Convertible bonds in jump-diffusion models

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Preliminary version

joint work with Pavel Gapeev and Kees van Schaik

Convertible callable bonds

A convertible bond is a hybrid between a standard corporate bond and a stock

- The holder can convert this bond into $\gamma \in \mathbb{R}_+$ stocks
- The issuing firm can recall the bond prematurely by paying the amount $K \in \mathbb{R}_+$
- Otherwise, at T the bondholder receives the amount 1, assume $1 < K$

Brennan and Schwartz (1977), Ingersoll (1977) Davis and Lischka (2002), Grau,Forsyth, and Vetzal (2004) Bühler and Koziol (2004), Sirbu, Pikovsky, and Shreve (2004), Heritage, Leobacher, and Rogers (2004), . . .

Explicite solution for the perpetual case.

Gapeev and K. (2005), reduced form model with exponential jumps (Kou model)

$$S_t = s\,\mathrm{exp}\left((r-\delta)t - \frac{\eta^2}{2}t + \eta\, W_t + \sum_{i=1}^{N_t} Y_i - at\right)$$

r interest rate, δ: dividend rate, $0 \leq \delta < r$
N_t Poisson process with rate $\lambda > 0$, $a = \lambda \int_{\mathbb{R}} (e^y - 1) F_Y(dy)$.

If the holder converts the bond into γ stocks at time t, the total (discounted) payoff is given by

$$L_t = c\int_0^t e^{-ru}\,du + e^{-rt}\,\gamma S_t,$$

If issuer recalls at time t the total (discounted) payoff is

$$U_t = c\int_0^t e^{-ru}\,du + e^{-rt}\,\mathrm{max}\{K, \gamma S_t\} \qquad \text{(no call protection)}$$

An arbitrage-free price is given by

$$
\begin{aligned}
V_*(s) &= \inf_{\sigma \in \mathcal{T}_{[0,\infty]}} \sup_{\tau \in \mathcal{T}_{[0,\infty]}} E_s[L_\tau I(\tau \le \sigma) + U_\sigma I(\sigma < \tau)] \\
&= \sup_{\tau \in \mathcal{T}_{[0,\infty]}} \inf_{\sigma \in \mathcal{T}_{[0,\infty]}} E_s[L_\tau I(\tau \le \sigma) + U_\sigma I(\sigma < \tau)],
\end{aligned}
$$

where $S_0 = s$. We have

$$
\gamma s \le V_*(s) \le K \vee \gamma s, \quad \forall s \ge 0.
$$

It turns out that optimal stopping strategies are of the following form

$$
\tau_* = \inf\{t \ge 0 \mid \gamma S_t \ge B_*\}
$$

$$
\sigma_* = \inf\{t \ge 0 \mid \gamma S_t \ge A_*\}
$$

Demonstration of the main idea

$$\tilde{V}(s) + a\tilde{V}'(s) + b\tilde{V}''(s) + c\int_0^\infty \alpha e^{-\alpha u}\left[\tilde{V}(s+u) - \tilde{V}(s)\right] du, \;\; \forall s \le s_0$$

and $\tilde{V}(s) = f(s), \;\forall s \ge s_0$ ("integro-differential equation")

$$\text{Substitute:} \quad H(s) := \int_0^\infty \alpha e^{-\alpha u}\tilde{V}(s+u)\, du$$

We have

$$H(s) = \alpha h\tilde{V}(s) + (1-\alpha h)H(s+h) + o(h), \quad h \to 0, \text{ resp.}$$

$$\alpha(H(s+h) - \tilde{V}(s)) = \frac{H(s+h) - H(s)}{h} + o(1), \quad h \to 0.$$

$$\text{that is} \quad \tilde{V}(s) = H(s) - \frac{H'(s)}{\alpha}$$

$\Leftrightarrow$ third-order differential equation with boundary conditions $H(s_0) := \int_0^\infty \alpha e^{-\alpha u}\tilde{V}(s_0+u)\, du, \; H'(s_0) = \dots$

$$(1+c)\left[H(s) - \frac{H'(s)}{\alpha}\right] + a\left[H'(s) - \frac{H''(s)}{\alpha}\right]$$

