Portfolio Theory with a Drift

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Chapter 1

Introduction

- Markowitz framework
  - asset returns are normally distributed
  - with constant volatilities
  - and constant correlations

- *Even within this framework* the volatility is not a suitable risk measure!

Chapter 2

Markowitz in a Nut Shell

- Goal of portfolio management
  - optimize *risk adjusted performance measures* (abbreviated "*RAPM*")

$$\frac{\text{(expected) portfolio return}}{\text{portfolio risk}} \tag{2.1}$$

- Portfolio V
  - consisting of holdings N_i in M risky assets (the *risk factors*)
  - having values V_i, volatilities σ_i, expected returns R_i and correlations ρ_{ij}
- In Markowitz theory: portfolio risk = the portfolio *volatility*

$$\sigma_V = \sqrt{\mathbf{w}^T \mathbf{C} \mathbf{w}} \tag{2.2}$$

  - $\mathbf{C}$ = covariance matrix with elements $C_{ij} = \sigma_i \rho_{ij} \sigma_j$ for $i, j = 1, \ldots M$
  - $\mathbf{w}$ = vector of asset weights, $w_i = N_i V_i / V$ for $i = 1, \ldots M$.
- Optimal investment strategy
  - yields highest expected return for the risk incurred
  - invest in the so called *Market Portfolio* having maximum *Sharpe Ratio*

$$\gamma_m = \frac{\widehat{R}_m}{\sigma_m} \tag{2.3}$$

σ_m = volatility of market portfolio
$\widehat{R}_m := R_m - r_f$ expected *excess* return (above the risk free rate r_f)

- Even if $\sigma_m \neq$ risk preference σ_{required} invest in this portfolio, although not all of the money
  - If $\sigma_{\text{required}} < \sigma_m$ invest a percentage $w < 1$ of the capital and the rest risk free
  - if $\sigma_{\text{required}} > \sigma_m$ borrow money and invest a percentage $w > 1$ of the capital (*leveraged investment*).
  - investing or borrowing in the money market produces neitehr any additional excess return nor any additional volatility
- The expected return of this strategy as a function of the risk (the volatility) is a straight line with slope γ_m, the *Capital Market Line.*
- Within this framework all sorts of optimizations, for example [3]:

$$\mathbf{w}_m = \frac{\mathbf{C}^{-1}\widehat{\mathbf{R}}}{\mathbf{1}^T\mathbf{C}^{-1}\widehat{\mathbf{R}}} \quad , \quad \widehat{R}_m = \frac{\widehat{\mathbf{R}}^T\mathbf{C}^{-1}\widehat{\mathbf{R}}}{\mathbf{1}^T\mathbf{C}^{-1}\widehat{\mathbf{R}}} \quad , \quad \sigma_m^2 = \frac{\widehat{R}_m}{\mathbf{1}^T\mathbf{C}^{-1}\widehat{\mathbf{R}}} \tag{2.4}$$

– with $\widehat{\mathbf{R}}$ the vector of the asset's expected excess returns R_i

Chapter 3

A Better Risk Measure

- *Risk* is a *potential loss*, not to be exceeded over a certain time horizon δt (the *holding period*) with a specified probability c (the *confidence*)
- For normally distributed asset returns this leads to (the Delta-Normal approx-

imation of) the *Value at Risk*

$$\mathrm{VaR}_{c,\delta t}(R_V) \approx |Q_{1-c}|\, V\sqrt{\delta t}\sqrt{\sum_{k,l=1}^{M} w_k \sigma_i \rho_{ij} \sigma_j w_l} - V\delta t \sum_{k=1}^{M} w_k R_k$$

- with Q_{1-c} the $(1-c)$ percentile of the standard normal distribution.
- more compactly using obvious vector notation:

$$\begin{aligned} \mathrm{VaR}_V(c) &\approx |Q_{1-c}|\, V\sqrt{\delta t}\sqrt{\mathbf{w}^T\mathbf{C}\mathbf{w}} - V\delta t\ \mathbf{w}^T\mathbf{R} \\ &= |Q_{1-c}|\, V\sqrt{\delta t}\sigma_V - V\delta t\ R_V \end{aligned} \quad (3.1)$$

- with covariance matrix C having elements $C_{ij} = \sigma_i \rho_{ij} \sigma_j$ for $i, j = 1, \ldots M$

• Several approximations necessary

1. normally ditributed returns with constant distribution parameters
2. linear relation between instrument prices and market parameters
3. **market parameter drifts are neglected**

- Neglect of the drift is in direct contradiction to the concept of the Capital Market Line
  - An investor can achieve *any* (also *arbitrarily small*) desired volatility
    - by distributing money between the market portfolio and the (risk free) money market account.
  - An arbitrarily small volatility is not large compared to the drift!
- Therefore: **the drift must not be neglected as soon as the money market comes into play!**
- Percentage invested in the risky assets

$$w = \sum_{i=1}^{M} w_i = \mathbf{w}^T \mathbf{1}$$

- Percentage invested in money market: $w_f = 1 - w$ yielding the risk free contribution $w_f r_f$ to the drift

$$\begin{aligned}
\mathrm{VaR}_V(c) &\approx |Q_{1-c}|\, V\sqrt{\delta t}\sqrt{\mathbf{w}^T\mathbf{C}\mathbf{w}} - V\delta t\left[\mathbf{w}^T\mathbf{R} + w_f r_f\right] \\
&= |Q_{1-c}|\, V\sqrt{\delta t}\sqrt{\mathbf{w}^T\mathbf{C}\mathbf{w}} - V\delta t\left[\mathbf{w}^T\mathbf{R} + (1-\mathbf{w}^T\mathbf{1})r_f\right] \\
&= |Q_{1-c}|\, V\sqrt{\delta t}\sqrt{\mathbf{w}^T\mathbf{C}\mathbf{w}} - V\delta t\left[\mathbf{w}^T(\mathbf{R}-\mathbf{1}r_f) + r_f\right]
\end{aligned}$$

