Bridging the Gap between Credit Risk+ and Merton-style Portfolio Models

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Bridging the Gap between CR+ and Merton-style models

- **Generalised CreditRisk+ Framework**
- **Analytically Tractable Factor Distributions**
- **General Formula for Risk Contributions**
- **Merton-style Models and the Multivariate Vasicek Model**

Generalised CreditRisk+ framework

- Pure default-or-survival model
- Fixed time horizon
- Discrete loss distribution

- **Input data:**

  ΔL : Discretisation constant

  $L_A = \Delta L \cdot \nu_A$: Loss at default of obligor A (incl. recovery rate, collateral, netting)

  $\nu_A \in \boldsymbol{N}$

  p_A : Probability of default (PD) of A over time horizon

- **Aim:**

  Calculate distribution ($\equiv$ probability generating function, **PGF**) of discretised portfolio loss:

$$\text{PortfolioLoss} = \Delta L \cdot L = \Delta L \cdot \sum_A \nu_A \cdot N_A$$

$$G(z) = E\left(z^L\right) = \sum_{n=0}^{\infty} P(L = n) \cdot z^n$$

N_A: Default indicator variables

L : Discretised portfolio loss

■ 1. Step: Poisson approximation

- Consider 2 obligors, N_A and N_B independent **Bernoulli** variables:

$$G_{AB}^{Bernoulli}(z) = \prod_{i=A,B}\left(1-p_i+p_i z^{\nu_i}\right)$$
$$= \prod_i\left(1-p_i(z^{\nu_i}-1)\right)$$

- N_A and N_B independent **Poisson** variables:

$$G_{AB}(z) = \exp\left\{p_A\left(z^{\nu_A}-1\right)+p_B\left(z^{\nu_B}-1\right)\right\}$$
$$= G_{AB}^{Bernoulli}(z) + \mathbf{O}(p^2)$$

Poisson approx. turns out to work excellent in practical applications

■ 2. Step: Credit drivers and conditional independence

- Introduce **K**-dimensional vector $\vec{\gamma}$ of credit factors
  (= potential "states of the world" at the time horizon)
- Assumption: Credit factors completely explain dynamics of the p_A

  ⇒ **Conditional on** $\vec{\gamma}$**,** all default events are **independent**

- Typically: Associate γ_k with a certain industry sector
- Linear model for **conditional** PDs:

$$p_A(\gamma) = p_A \cdot \sum_{k=1}^{K} g_A^k \cdot \gamma_k\,, \quad \sum_{k=1}^{K} g_A^k = 1$$

- Assume a **multivariate distribution** for $\{\gamma_k \geq 0\}$ with

$$E(\gamma_k) = 1, \qquad Cov(\vec{\gamma}) = \boldsymbol{\sigma}$$

■ Putting things together...

A) PGF conditional on $\vec{\gamma}$

$$G_{\gamma}(z) = \exp\left(\sum_{k=1}^{K} \gamma_k P_k(z)\right)$$
$$\text{with} \quad P_k(z) = \sum_{A} g_A^k \, p_A \left(z^{\nu_A} - 1\right)$$
$$(\text{"}\textbf{Sector polynomial} \text{ for sector k"})$$

Remark: Same equation if exposures are independently stochastic (cf. Tasche 2003)

Generalised CR+ Framework

■ Putting things together...

B) Averaging over possible states of the economy

$$G(z) = E^{\gamma}\left(\exp\left(\vec{\gamma} \cdot \vec{P}(z)\right)\right) = M_{\gamma}\left(\vec{P}(z)\right)$$

⟷ $M_{\gamma}(\vec{t}) = E^{\gamma}\left(\exp(\vec{\gamma} \cdot \vec{t})\right)$ Moment generating function (**MGF**) of $\vec{\gamma}$

PGF of L = MGF of $\vec{\gamma}$ at particular „point“ $\vec{t} = \vec{P}(z)$!

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- **Standard CreditRisk+:**
  Consider $\vec{\gamma}$ as <u>independent and gamma-distributed</u>

$$M_{\gamma}^{CR+}(\vec{t}) = \prod_{k=1}^{K} \left(1 - \sigma_{kk} t_k\right)^{-\frac{1}{\sigma_{kk}}}$$

Write PGF as

$$G^{CR+}(z) = \exp\left(-\sum_{k=1}^{K} \frac{1}{\sigma_{kk}} \log\left(1 - \sigma_{kk} P_k(z)\right)\right)$$

