction
- obtain a scenario at the liability side

Liability Side

Senior

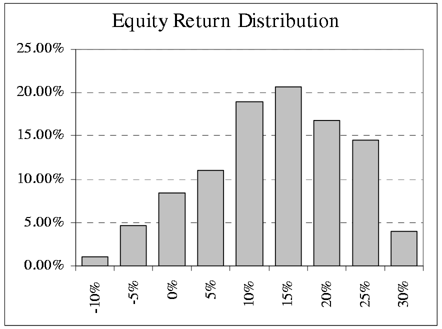
Mezzanine

Equity

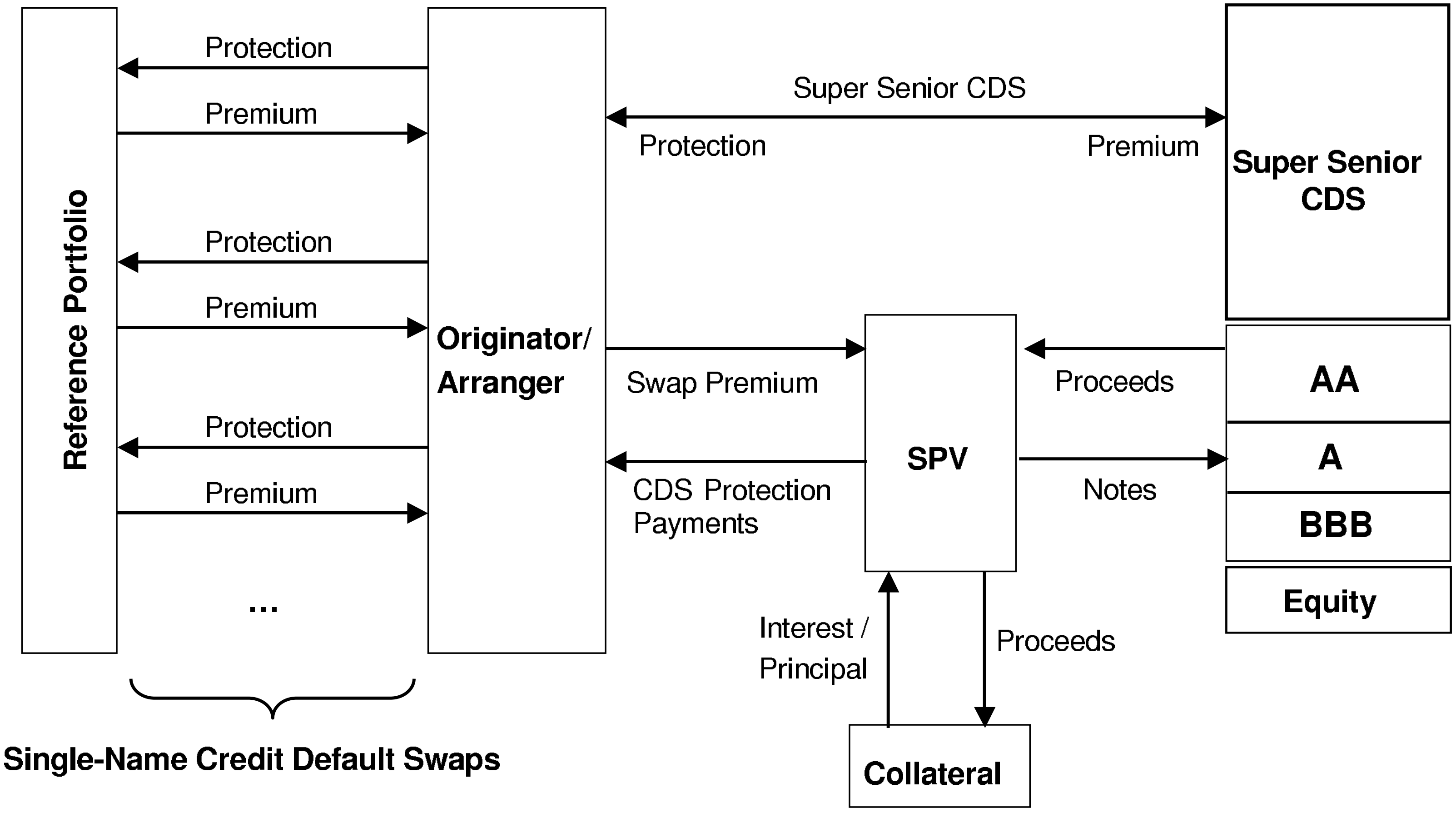


Example:

obtain the equity return distribution of a CDO by transforming 100,000 asset scenarios and then aggregating accordingly