- Thus, the Value at Risk for the total investment is

$$\begin{aligned}
\mathrm{VaR}_V(c) &\approx |Q_{1-c}|\, V\sqrt{\delta t}\sqrt{\mathbf{w}^T\mathbf{C}\mathbf{w}} - V\delta t[\mathbf{w}^T\widehat{\mathbf{R}} + r_f] \\
&= |Q_{1-c}|\, V\sqrt{\delta t}\;\sigma_V - V\delta t R_V
\end{aligned} \tag{3.2}$$

- Percentile Q_{1-c} and holding period δt do *not* enter as common factors.
  - The smaller the quantile (the larger the confidence) the larger is the fluctuations' influence
  - The larger the holding period, the larger is the drift effect.
  - Therefore, c and δt can *not* be eliminated from the risk definition.

- Divide the VaR by V and δt to generate dimensionless RAPMs.
  - Another motivation for dividing by $V\delta t$: write the VaR directly in terms of annualized portfolio returns r_V:

$$\mathrm{VaR}_V(c) \approx |Q_{1-c}|\, V\delta t\sqrt{\mathrm{var}\,[r_V]} - V\delta t\ \mathrm{E}\,[r_V] \tag{3.3}$$

    * This form can be derived by observing that[1]

$$\sigma_V^2 \equiv \frac{1}{\delta t}\mathrm{var}\,[\delta \ln(V)] = \delta t\ \mathrm{var}\,[r_V]$$

  - Thus, our risk measure is the risk *per unit of time and per monetary unit*

[1] Since $\delta\ln(V) = \ln V(t+\delta t) - \ln V(t+\delta t) = \ln\left[\frac{V(t+\delta t)}{V(t)}\right]$ and by the very definition of a return we have $V(t+\delta t) = V(t)e^{r_V(t)\delta t}$, i.e., $\delta\ln(V) = r_V(t)\delta t$.

invested

$$
\begin{aligned}
\eta_V &\equiv \frac{\mathrm{VaR}_V(c)}{V\ \delta t} \\
&= |Q_{1-c}| \sqrt{\mathrm{var}\left[r_V\right]} - \mathrm{E}\left[r_V\right] \\
&= q\sigma_V - R_V \\
&= q\sqrt{\mathbf{w}^T\mathbf{C}\mathbf{w}} - \mathbf{w}^T\widehat{\mathbf{R}} - r_f
\end{aligned} \tag{3.4}
$$

with the abbreviation

$$
q \equiv |Q_{1-c}| / \sqrt{\delta t} \tag{3.5}
$$

- The expected investment return R_V (which includes the risk free earnings!) influences the risk.
  - Whenever one is not fully invested, **the risk free money market influences the investment risk**,

Chapter 4

The Capital Market Line with Drift

- Assume that for any given *investment universe* (i.e., for any given set of M risky assets) there is a fully invested *optimal portfolio* with return R_m and risk η_m.
  - if risk $\eta_m \neq \eta_{\text{required}}$ still invest in the optimal portfolio, although with a percentage $w \neq 1$
- The expected return of such an investment in the money market and the optimal

portfolio is

$$R_V = wR_m + (1 - w)r_f \quad \text{with } w := \mathbf{w}^T\mathbf{1} \tag{4.1}$$

- With this R_V, the risk, as defined in Equation 3.4, is

$$\begin{aligned}\eta_V &\equiv q\sigma_V - R_V \\ &= q\sigma_V - wR_m - (1 - w)r_f\end{aligned}$$

  - showing explicitly how the risk free rate r_f influences the risk

- Now write $\sigma_V = w\sigma_m$ since the risk free return has no volatility

$$\begin{aligned}\eta_V &= w\left(q\sigma_m - R_m\right) - (1 - w)r_f \\ &= w\eta_m - (1 - w)r_f \qquad (4.2)\end{aligned}$$

- Intuitively:

  - The part w invested in the optimal portfolio contributes the risk of the optimal portfolio

  - The part $(1 - w)$ invested in the money market reduces the risk by its expected (risk free) return.

- Solving for the leverage w, we find that the extent of investment in risky asset is

$$w = \frac{\eta_V + r_f}{\eta_m + r_f} \tag{4.3}$$

- Inserting this w into Equation 4.1 allows us to write the investment return as a function of the investment risk

$$\begin{aligned} R_V &= \frac{\eta_V + r_f}{\eta_m + r_f} R_m + (1 - \frac{\eta_V + r_f}{\eta_m + r_f})\ r_f \\ &= \frac{\eta_V + r_f}{\eta_m + r_f} R_m + \frac{\eta_m - \eta_V}{\eta_m + r_f}\ r_f \\ &= \frac{(R_m - r_f)\,\eta_V + (R_m + \eta_m)\,r_f}{\eta_m + r_f} \end{aligned}$$

  or

$$R_V = \frac{R_m - r_f}{\eta_m + r_f} \cdot \eta_V + \frac{\eta_m + R_m}{\eta_m + r_f} \cdot r_f \tag{4.4}$$