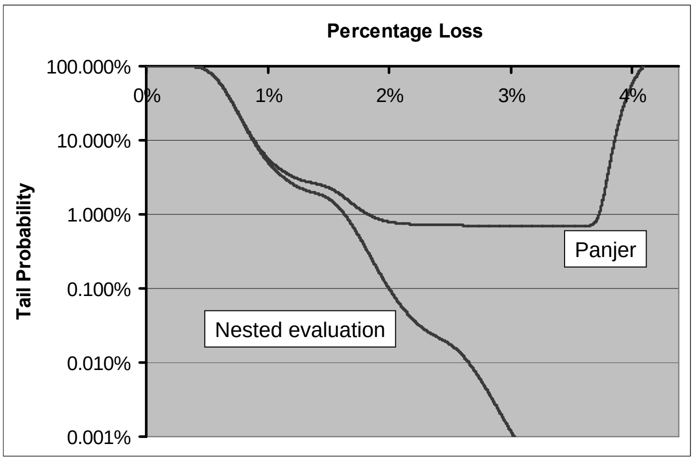
How to evaluate right hand side?

Idea of "nested evaluation":

$$G^{CR+}(z) = \exp\left(-\sum_{k=1}^{K} \frac{1}{\sigma_{kk}} \log\left(1 - \sigma_{kk} P_k(z)\right)\right)$$

- Approximate power series as polynomials of fixed degree N_{max} (i.e. store coefficients in arrays of real numbers)
- "Teach" computer **elementary operations** on polynomials as well as **Exp**(P(z)) and **Log**(P(z)) ($\Leftrightarrow$ simple recursion rules)
- $\Rightarrow$ Equation can be evaluated term-by-term as if $P_k(z)$ were ordinary real numbers!
- Numerically very fast and stable ! (As opposed to standard approach (CSFB '97): Panjer recursion)

Breakdown of the Panjer Recursion
(1.4 mn Obligors, K = 65 Sectors, N_{max} = 20,000)



Use nested evaluation for a broader class of factor distributions

Consider distributions with

$$E(\gamma_k) = 1, \qquad \gamma_k \geq 0$$

If MGF is combination of **elementary operations** & **Exp** & **Log** :
Simply plug $M_\gamma(\vec{t})$ into

$$G(z) = M_\gamma(\vec{P}(z))$$

$\Rightarrow$ Evaluate by nested calculation

$\Rightarrow$ Now possible to include factor **dependence**!

Example: "Compound gamma" model (CGM)

Mixture distribution with MGF

$$M_\gamma^{CG}(\vec{t}) = \exp\left\{-\frac{1}{\hat{\sigma}^2}\log\left(1+\hat{\sigma}^2\sum_{k=1}^{K}\frac{1}{\beta_k}\log(1-\beta_k t_k)\right)\right\}$$

Covariance structure (for K=3):

$$\boldsymbol{\sigma} = \mathrm{Cov}(\vec{\gamma}) = \begin{pmatrix} \beta_1+\hat{\sigma}^2 & \hat{\sigma}^2 & \hat{\sigma}^2 \\ \vdots & \beta_2+\hat{\sigma}^2 & \hat{\sigma}^2 \\ \cdot & \cdots & \beta_3+\hat{\sigma}^2 \end{pmatrix} \Longrightarrow$$

Convenient default correlation structure

Limiting cases:

1. $\hat{\sigma} = 0 \quad \Rightarrow$ Independent gamma distribution
2. All $\beta_k = 0 \quad \Rightarrow$ K replicas of one single factor (Bürgisser et. al 1999)

Compound gamma model (cont'd):

- Only K+1 free parameters: Fitting to given input covariance structure necessary

Other tractable choices:

- Multivariate Gamma distribution
- Squared Gaussians
- …

In all cases discussed:

Marginals (close to) gamma distributed !

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Credit-Value-at-Risk (CVaR) as standard risk measure:

Let L = $\mathbf{l_q}$ be the q%-quantile of the portfolio loss distribution

$$CVaR_q = l_q - E(L)$$

Additive, risk-adjusted breakdown of risk measure is key:

- Portfolio analysis (concentration and bulk risks)
- Capital allocation
- Pricing

Define **CVaR contribution** of obligor A as conditional expected loss:

$$CVaR_A^q = \nu_A \, E(N_A \mid L = l_q) - p_A \nu_A$$

Breakdown of **tail risk**!