Synthetic transaction based on credit default swaps (CSO)



Comonotonicity in Default Quote Paths

Starting point: default times vector $\boldsymbol{\tau} = (\tau_1, ..., \tau_m)$ and default quote path $(L_t)_{0\le t\le T}$ with

$$L_t \ = \ \frac{1}{m}\sum_{k=1}^{m} \mathbf{1}_{\{\tau_k \le t\}} \tag{1}$$

Intertemporal dependence of jumps $L_t - L_s$ for times $s < t$, sampled at times

$$0 = t_0 < t_1 < \cdots < t_{q-1} < t_q = T$$

is captured by some copula function

$$C : [0,1]^m \to [0,1],\ (u_1, ..., u_m) \mapsto C(u_1, ..., u_m)$$

(see SKLAR's theorem) such that

$$\mathbb{P}[L_{t_1} \le x_1, ..., L_{t_q} \le x_q] \ = \ C(\mathbb{P}[L_{t_1} \le x_1], ..., \mathbb{P}[L_{t_q} \le x_q]) \tag{2}$$

Note that always $x_i = k_i/m$ with $k_i \in \{1, ..., m\}$ and that the copula C not unique!

Definition 1 *A default quote path* $(L_t)_{0\leq t\leq T}$ *is called comonotonic if*

$$\mathbb{P}[L_{t_1} \leq x_1, ..., L_{t_q} \leq x_q] \;=\; \tilde{C}(\mathbb{P}[L_{t_1} \leq x_1], ..., \mathbb{P}[L_{t_q} \leq x_q])$$

where $C = \tilde{C}$ *is the comonotonic copula defined by*

$$\tilde{C}(u_1, ..., u_m) \;=\; \min\{u_1, ..., u_m\} \qquad (u_1, ..., u_m \in [0, 1]).$$

Note that $\tilde{C}$ *sometimes is called the upper Frechet copula. It is the copula inducing the strongest possible multivariate depencency*

Next, we will construct the comonotone version of any given default quote path:

Define the following probabilities w.r.t. any given portfolio model,

$$p_{t,k} \;=\; \mathbb{P}[L_t = k/m], \tag{3}$$

where m denotes the number of names in the portfolio and L_t is from (1)

Given theses probabilities for a discrete time grid $t = t_1, ..., t_q$,

$$(p_{t,k})_{t=t_1,...,t_q;\ k=0,1,2,...,m} \in \mathbb{R}^{q\times(m+1)},$$

we write $[z] = \max\{k \in \mathbb{N}_0 : k \leq z\}$ for the greatest nonnegative integer below $z \geq 0$ and define for any time $t_j > 0$ a distribution function

$$G_{t_j}(x) \ = \ \sum_{k=0}^{[mx]} p_{t_j,k} \qquad (x \in [0,1]) \tag{4}$$

G_{t_j} is a step function due to the finite granularity m of the portfolio

We now compare the original default quote path

$$\boldsymbol{L} \ = \ (L_{t_1}, ..., L_{t_q}) \ = \ \left(\frac{1}{m}\sum_{k=1}^{m} \mathbf{1}_{\{\tau_k \leq t_1\}}, ..., \frac{1}{m}\sum_{k=1}^{m} \mathbf{1}_{\{\tau_k \leq t_q\}}\right),$$