- Thus, the expected return R_V as a function of the risk η_V is a straight line.
  - This is the *capital market line* when drift effects are considered.
  - Its slope is *not* given by the Sharpe Ratio but rather by the[1] *Deutsch Ratio*™

$$\Gamma_m \equiv \frac{R_m - r_f}{\eta_m + r_f} = \frac{\widehat{R}_m}{\eta_m + r_f} = \frac{\widehat{R}_m}{q\sigma_m - \widehat{R}_m} \tag{4.5}$$

- The Deutsch Ratio™ is not only defined for the optimal portfolio V_m but for any portfolio[2]:

$$\Gamma_V \equiv \frac{R_V - r_f}{\eta_V + r_f} = \frac{\widehat{R}_V}{q\sigma_V - \widehat{R}_V} \tag{4.6}$$

[1] Deutsch Ratio™ is a Trademark of **d-fine** GmbH, Frankfurt, Germany.
[2] For all investments on the Capital Market Line we of course have $\Gamma_V = \Gamma_m$.

Chapter 5

The Risk of the *Excess* Returns

- The numerator in the Deutsch Ratio^{TM} (as in the Sharpe Rato) is the expection of the investment's *excess* returns.
- To achieve a just as natural interpretation for the denominator of the Deutsch Ratio^{TM}:
  - The VaR in 3.2 or 3.3 is the quantile of the distribution of the investment

returns r_V.

- The investment's *excess* returns $\widehat{r}_V$ are obtained by simply subtracting the *constant* risk free rate r_f from *each* r_V.
- The distribution of the *excess* returns $\widehat{r}_V$ has exactly the same shape as the distribution of the returns r_V, simply shifted by $-r_f$.
- Thus, the VaR of the *excess* returns, i.e. the quantile of their distribution, is

$$\mathrm{VaR}_c(\widehat{r}_V) = \mathrm{VaR}_c(r_V) + V\ \delta t\ r_f \tag{5.1}$$

- Or directly from 3.3 by observing that r_f only adds to the expected return not to the variance:

$$\begin{aligned}
\mathrm{VaR}_c(\widehat{r}_V) &= |Q_{1-c}|\, V\delta t \sqrt{\mathrm{var}\left[\widehat{r}_V\right]} - V\delta t\ \mathrm{E}\left[\widehat{r}_V\right] \\
&= |Q_{1-c}|\, V\delta t \sqrt{\mathrm{var}\left[r_V\right]} - V\delta t\ \mathrm{E}\left[r_V - r_f\right] \\
&= \underbrace{|Q_{1-c}|\, V\delta t \sqrt{\mathrm{var}\left[r_V\right]} - V\delta t\ \mathrm{E}\left[r_V\right]}_{\mathrm{VaR}_c(r_V)} + V\delta t\ r_f
\end{aligned}$$

- Dividing by $V\ \delta t$ yields the risk measure $\widehat{\eta}$ of the *excess* returns in the same units as η

$$\begin{aligned} \widehat{\eta}_V &\equiv \frac{\mathrm{VaR}_c(\widehat{r}_V)}{V\delta t} \\ &= |Q_{1-c}|\sqrt{\mathrm{var}\left[\widehat{r}_V\right]} - \mathrm{E}\left[\widehat{r}_V\right] \\ &= q\sigma_V - \widehat{R}_V \\ &= q\sqrt{\mathbf{w}^T\mathbf{C}\mathbf{w}} - \mathbf{w}^T\widehat{\mathbf{R}} \\ &= \eta_V + r_f \end{aligned} \tag{5.2}$$

- Thus, the denominator of the Deutsch Ratio™ is the *risk of the excess returns.*
- The Deutsch Ratio™ is simply the expected *excess* return divided by the risk of the *excess* returns:

$$\Gamma_V \equiv \frac{\widehat{R}_V}{\widehat{\eta}_V} = \frac{\text{expected excess return}}{\text{risk of excess returns}} \tag{5.3}$$

It is much more natural to *always* work with *excess* returns, not only regarding expectations, but also regarding *risk.*

- Example of how much more natural excess returns are:
  - The risk η_V in Eq. 4.2 is not zero for $w = 0$ but rather for

$$w_0 := \frac{r_f}{\eta_m + r_f} \tag{5.4}$$

  - A percentage $w < w_0$ has "negative risk".
  - The fluctuations of the investment value are so small compared to the (positive) drift, that the investment return at the border of the confidence interval (i.e. the quantile of the return distribution belonging to confidence c) is still positive.
  - In other words: the potential loss incurred by the fluctuations is still less than the expected return.
  - For instance, for $w = 0 < w_0$ we get $R_V = r_f$ and $\eta_V = -r_f$.
    * The whole investment is in the money market.
    * The expected return is the money market return without any fluctuations.

    * The risk for any confidence c, i.e. the potential loss, is the negative of the one and only P&L value r_f.
    * However, it might seem awkward that the so called risk *free* investment should have a non-zero risk (and a negative one at that!).
  - This does *not* happen, if one uses the risk of the *excess* returns!
    * Since $\widehat{\eta}_V = \eta_V + r_f$, this risk measure is exactly zero when everything is invested risk free.