CVaR contributions (cont'd)

Generalise result of Tasche & Haaf 2002:

If <u>conditional on $\{\gamma_k\}$</u> N_A independent & Poisson distributed
and $p_A(\vec{\gamma})$ <u>linear</u> in $\vec{\gamma}$:

CVaR contributions depend on partial derivatives of MGF

$$CVaR_A^q = p_A \nu_A \frac{\sum_{k=1}^{K} g_A^k \cdot G_k[l_q - \nu_A]}{G[l_q]} - p_A \nu_A$$

$$\text{where } G_k(z) := \frac{\partial}{\partial t_k} M_{\gamma}\left(\vec{t} = \vec{P}(z)\right) \text{ is also PGF}$$

(l_q - ν_A)-th coefficient

Risk Contributions

CVaR contributions (cont'd)

Result can be written as

Size and sector dependent "**penalty factor**"

Reflects portfolio characteristics !

$$\mathrm{CVaR}_A^q = \underbrace{\mathrm{EL}_A}_{p_A \nu_A} \cdot \sum_k g_A^k \; F_k(\nu_A)$$

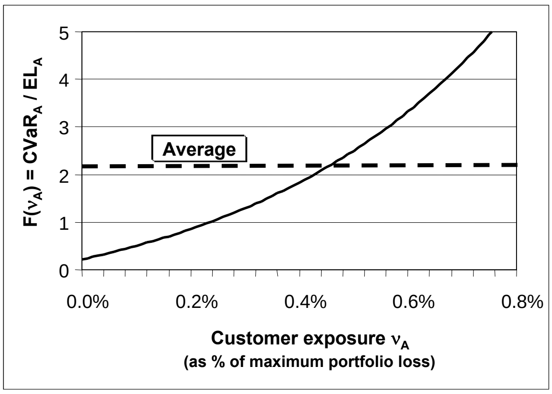
Linear in p_A
(Poisson approx.)

with $$F_k(\nu_A) = \frac{G_k[l_q - \nu_A]}{G[l_q]} - 1$$

Very convenient formula for pricing & what-if calculations

Merton models and the MVM

Penalty factor curves have exponential shape



Result from a 1-factor model for a sample portfolio

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Merton-style models…

… offer a causal model for the dynamics of the PDs
… are in wide-spread use (even Basel II risk weights from 1-factor version)

- Asset return A_A of obligor A at horizon

$$A_A = \vec{b}_A \cdot \vec{X} + \sqrt{1 - R_A^2}\,\varepsilon_A$$
$$\text{where} \quad R_A^2 = \vec{b}_A \cdot \mathbf{\Sigma} \cdot \vec{b}_A$$

- $\{X_k\}$ systematic risk factors, $N(0,\Sigma)$ standard normal distributed,
- ε_A independent, N(0,1) distrib., "idiosyncratic return"

$\Rightarrow$ A_A also N(0,1) distributed

Merton-style models (cont'd)

- **Default** occurs, if A_A falls below **threshold** c_A
- Set $c_A = N^{-1}(p_A)$
- Conditional on $\{X_k\}$ defaults are – as in CR+ - independent
- Conditional PD:

$$p_A(\vec{X}) = \mathrm{Prob}\left(A_A \le c_A \mid \vec{X}\right) = N\left(\frac{N^{-1}(p_A) - \vec{b}_A \cdot \vec{X}}{\sqrt{1 - R_A^2}}\right)$$

Remember: $p_A(\vec{\gamma})$ **linear** in CR+ framework!

Merton-style models (cont'd)

- Averaging over "states-of-the-world" $\vec{X}$ generates

  Vasicek distributed $p_A(\vec{X})$
- Mean:

$$E\left(p_A(\vec{X})\right) = p_A$$

- Relative std. dev. :

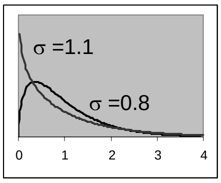
$$\sigma_A^{rel} = \frac{\sigma(p_A(\vec{X}))}{p_A} = \sqrt{\frac{N_2\left(N^{-1}(p_A), N^{-1}(p_A), R_A^2\right)}{p_A^2} - 1}$$

Remember: $\sigma_A^{rel} = \vec{g}_A \cdot \sigma \cdot \vec{g}_A$ **constant** in CR+ framework!

What are the "gaps" between Merton- and CR+-style models ?

Gap 1: Level of PD volatility

- Typical values (>10%) for R_A^2 imply $\sigma_A^{rel} > 1$
- **Problem**: Gamma distribution with $\sigma_{kk} > 1$ has **singularity** at $\gamma_k = 0 \Rightarrow$ Unrealistic for a PD distribution