$$+b\left[H''(s) - \frac{H'''(s)}{\alpha}\right] + cH(s) = 0$$

In the continuation region the value function V satisfies the following integro-differential equation (written down for positive jumps)

$$-(r+\lambda)\,V(s)+(r+\zeta-\delta)s\,V'(s)+\frac{\eta^2}{2}s^2\,V''(s)$$

$$+\alpha\lambda s^{\alpha}\left(\int_s^{A_*}V(z)\,\frac{dz}{z^{\alpha+1}}-\frac{K^{1-\alpha}}{\alpha(1-\alpha)}+\frac{KA_*^{-\alpha}}{\alpha}\right)+c=0$$

with $\alpha=1/\theta$, $\zeta=-\lambda\theta/(1-\theta)$. With the substitution

$$G(s)=\ -\int_s^{A_*}V(z)\,\frac{dz}{z^{\alpha+1}}+\frac{K^{1-\alpha}}{\alpha(1-\alpha)}-\frac{KA_*^{-\alpha}}{\alpha}$$

we can reduce it to the following (third-order) ordinary differential equation (possible as jumps are exponentially distributed)

$$\frac{\eta^2s^3}{2}G'''(s)+[\eta^2(\alpha+1)+r+\zeta-\delta]s^2\,G''(s)$$

$$+\left[(\alpha+1)\left(\frac{\eta^2\alpha}{2}+r+\zeta-\delta\right)-(r+\lambda)\right]s\,G'(s)-\alpha\lambda\,G(s)=-s^{-}$$

Solutions.

Case I: Brownian motion, no jumps

with standing assumption $K \geq \frac{c}{r}$ we have two regions depending on K (recall price)

$\mathbf{K} > \underbrace{\frac{\gamma_1 \delta}{(\gamma_1 - 1)r}}_{>1} \frac{c}{\delta}$: holder converts

$\mathbf{K} \leq \underbrace{\frac{\gamma_1 \delta}{(\gamma_1 - 1)r}}_{>1} \frac{c}{\delta}$: no stopping before conversion value γS_t hits recall price K

Case II: Brownian motion & posivtive jumps. If

$K < \frac{\gamma_1 \gamma_2}{(\gamma_1 - 1)(\gamma_2 - 1) + \alpha - 1} \frac{c}{r}$ then the issuer recalls (even if $K > \frac{c}{r}$)

Case III: Positive jumps, no diffusion part

$$K < \underbrace{\frac{\alpha - 1}{\alpha} \frac{\gamma_1 \delta}{(\gamma_1 - 1) r}}_{>1} \frac{c}{\delta} \text{ issuer recalls}$$

$$K > \underbrace{\frac{\alpha - 1}{\alpha} \frac{\gamma_1 \delta}{(\gamma_1 - 1) r}}_{>1} \frac{c}{\delta} \text{ holder converts}$$