- Another example:
  - The Capital Market Line, Eq. 4.4, has on *offset*. For risk $\eta_V = 0$ (attainable with $w = w_0$) the expected return is

$$R_V = \frac{\eta_m + R_m}{\eta_m + r_f} r_f$$

  - This is larger than r_f as long as the return R_m of the optimal portfolio is larger than r_f

    * which usually is the case as a compensation for the risk of the optimal portfolio.
  - Thus, even for $\eta_V = 0$ the expected return is larger than the risk free return.
    * This is the compensation for the risk of incurring a loss greater than the border of the confidence interval.
    * $\eta_V = 0$ only means that the negative influence of the fluctuations just compensates the positive influence of the drift. But it does not mean that there are no fluctuations[1]!

- All these things disappear, as soon as one uses excess returns throughout.
- Subtracting r_f from Equation 4.4 yields the capital market line in terms of *excess*

[1] This is the fundamental difference to a risk measure based solely on the volatility like, e.g., Eq. 2.2. If such a risk measure is zero then there are no fluctuations at all by definition.

returns:

$$\begin{aligned}
\widehat{R}_V &= R_V - r_f \\
&= \frac{R_m - r_f}{\eta_m + r_f} \cdot \eta_V + \frac{\eta_m + R_m - (\eta_m + r_f)}{\eta_m + r_f} \cdot r_f \\
&= \frac{R_m - r_f}{\eta_m + r_f} \cdot (\eta_V + r_f)
\end{aligned}$$

- Thus, the Capital Market line can be written very elegantly as

$$\widehat{R}_V = \Gamma_m \widehat{\eta}_V \quad \text{with} \quad \Gamma_m = \frac{\widehat{R}_m}{\widehat{\eta}_m} \tag{5.5}$$

  - No more offset!
  - The expected excess return is zero when the excess return risk is zero.
  - And the excess return risk is zero when everything is invested risk free.

5.1 The Condition for Positive Risk of Excess Returns

- Even the *excess* return risk $\widehat{\eta}_V$ could become negative
  - when the confidence c is chosen so low[2] and/or
  - the holding period δt so large that

$$q < \widehat{R}_V / \sigma_V$$

  - The latter happens to be the traditional Sharpe Ratio γ!
- Thus, we have the following requirement for sensible parameters guaranteeing positive risk of excess returns:

$$\frac{|Q_{1-c}|}{\sqrt{\delta t}} \equiv q \overset{!}{>} \gamma_V \equiv \frac{\widehat{R}_V}{\sigma_V} \tag{5.6}$$

[2]For instance for a confidence of $c = 50\%$ the percentile of the standard normal distribution is $Q_{1-c} = 0$ and the "risks" are always $\widehat{\eta}_V = -\widehat{R}_V$ and $\eta_V = -R_V$ no matter how large the fluctuations are! This clearly shows that some parameter choices are utterly senseless!

- The $1-c$ quantile of the standard normal distribution is negative for any confidence $c > 50\%$, i.e.,

$$|Q_{1-c}| = -Q_{1-c} \quad \forall c > 50\%$$

- For such confidence levels we therefore obtain the requirement

$$\begin{aligned} -Q_{1-c} &\overset{!}{>} \frac{\widehat{R}_V}{\sigma_V}\sqrt{\delta t} \\ 1-c &< \mathrm{N}\left(-\frac{\widehat{R}_V}{\sigma_V}\sqrt{\delta t}\right) \\ c &> 1 - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\sqrt{\delta t}\widehat{R}_V/\sigma_V} e^{-x^2/2}dx \end{aligned} \tag{5.7}$$

- This is not much of a restriction in practice, see Figure 5.1.
  - Only high expected excess return and *simultaneously* low volatility require a confidence c larger than ca. 70%.

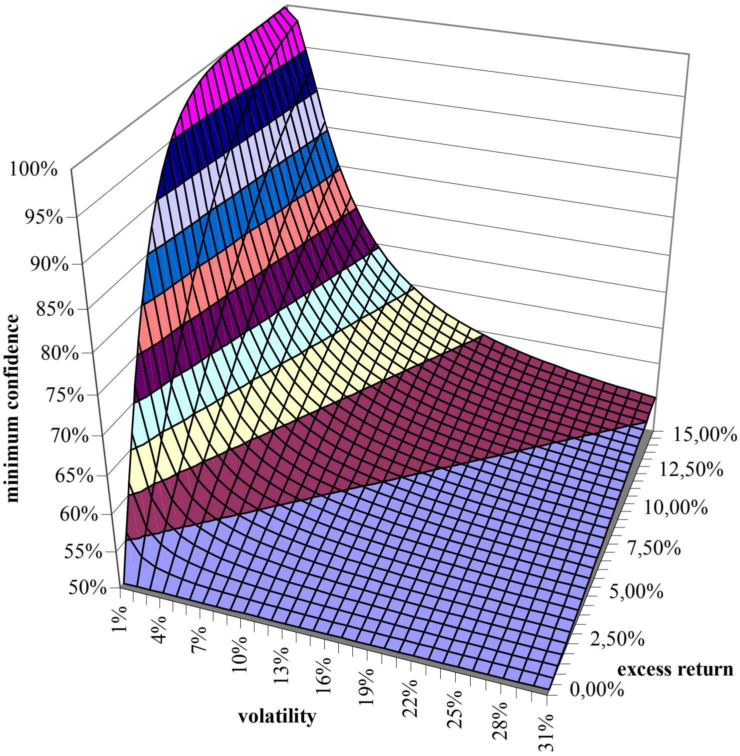

Figure 5.1: Minimum required confidence level to guarantee positive excess return risk for a portfolio with a holding period of 30 days as a function of the portfolio's volatility and expected excess return.