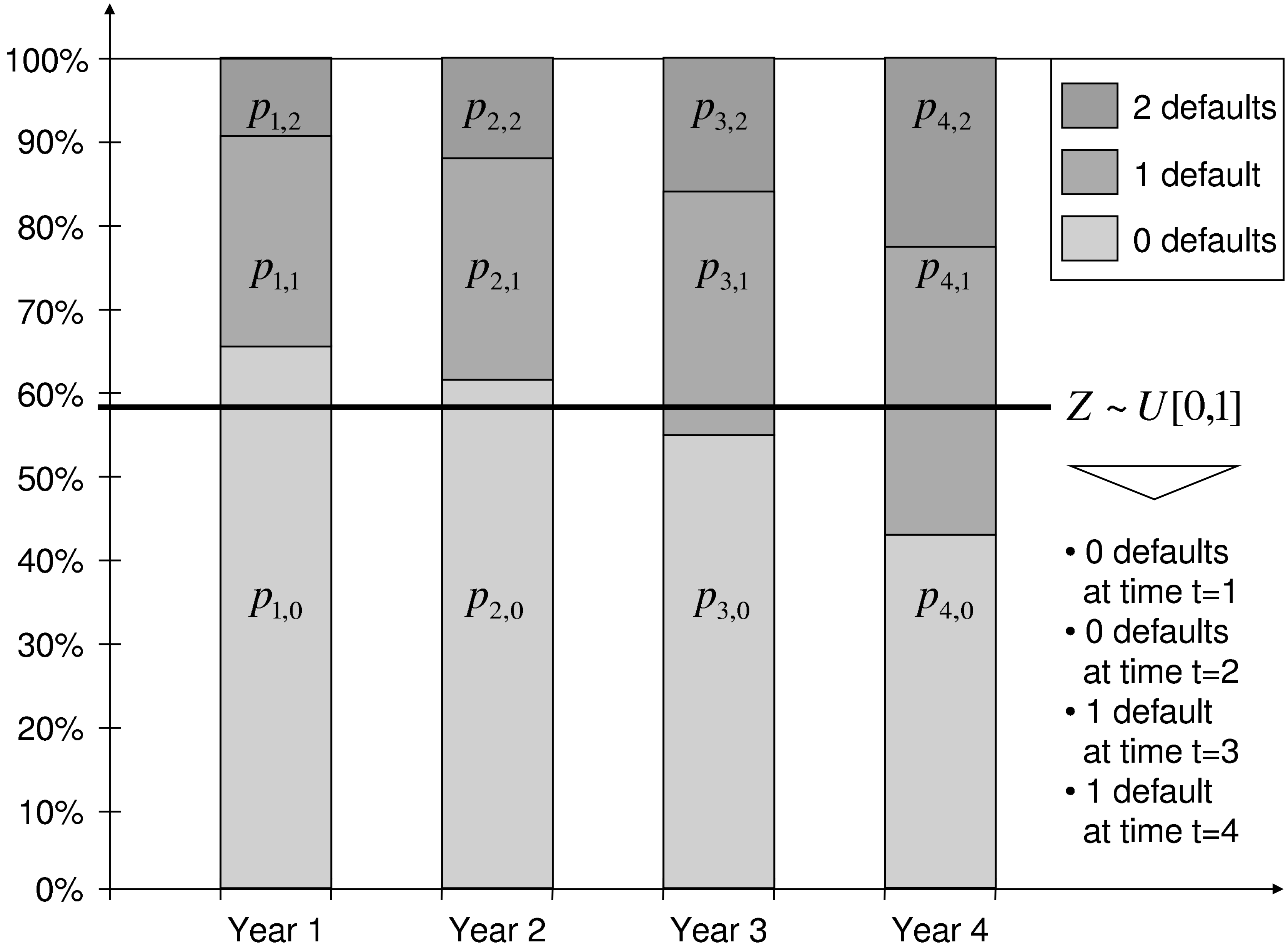
determined by the chosen portfolio model in line with (1) with the default quote path

$$\tilde{\boldsymbol{L}} \ = \ (\tilde{L}_{t_1}, ..., \tilde{L}_{t_q}) \ = \ \left(G_{t_1}^{-1}(Z), ..., G_{t_q}^{-1}(Z)\right) \qquad (Z \sim U[0,1])$$

Proposition 1 *The marginal distributions of* $\boldsymbol{L}$ *and* $\tilde{\boldsymbol{L}}$ *coincide and* $\tilde{\boldsymbol{L}}$ *is comonotonic*

From Proposition 1 we can derive a simple simulation scheme for $\tilde{\boldsymbol{L}}$:

- Simulate $Z \sim U[0, 1]$
- Compare Z with the ‘partition of unity’ applied in (4)