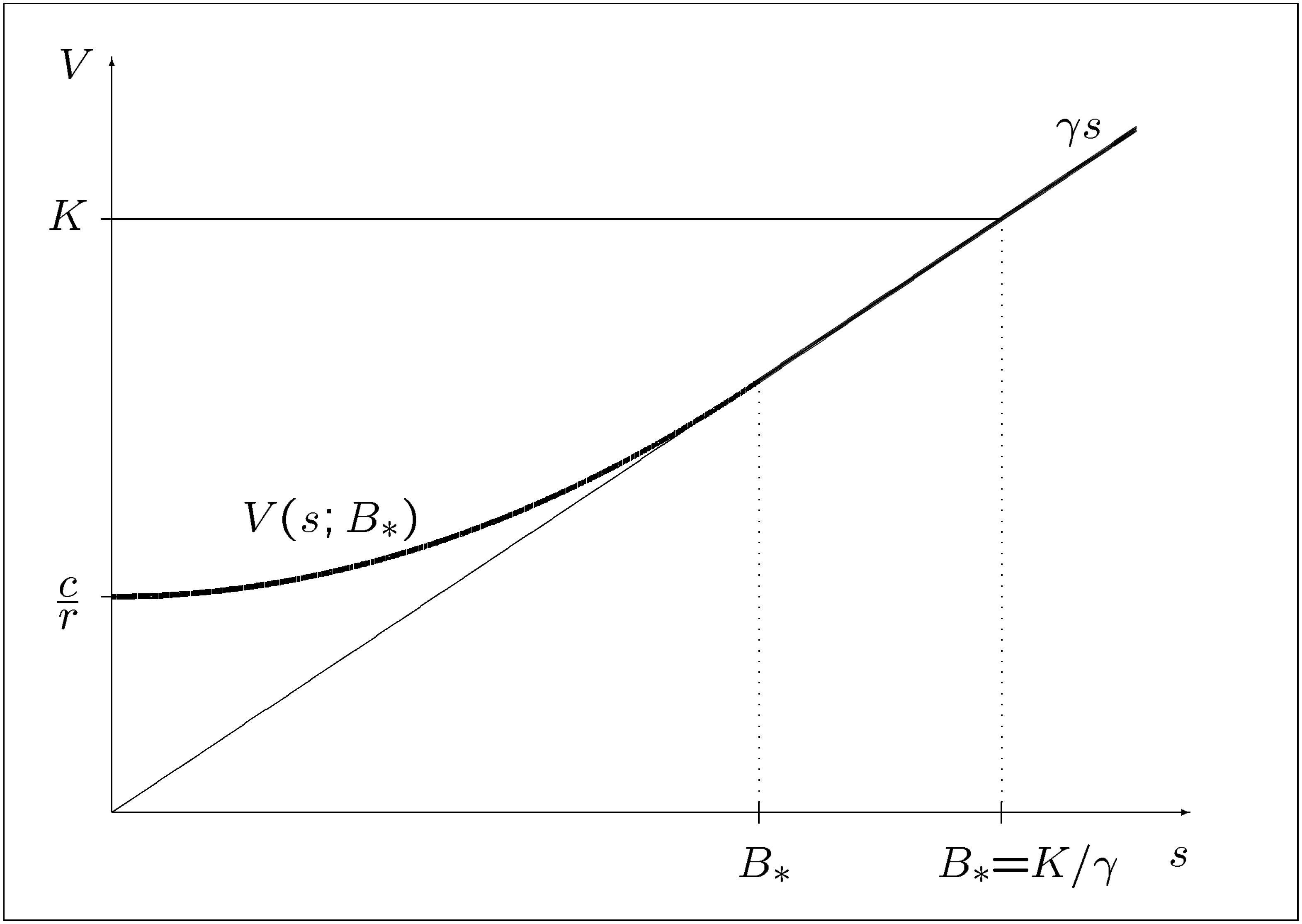
only for a single K nobody stops !