  - Such portfolios are unrealistic, since for low volatilities the expected portfolio return should approach the risk free return.

- For confidence levels above the required lower bound the Deutsch Ratio^{TM} behaves nicely, see Figure 5.2.

- If the confidence is below the lower bound,

  - the risk of excess returns goes from positive values through zero to negative values
  - this generates artificial poles of the Deutsch Ratio^{TM}
  - leading to spurious optimization results strongly dependent on the parameter choice, see Figure 5.3.

- One should always make sure that the confidence is above the required minimum level.

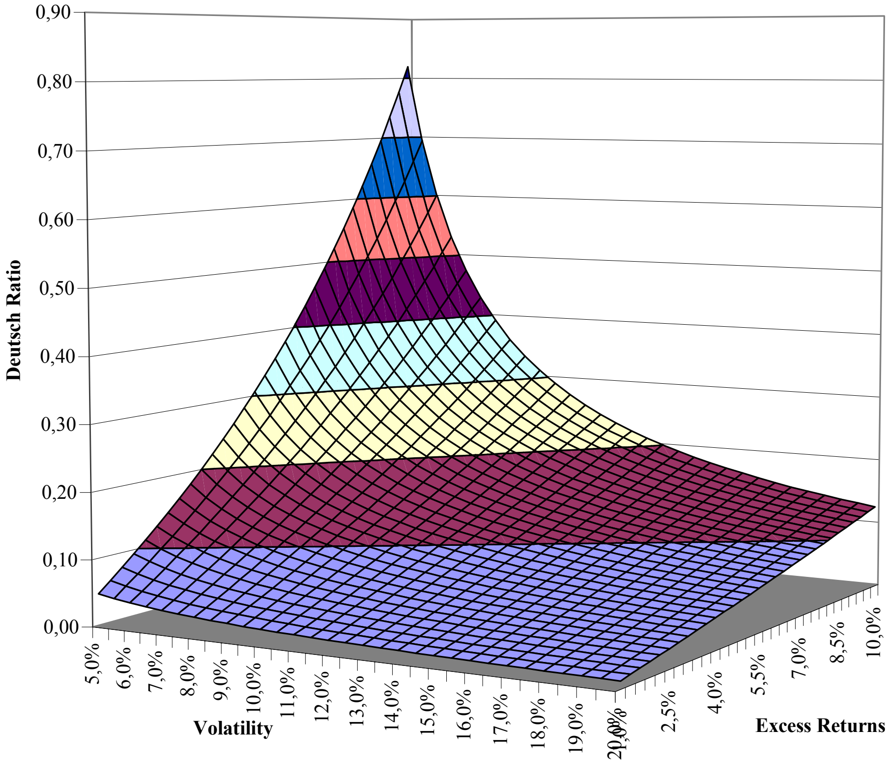

Figure 5.2: Deutsch Ratio for 90% confidence of a portfolio with a holding period of 30 days as a function of the portfolio's volatility and expected excess return. For the most extreme combination of volatility and excess return in the picture ($\widehat{R}_V = 10\%, \sigma_V = 5\%$), the minimum required confidence level according to Eq. 5.7 is $71,8\%$.

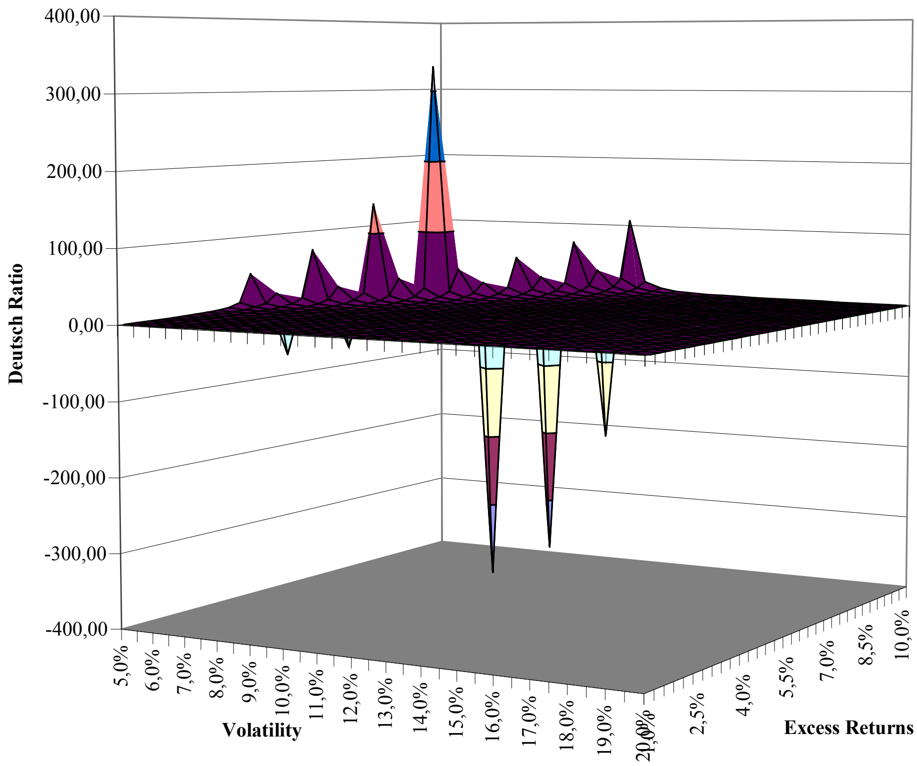

Figure 5.3: Deutsch Ratio for the same situation as in figure 5.2. The only difference is that instead of 90% the confidence was now chosen to be $c = 60\%$ which is well below the $71,8\%$ required by Eq. 5.7.