- 0 defaults at time t=1
- 0 defaults at time t=2
- 1 default at time t=3
- 1 default at time t=4

Comonotonic Default Timing

Denote by $\tau_{n:m}$ the time until the n-th default out of m obligors w.r.t. $\boldsymbol{L}$,

$$\tau_{n:m} \;=\; \inf\{t \geq 0 \;:\; L_t \geq n/m\}$$

and write $\tilde{\tau}_{n:m}$ for the time until the n-th default w.r.t. $\tilde{\boldsymbol{L}}$,

$$\tilde{\tau}_{n:m} \;=\; \inf\{t \geq 0 \;:\; G_t^{-1}(Z) \geq n/m\} \qquad (Z \sim U[0,1])$$

We now come to a very useful result

Proposition 2 *The distributions of* $\tau_{n:m}$ *and* $\tilde{\tau}_{n:m}$ *agree,*

$$\mathbb{P}[\tau_{n:m} \leq t] \;=\; \mathbb{P}[\tilde{\tau}_{n:m} \leq t] \qquad \textit{for all } t \geq 0.$$

Default timing in the comonotonic approach therefore coincides with default timing according to the original model in line with (1)

Proof. We start with the comonotonic path $\tilde{\boldsymbol{L}}$. Define numbers

$$q_{t,n:m} \;=\; \sum_{k=0}^{n-1} p_{t,k} \qquad (t \geq 0;\; n = 1, ..., m)$$

Note that $t \mapsto q_{t,n:m}$ is a strictly decreasing function

Observing n out of m defaults at time t in a comonotonic simulation is equivalent to

$$q_{t,n:m} \leq Z < q_{t,(n+1):m} \tag{5}$$

Therefore, the n-th default time agrees with the first time t s.t. $Z \;\geq\; q_{t,n:m}$ for $Z \sim U[0,1]$,

$$\tilde{\tau}_{n:m} \;=\; \inf\{t \geq 0 \;:\; Z \geq q_{t,n:m}\}$$

We turn our attention to the original $\boldsymbol{L}$

The statement $\tau_{n:m} \leq t$ means that at least n out of m defaults occurred before or at time t

The likelihood for n or more defaults in the time interval $[0, t]$ is given by

$$\begin{aligned}\mathbb{P}[\#\text{defaults} \in \{n, n+1, ..., m\} \text{ until time } t] &= \sum_{i=n}^{m} \mathbb{P}[L_t = i/m] \\ &= \sum_{i=n}^{m} p_{t,i} \; = \; 1 - q_{t,n:m}\end{aligned}$$

Combining the pieces we have yield the conclusion

$$\mathbb{P}[\tau_{n:m} \leq t] \; = \; 1 - q_{t,n:m} \; = \; \mathbb{P}[Z \geq q_{t,n:m}] \; = \; \mathbb{P}[\tilde{\tau}_{n:m} \leq t].$$

Therefore, the n-th default times exactly match in distribution $\Box$

Implications of Comonotonicity

We illustrate implications and benefits by means of a one-factor model,

$$p_{t,k} \;=\; \mathbb{P}[L_t = k/m] \;=\; \binom{m}{k} \int\limits_{-\infty}^{\infty} p(t,y)^k (1 - p(t,y))^{m-k} dN(y), \tag{6}$$

$$p(t,Y) \;=\; N\left[\frac{N^{-1}(p_t) - \sqrt{\varrho}\,Y}{\sqrt{1-\varrho}}\right] \tag{7}$$

where $Y \sim N(0,1)$, $(p_t)_{t\geq 0}$ is a uniform credit curve, and ϱ denotes the uniform asset correlation between (uniformly factored) latent variables

$$r_i \;=\; \sqrt{\varrho}\,Y + \sqrt{1-\varrho}\,\varepsilon_i \quad (i = 1, ..., m) \tag{8}$$

triggering default indicators leading to default quotes

$$L_t \;=\; \frac{1}{m}\sum_{i=1}^{m} \mathbf{1}_{\{r_i < N^{-1}[p_t]\}} \qquad (Y, \varepsilon_1, ..., \varepsilon_m \sim N(0,1) \text{ iid}) \tag{9}$$

Comparing (1) and (9) suggests to define the default time of any obligor i by

$$\tau_i \;=\; F^{-1}(N[r_i]) \qquad (i = 1, ..., m) \tag{10}$$

where r_i is defined in (8) and $F(t) = \mathbb{P}[\tau_i \leq t] = p_t$ (distribution function of τ_i on $[0, \infty)$)

In the original model we have to simulate $(m+1)$ variables $Y, \varepsilon_1, ..., \varepsilon_m \sim N(0,1)$

In the comonotonic approach we only have to simulate one variable $Z \sim U[0,1]$

Both methods give the same n-th to default times.