Form of solution

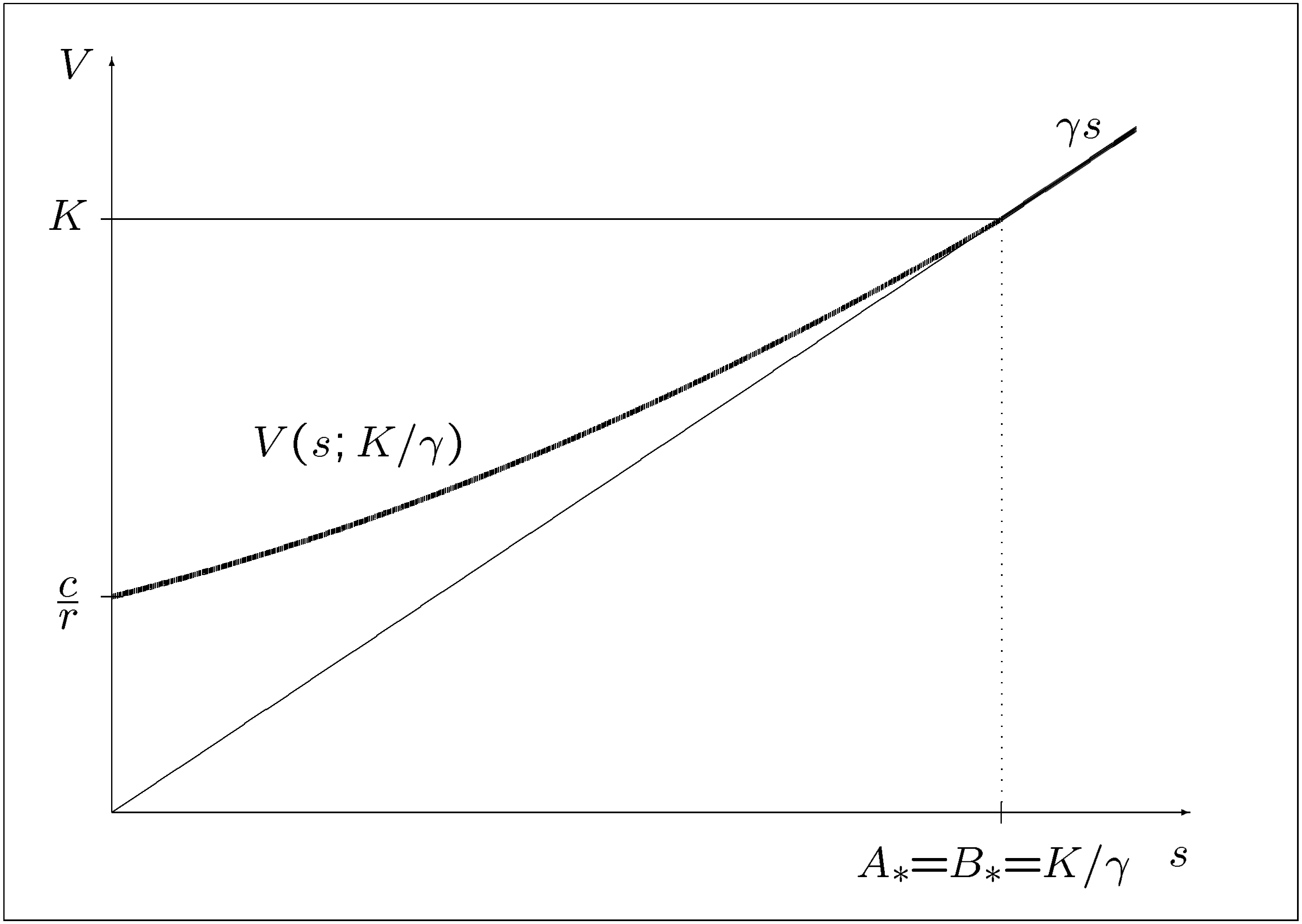
$$V(s; A_*) = -\frac{1}{\beta_1 - \beta_2} \gamma_2 \left(K - \frac{c}{r} \right) \left(\frac{s}{A_*}\right)^{\gamma_1} + \frac{1}{\beta_1 - \beta_2} \gamma_1 \left(K - \frac{c}{r} \right) \left(\frac{s}{A_*}\right)^{\gamma_2}$$