Chapter 6

Interpretation of the Deutsch Ratio™

- Most intuitive interpretation: Ratio of the expectation to the risk of the *excess* returns.
- Deutsch Ratio™ and Market Price of Risk

- The Capital Market Line. 5.5 can be read as:
- For each additional unit of risk $\widehat{\eta}_V$, the expected return of the total investment increases by Γ_m.
- The Deutsch Ratio$^{\mathrm{TM}}$ of the optimal portfolio is therefore the *market price of risk* of the investment universe when drift effects are taken into account.

- Deutsch Ratio$^{\mathrm{TM}}$ and Sharpe Ratio
  - The reciprocal of Eq. 4.6 yields the relation between the Deutsch Ratio$^{\mathrm{TM}}$ and the Sharpe Ratio

$$\Gamma_V^{-1} = q\,\gamma_V^{-1} - 1 \Longleftrightarrow \Gamma_V = \frac{\gamma_V}{q - \gamma_V} \quad \text{with} \quad \gamma_V \equiv \frac{R_V - r_f}{\sigma_V} \tag{6.1}$$

  - This explicitly shows that $-\infty < \Gamma_V < \infty$ only for $q > \gamma_V$

- Deutsch Ratio$^{\mathrm{TM}}$ and Risk Adjusted Performance
  - RAPM is defined as expected (excess) return over the holding period δt per risk over that holding period

- Without drift:

$$
\begin{aligned}
\mathrm{RAPM}\,(c,\delta t)_{\text{drift neglected}} &= \frac{R_V\delta t - r_f\delta t}{\mathrm{VaR}_V(c)/V} \\
&\approx \frac{R_V\delta t - r_f\delta t}{|Q_{1-c}|\sqrt{\delta t}\sigma_V} \\
&= \frac{\sqrt{\delta t}}{|Q_{1-c}|}\frac{R_V - r_f}{\sigma_V} \\
&= \frac{1}{q}\gamma_V
\end{aligned}
$$

- The Sharpe Ratio has to be “scaled” with the factor $1/q = \sqrt{\delta t}/\,|Q_{1-c}|$ to qualify for a (dimensionless) RAPM
- The Deutsch Ratio^{TM}, on the other hand, is directly a RAPM (all factors

δt cancel):

$$\begin{aligned}\mathrm{RAPM}\,(c,\delta t)_{\text{risk as Eq.3.4}} &= \frac{R_V\delta t - r_f\delta t}{\mathrm{VaR}_V(c)/V + r_f\delta t}\\ &\approx \frac{R_V\delta t - r_f\delta t}{\eta_V\delta t + r_f\delta t}\\ &= \Gamma_V\end{aligned}$$

- Comparing the Deutsch Ratio^{TM} with the Sharpe Ratio numerically,
  - either compare the two resulting RAPMs,
  - or compare γ_V with the scaled Deutsch Ratio^{TM} $q\Gamma_V$ (see Figure 6.1):

$$q\Gamma_V = \frac{1}{\gamma_V^{-1} - q^{-1}}$$

  - Thus, the larger q is compared to the Sharpe Ratio, the closer (apart from the scaling) the Sharpe Ratio and the Deutsch Ratio^{TM} become.

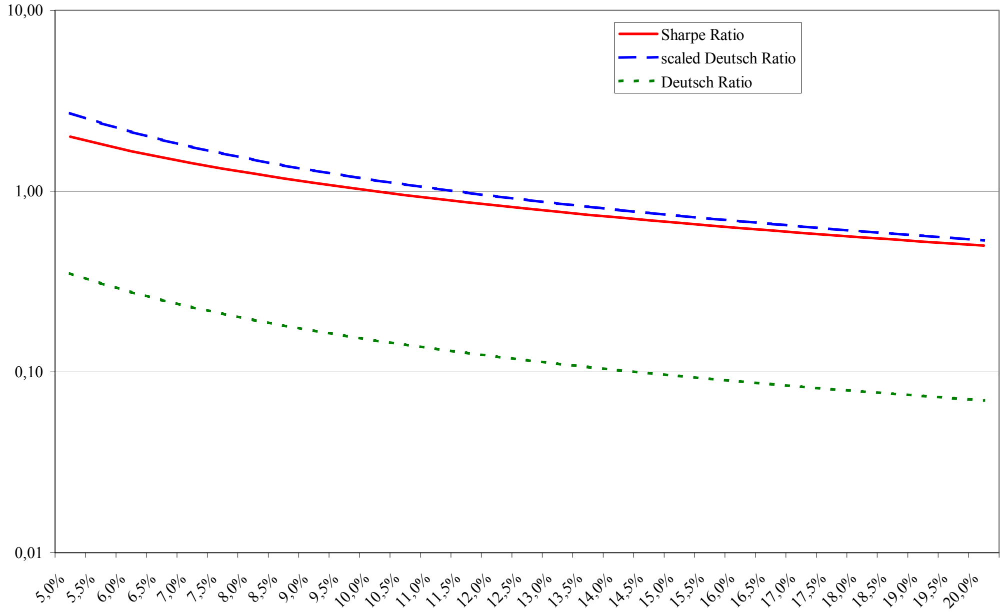

Figure 6.1: Comparison of Sharpe and Deutsch Ratio™ for 90% confidence of a portfolio with a holding period of 10 days as a function of the portfolio's volatility for a constant expected excess return of 10% (note the logarithmic scale).