Comontonicity, however, works only for a homogenous basket!!

The comonotonic approach does not capture the fact that in general defaults follow a multinomial pattern.

Again we illustrate this by means of the one-factor model with default quotes L_t from (9)

Given a time grid $t_1, ..., t_q$, the corresponding default quote *jumps* are given by

$$S_j \; = \; L_{t_j} - L_{t_{j-1}} \qquad (j = 1, ..., q; \; L_0 = 0). \tag{11}$$

Additionally we define a ‘residual jump’

$$S_\infty \; = \; 1 - L_{t_q}$$

for the time period $[t_q, \infty)$

Based on Equation (7) we define functions

$$g_1(y) \;=\; p(t_1, y), \quad g_j(y) \;=\; p(t_j, y) - p(t_{j-1}, y), \quad g_\infty(y) = 1 - p(t_q, y), \tag{12}$$

where y denotes a realization of the factor Y

Obligor $k \in \{1, ..., m\}$ admits $q+1$ possible time intervals capturing her/his default time,

$$[0, t_1), \quad [t_1, t_2), \quad ..., \quad [t_{q-1}, t_q), \quad \text{or the residual period} \quad [t_q, \infty)$$

Denoting the default time of obligor k by τ_k we see that

$$\mathbb{P}[t_{j-1} \le \tau_k < t_j \mid Y = y] \;=\; g_j(y) \quad \text{and} \quad \mathbb{P}[t_q \le \tau_k < \infty \mid Y = y] \;=\; g_\infty(y) \tag{13}$$

Note that by construction we have

$$\sum_{j=1}^{q} g_j(y) + g_\infty(y) \;=\; 1 \qquad (\forall\, y \in \mathbb{R}) \tag{14}$$

Proposition 3 *The distribution of the jump vector* $m \times (S_1, ..., S_q, S_\infty)$ *conditional on* $Y = y$ *follows a multinomial distribution* $M[m; g_1(y), ..., g_q(y), g_\infty(y)]$

Proof. Given $Y = y$, the vector $(\mathbf{1}_{\{\tau_k \in [0,t_1)\}}, ...\mathbf{1}_{\{\tau_k \in [t_{q-1},t_q)\}}, \mathbf{1}_{\{\tau_k \in [t_q,\infty)\}})$ is multivariate Bernoulli with occurence probabilities $g_1(y), ..., g_q(y), g_\infty(y)$; see also (14)

Due to the independence of default times conditional on Y,

$$\sum_{k=1}^{m} (\mathbf{1}_{\{\tau_k \in [0,t_1)\}}, ...\mathbf{1}_{\{\tau_k \in [t_{q-1},t_q)\}}, \mathbf{1}_{\{\tau_k \in [t_q,\infty)\}})$$

given $Y = y$ is a convolution of (independent) multivariate Bernoulli distributions and therefore must follow a multinomial distribution $M[m; g_1(y), ..., g_q(y), g_\infty(y)]$

See Equation (8) regarding conditional independence □

Conclusions:

- First analysis of the dependency of defaults in different time buckets
- Comonotonicity provides a powerful tool for efficient and variance-reducing simulation of n-th-to-default times in baskets and CDOs
- In copula models the dependency of default events in different time slices are close to the comonotonic copula, especially in a one-factor normal copula.
- However, one has to keep in mind that comonotonicity (as a side effect) leads to a lack of precision if the underlying assets are too far from being ‘exchangeable’
- Additionally we pointed out that the multinomial distribution governs the time dependency (in a one-factor) copula model)
- Implication for using asset correlation for the copula of default time models (which is currently an industry standard) requires further investigation, cf Fingers (2000).

Literature Remarks

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