where

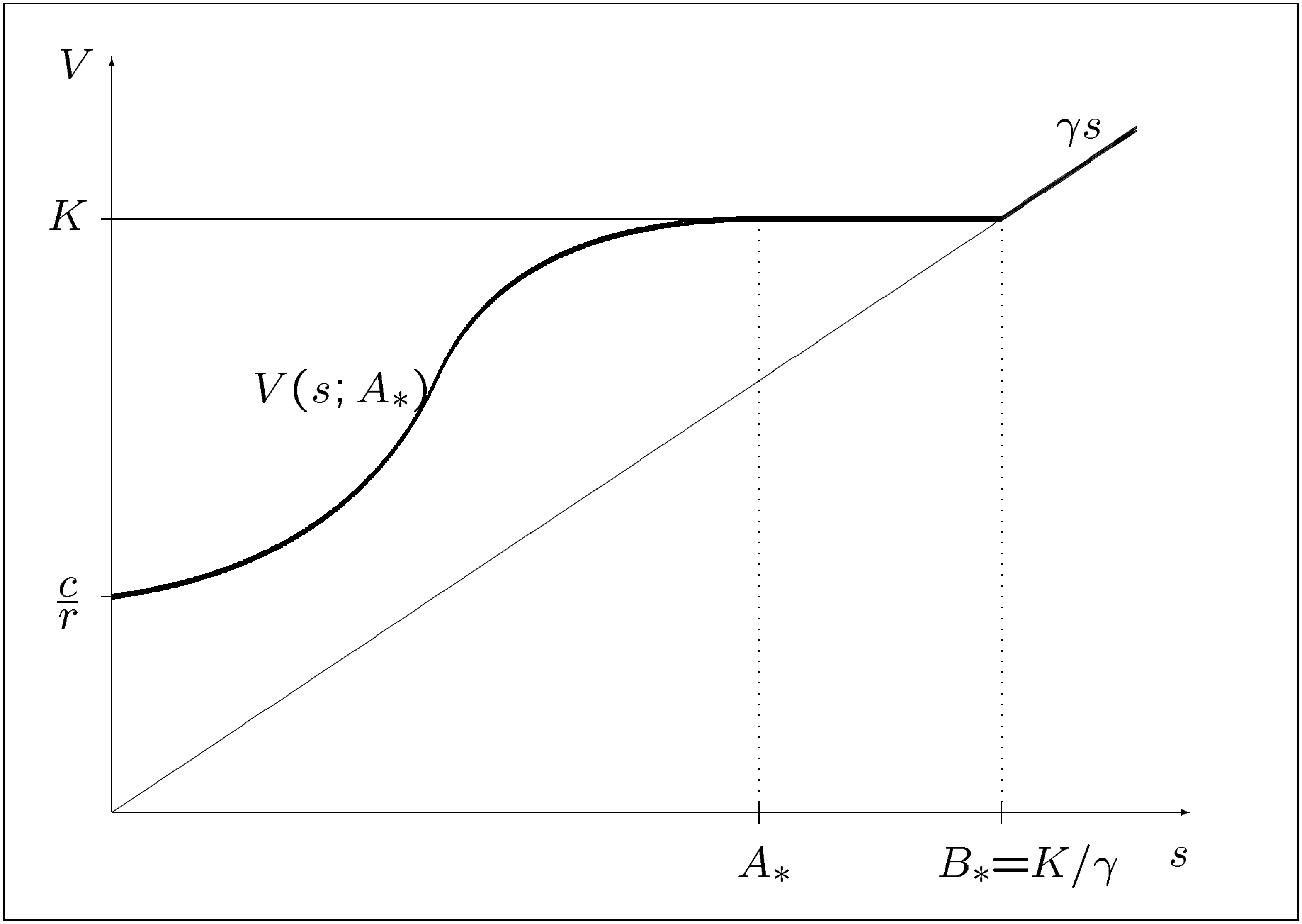
$$A_* = K \left(\frac{(1 - \alpha) \gamma_1 \gamma_2}{\beta_1 \beta_2 K} \left(K - \frac{c}{r} \right) \right)^{1/\alpha} .$$