Chapter 7

The Market Portfolio with Drift

- Up to now we *assumed* that there exists an optimal portfolio.
- We will now *construct* it.
  - Search for *any* portfolio with maximum Deutsch Ratio
  - Then scale its weights such that it is fully invested.

- Find the so-called *characteristic portfolio* [3] with respect to the excess return.
  - Take the vector $\widehat{\mathbf{R}}$ of the asset's excess returns as the so-called *attribute vector*.
  - The portfolio's excess return $\widehat{R}_V = \mathbf{w}^T\widehat{\mathbf{R}}$ is the portfolio exposure to that attribute [3].
  - The characteristic portfolio V_R for attribute $\widehat{\mathbf{R}}$ is the minimum risk portfolio with exposure $\widehat{R}_V = 1$.
- Thus, the characteristic portfolio V_R for attribute $\widehat{\mathbf{R}}$ has to minimize the risk

$$\widehat{\eta}_R = q\underbrace{\sqrt{\mathbf{w}_R^T\mathbf{C}\mathbf{w}_R}}_{\sigma_R} - \underbrace{\mathbf{w}_R^T\widehat{\mathbf{R}}}_{\widehat{R}_R} \tag{7.1}$$

under the constraint

$$\mathbf{w}_R^T\widehat{\mathbf{R}} \stackrel{!}{=} 1 \tag{7.2}$$

- In other words:

– for *constant* $\widehat{R}_V = 1$, we construct the portfolio with minimum risk $\widehat{\eta}_V$ or equivalently with maximum Ratio $1/\widehat{\eta}_V$.
– Since $\widehat{R}_V = 1$ by construction, $1/\widehat{\eta}_V$ is the Deutsch Ratio™ $\widehat{R}_V/\widehat{\eta}_V$.
– Therefore, this portfolio will have maximum Deutsch Ratio™.

- The Lagrangian for this optimization problem is

$$\mathcal{L} = \underbrace{q\sqrt{\mathbf{w}_R^T \mathbf{C} \mathbf{w}_R} - \mathbf{w}_R^T \widehat{\mathbf{R}}}_{\text{To be Minimized}} - \lambda \underbrace{\left[\mathbf{w}_R^T \widehat{\mathbf{R}} - 1\right]}_{\text{Constraint}}$$

- To find the optimal weights $\mathbf{w}_R$ we set the derivative equal to zero:

$$0 \stackrel{!}{=} \frac{\partial \mathcal{L}}{\partial \mathbf{w}_R^T} = q \frac{\mathbf{C} \mathbf{w}_R}{\sqrt{\mathbf{w}_R^T \mathbf{C} \mathbf{w}_R}} - (1+\lambda)\widehat{\mathbf{R}} \tag{7.3}$$

- Multiplying from the left by $\mathbf{w}_R^T$ and observing Eq. 7.2 yields the Lagrange multiplier.

$$
\begin{aligned}
0 &= q\frac{\mathbf{w}_R^T \mathbf{C} \mathbf{w}_R}{\sqrt{\mathbf{w}_R^T \mathbf{C} \mathbf{w}_R}} - (1+\lambda)\underbrace{\mathbf{w}_R^T \widehat{\mathbf{R}}}_{1} \\
1+\lambda &= q\sqrt{\mathbf{w}_R^T \mathbf{C} \mathbf{w}_R} = q\sigma_R
\end{aligned}
$$

- With this λ, the weights in Equation 7.3 become

$$
\begin{aligned}
\mathbf{w}_R &= \frac{1}{q}\sqrt{\mathbf{w}_R^T \mathbf{C} \mathbf{w}_R}(1+\lambda)\mathbf{C}^{-1}\widehat{\mathbf{R}} \\
&= \sigma_R^2 \mathbf{C}^{-1}\widehat{\mathbf{R}} \qquad (7.4)
\end{aligned}
$$

  – Note at this point that q has cancelled! Thus, we have the reassuring result

The weights of the optimal portfolio which maximizes the Deutsch Ratio™ are independent of an investors holding period δt and confidence level c.

- Now, left-multiply by $\widehat{\mathbf{R}}^T$, exploiting Constraint 7.2 again:

$$\sigma_R^2 \, \widehat{\mathbf{R}}^T \mathbf{C}^{-1} \widehat{\mathbf{R}} = \widehat{\mathbf{R}}^T \mathbf{w}_R = 1$$

  - This directly yields the variance of the characteristic portfolio:

$$\sigma_R^2 = \frac{1}{\widehat{\mathbf{R}}^T \mathbf{C}^{-1} \widehat{\mathbf{R}}}$$

  - Thus, the weights in Eq. 7.4 are explicitly

$$\mathbf{w}_R = \frac{\mathbf{C}^{-1} \widehat{\mathbf{R}}}{\widehat{\mathbf{R}}^T \mathbf{C}^{-1} \widehat{\mathbf{R}}} \tag{7.5}$$

- This portfolio has an excess return of 1, i.e., of 100%.
- Therefore it usually contains significant leverage.
- The degree of investment in risky assets is

$$w = \mathbf{1}^T \mathbf{w}_R = \frac{\mathbf{1}^T \mathbf{C}^{-1} \widehat{\mathbf{R}}}{\widehat{\mathbf{R}}^T \mathbf{C}^{-1} \widehat{\mathbf{R}}}$$