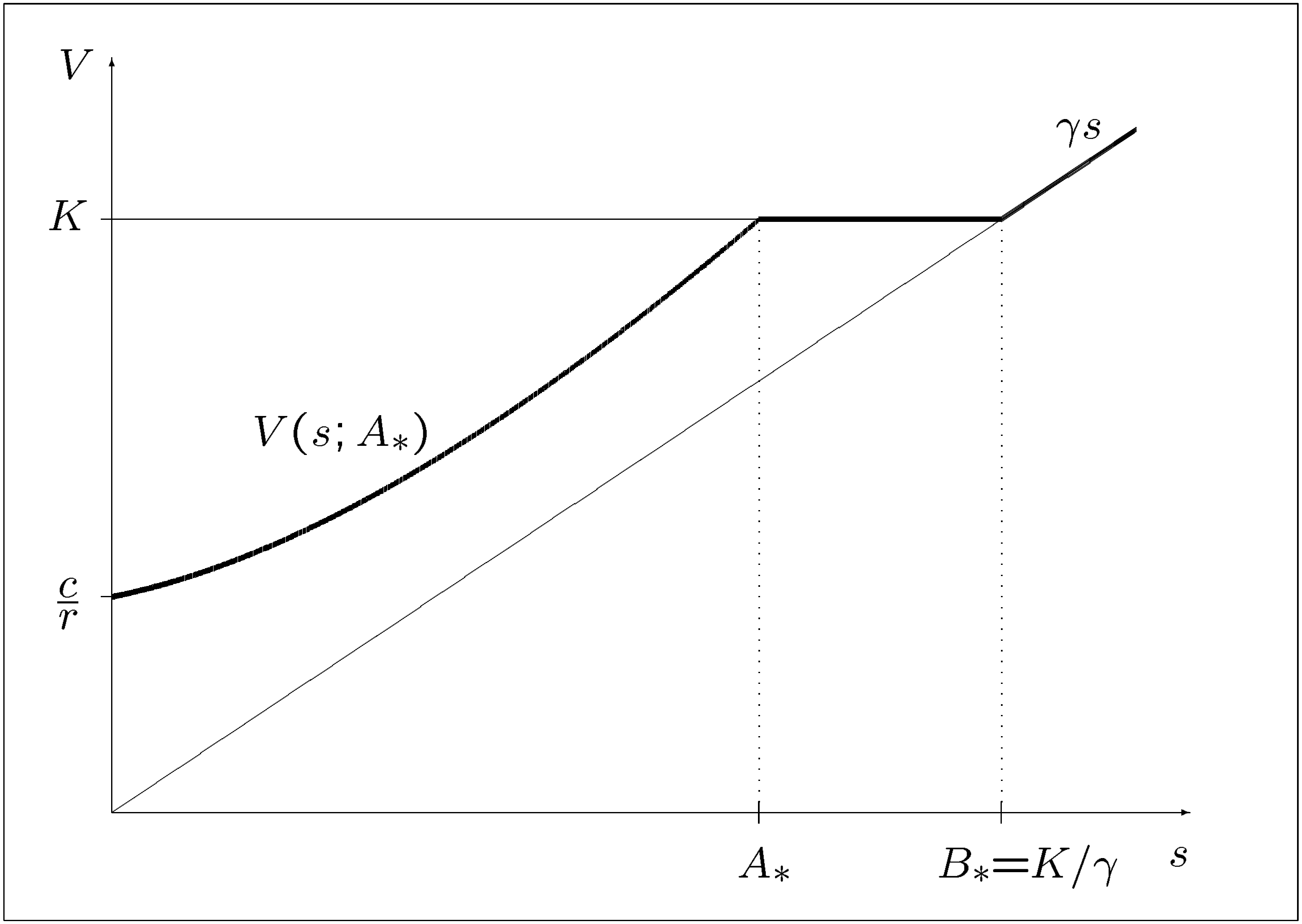
Case I. The value function $V_*(s)$ in case of
$B_* < A_* = K/\gamma$ (holder stops)



Case I. The value function $V_*(s)$ in case of $A_* = B_* = K/\gamma$ (no early stopping)



Case II. The value function $V_*(s)$ in case of a diffusion & positive jumps $\rightsquigarrow A_* < B_* = K/\gamma$ can occur – with smooth fit



Case III. The value function $V_*(s)$ in case of a pure jump process $\rightsquigarrow$ again $A_* < B_* = K/\gamma$, but without smooth fit

- Perpetual case was the easiest (as the state variable "time" drops out)

- Nonperpetual case (i.e. bond expires at $T \in \mathbb{R}_+$) is in most cases too difficult to obtain explicit solutions...

- Therefore: **canadization** procedure. Idea by Carr (1998). Method to approximate nonperpetual stopping problem.

- Game version in a jump-diffusion model: PhD project of van Schaik

Canadization procedure for convertible bonds

- Instead of the deterministic maturity T consider the random maturity T_n, where $T_k := \sum_{i=1}^{n} \tau_i, \;\; k = 1, \ldots, n$
  n should be "large"
  τ_i i.i.d. **exponential** times (independent of S) with $E(\tau_i) = \frac{T}{n}$

- As n is large we have $T_n \approx T$, in terms of $E(T_n) = T$ and Var$(T_n) = \frac{T^2}{n}$

- Enlargement of filtration: $(\mathcal{F}_t^S)_{t \in \mathbb{R}_+} \rightsquigarrow (\tilde{\mathcal{F}}_t)_{t \in \mathbb{R}_+}$, where

$$\tilde{\mathcal{F}}_t := \mathcal{F}_t^S \vee \sigma\left(\{T_k \le t\}, \;\; k = 1, \ldots, n\right).$$

Given the information that exacly k exponentials have occured until t, i.e. knowing that $T_k \leq t < T_{k+1}$ the current value of the game only depends on S_t and k, but not on t anymore (because of the lack of memory of the exponentials). Define

$$v_k^{(n)}(s) := \inf_{\sigma \in \mathcal{T}_{[0,\infty]}} \sup_{\tau \in \mathcal{T}_{[0,\infty]}} \mathbb{E}_s \Big(\mathbf{1}_{\{\tau \leq \sigma \text{ and } \tau < T_k\}} e^{-r\tau} \gamma S_\tau$$

$$+ \mathbf{1}_{\{\sigma < \tau \wedge T_k\}} e^{-r\sigma} \max\{K, \gamma S_\sigma\} + \mathbf{1}_{\{T_k \leq \tau \wedge \sigma\}} e^{-rT_k} \max\left\{1, \gamma S_{T_k}\right\}\Big)$$

(where $\mathcal{T}_{[0,\infty]}$ is the set of stopping times w.r.t. $(\tilde{\mathcal{F}}_t)_{t \in \mathbb{R}_+}$).

$$\rightsquigarrow v^{(n)}(S_t, t, \omega) = v_k^{(n)}(S_t), \text{ if } t \in [T_{n-k}, T_{n-k+1}),$$

where $v^{(n)}$ is the value function for the bond with duration T_n which converges to the "true" value function for the bond with duration T. This generates a system of **ordinary** integro-differential equations.

Exercise boundaries are constant on the intervals $[T_k, T_{k+1})$, $k = 0, \dots, n-1$, i.e.

$$\tau = \inf\{t \geq 0 \, |S_t \geq B_*(k) \text{ for } t \in [T_k, T_{k+1})\},$$

$B_*(k) \in \mathbb{R}_+ \rightsquigarrow$ "partial" discretization of time.

We have the following recursive relation

$$v_k^{(n)}(s) = \inf_{\sigma \in \mathcal{T}_{[0,\infty]}} \sup_{\tau \in \mathcal{T}_{[0,\infty]}} \mathbb{E}_s \Big(\mathbf{1}_{\{\tau \le \sigma \text{ and } \tau < T_1\}} e^{-r\tau} \gamma S_\tau$$

$$+ \mathbf{1}_{\{\sigma < \tau \wedge T_1\}} e^{-r\sigma} \max\{K, \gamma S_\sigma\} + \mathbf{1}_{\{T_1 \le \tau \wedge \sigma\}} e^{-rT_1} v_{k-1}^{(n)} S_{T_1} \Big).$$

In the continuation region the value function $v_k^{(n)}$ satisfies the following integro-differential equation (written down for positive jumps)

$$-(r+\lambda)\, v_k^{(n)}(s) + (r+\zeta-\delta)s \left(v_k^{(n)}\right)'(s) + \frac{\eta^2}{2} s^2 \left(v_k^{(n)}\right)''(s)$$

$$+\alpha\lambda s^\alpha \left(\int_s^{A_*} v_k^{(n)}(z)\, \frac{dz}{z^{\alpha+1}} - \frac{K^{1-\alpha}}{\alpha(1-\alpha)} + \frac{K A_*^{-\alpha}}{\alpha} \right)$$

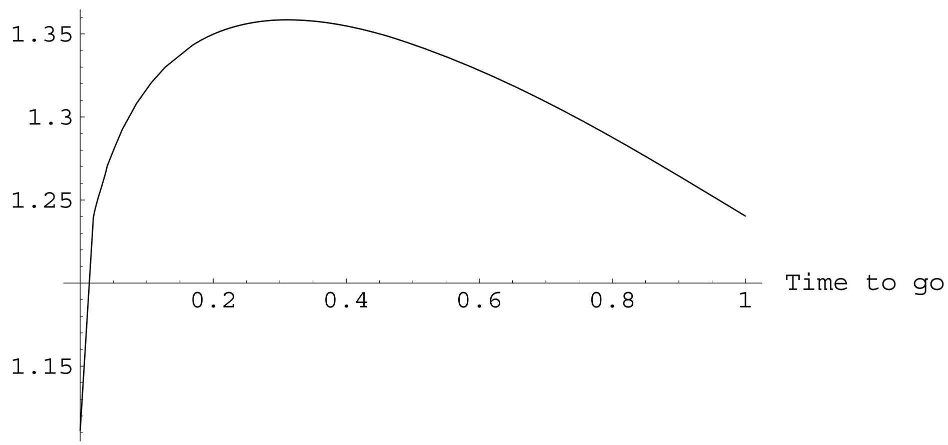
$$+\frac{n}{T} \left(v_{k-1}^{(n)}(s) - v_k^{(n)}(s) \right) = 0$$

with $\alpha = 1/\theta$, $\zeta = -\lambda\theta/(1-\theta)$.