- The *Market Portfolio* for our risk measure η is by definition the *fully invested* portfolio with maximum Deutsch Ratio™.
  - Simply divide the weights of the above chracteristic portfolio by w:

$$\begin{aligned}\mathbf{w}_m &= \frac{1}{w}\mathbf{w}_R \\ &= \frac{\widehat{\mathbf{R}}^T\mathbf{C}^{-1}\widehat{\mathbf{R}}}{\mathbf{1}^T\mathbf{C}^{-1}\widehat{\mathbf{R}}}\frac{\mathbf{C}^{-1}\widehat{\mathbf{R}}}{\widehat{\mathbf{R}}^T\mathbf{C}^{-1}\widehat{\mathbf{R}}} \\ &= \frac{\mathbf{C}^{-1}\widehat{\mathbf{R}}}{\mathbf{1}^T\mathbf{C}^{-1}\widehat{\mathbf{R}}}\end{aligned}$$

  - But these are exactly the same weights as the weights of the Market Portfolio in Markowitz theory!

The portfolio which maximizes the Deutsch Ratio™ also maximizes the Sharpe Ratio.

- Its expected excess return, volatility and its risk are

$$\begin{aligned}
\widehat{R}_m &= \widehat{\mathbf{R}}^T \mathbf{w}_m = \frac{\widehat{\mathbf{R}}^T \mathbf{C}^{-1} \widehat{\mathbf{R}}}{\mathbf{1}^T \mathbf{C}^{-1} \widehat{\mathbf{R}}} \\
\sigma_m^2 &= \mathbf{w}_m^T \mathbf{C} \mathbf{w}_m = \frac{\widehat{\mathbf{R}}^T \mathbf{C}^{-1} \widehat{\mathbf{R}}}{\left(\mathbf{1}^T \mathbf{C}^{-1} \widehat{\mathbf{R}}\right)^2} = \frac{\widehat{R}_m}{\mathbf{1}^T \mathbf{C}^{-1} \widehat{\mathbf{R}}} \\
\widehat{\eta}_m &= q\sigma_m - \widehat{R}_m = \frac{\sqrt{\widehat{\mathbf{R}}^T \mathbf{C}^{-1} \widehat{\mathbf{R}}}}{\mathbf{1}^T \mathbf{C}^{-1} \widehat{\mathbf{R}}} \left(q - \sqrt{\widehat{\mathbf{R}}^T \mathbf{C}^{-1} \widehat{\mathbf{R}}}\right)
\end{aligned} \tag{7.6}$$

- Although the portfolio to construct the Capital Market Line is the same as in Markowitz theory,
- any portfolio for a specific risk preference η_V is still dependent on δt and c.
  - Any portfolio on the Capital Market Line has to fulfill a constraint regarding the preferred *risk*
  - while the Market Portfolio itself only has to fulfill the “fully invested” constraint

- All these concepts are implemented in d-fine's *triple-α* portfolio optimization service
  - see www.triple-alpha.de and Figure 7.1

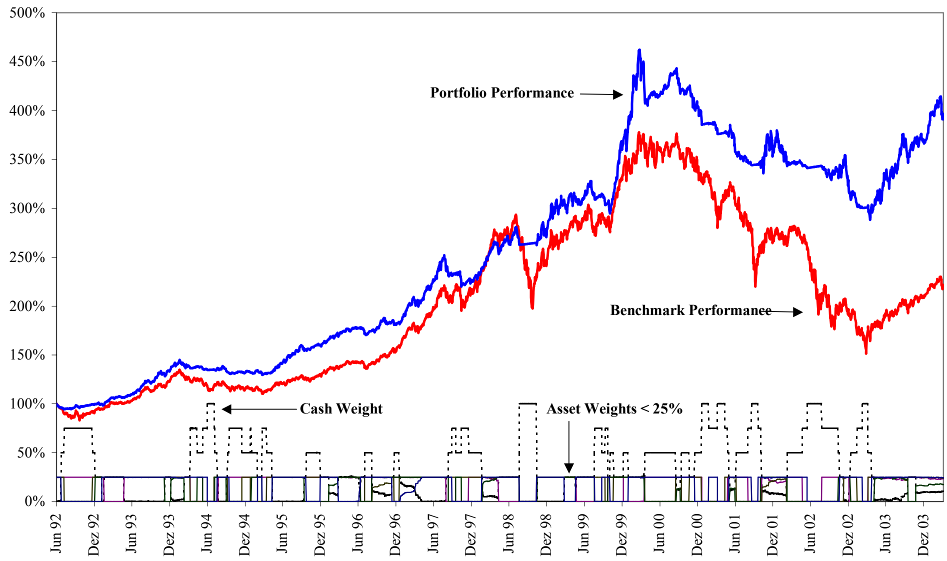

Figure 7.1: A typical portfolio performance over 12 years. The risky assets are ETFs on three DJ STOXX 600 Sector Indexes (Media, Healthcare and Technology) and an ETF on the MDAX. The Benchmark is the DJ STOXX 600 itself. The Deutsch Ratio™ was optimized under the constraints that no short selling was allowed and that the weight of each risky asset has to always be $\leq 25\%$. The risk was kept as close to the risk of the benchmark as possible within these constraints (but always lower than the benchmark risk). In addition each position was stop loss managed based on its current volatility. Transaction costs were fully taken into account. They are the reason for the relatively low trading frequency despite the fact that the optimizer was "allowed" to trade every single trading day. The annualized average Alpha over those 12 years is 6.93%.

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