Solution in case of $K = \infty$ (no recalling) or K large enough

(exercise boundary for holder and value function)

, kmax = 50, K = 1.188, recall = False, σ = 0.4, r = 0.3, d = 0.1, λ = 50, γ = 0.9



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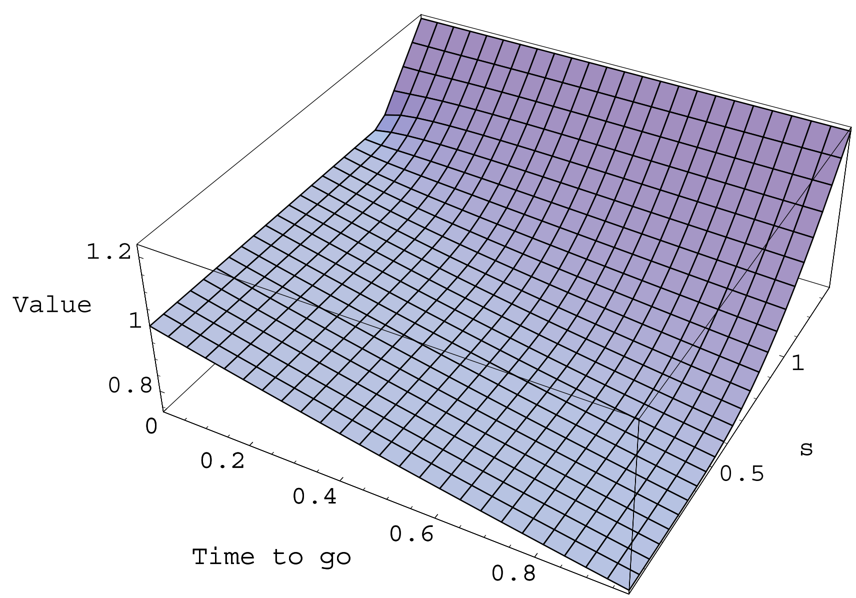
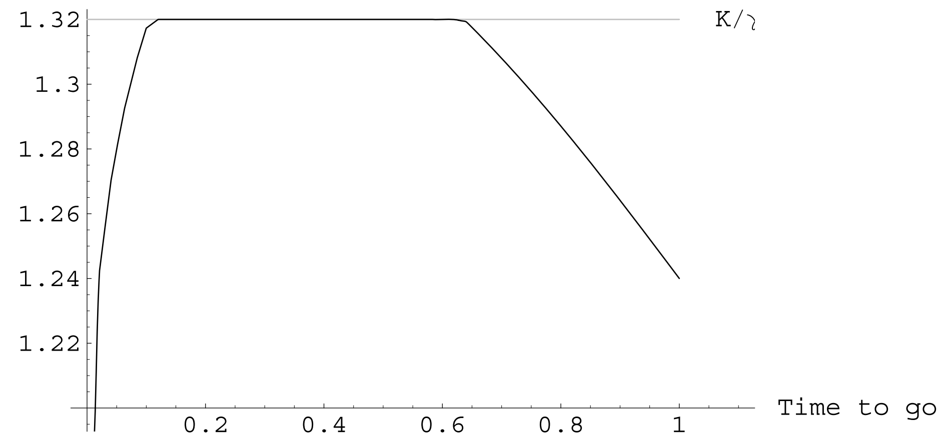


The same example but with smaller recall value K

(exercise boundary for holder and value function)

, kmax = 50, K = 1.188, recall = True, σ = 0.4, r = 0.3, d = 0.1, λ = 50, γ = 0.9



, kmax = 50, K = 1.188, recall = True, σ = 0.4, r = 0.3, d = 0.1, λ = 50, γ = 0.9