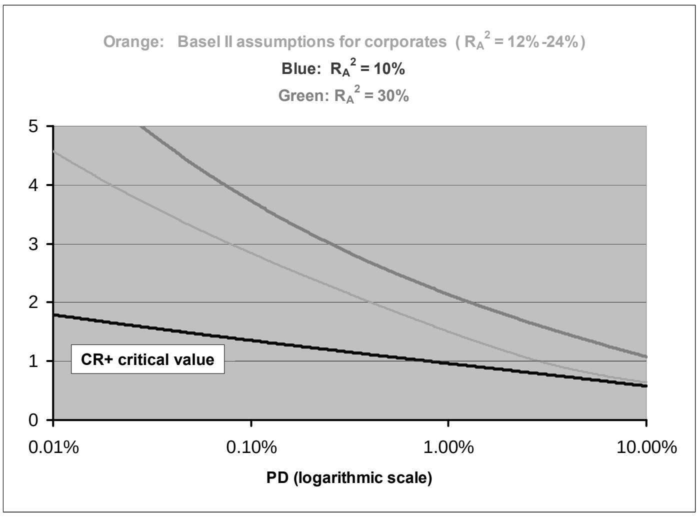


- Closely related: **Level of default correlation**
  Typically **higher** in Merton models

High systematic risk (R_A^2 or σ_{kk})

High PD volatility — High default correlation

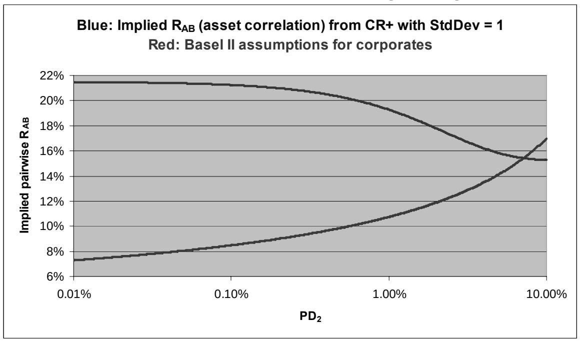
Relative standard deviation σ_A^{rel} as function of PD



CR+ implied asset correlation (defined via default correlation)

Two customers with $PD_1 = 1\%$, PD_2: x-axis

Generalised CR+ model with $\vec{g}_A \cdot \sigma \cdot \vec{g}_B = 1$



Merton :

$$\rho^{Def}_{AB} \approx \sqrt{p_A p_B}\left[\frac{N_2(N^{-1}(p_A),N^{-1}(p_B),R_{AB})}{p_A p_B} - 1\right]$$

$$R_{AB} = \vec{b}_A \cdot \mathbf{\Sigma} \cdot \vec{b}_B$$

CR+ :

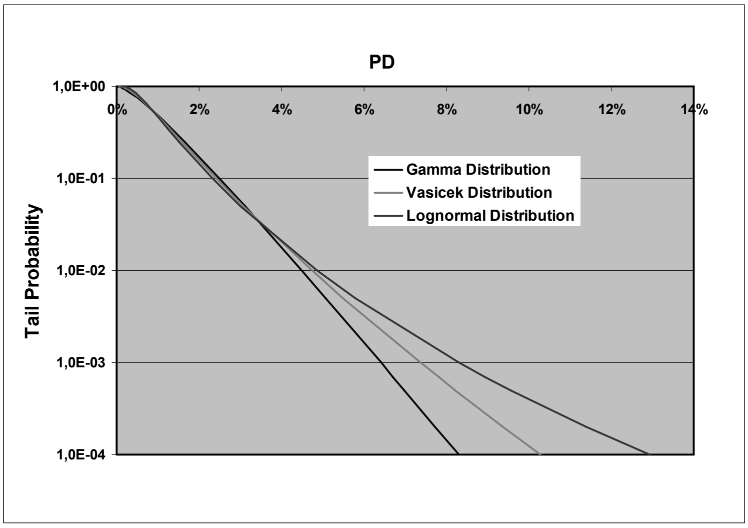
$$\rho^{Def}_{AB} \approx \sqrt{p_A p_B}\ \vec{g}_A \cdot \boldsymbol{\sigma} \cdot \vec{g}_B$$

What are the “gaps” between Merton- and CR+-style models ?

Gap 2: Tail behaviour of PD distribution

- Gamma dist. has much smaller tail than Vasicek (or log-normal)
- Tail behaviour of PD distribution drives dependence between obligor defaults
- ⇒ Strong impact on **tail of loss distribution** (see below)

Different PD distributions with $E(p_A)$ = 1.21% and σ_A^{rel}= 0.8



What are the “gaps” between Merton- and CR+-style models ?

Gap 3: Tail risk contributions

- Implementations of Merton-style models require Monte Carlo simulation
- Single simulated portfolio loss per Monte Carlo sweep
- Contributions to tail risk are here **difficult to calculate**
  ⇔ requires expectation values conditional on **rare** events !
- Closed form solution in CR+ framework

Multivariate Vasicek Model (MVM) – the "Bridge"

Back to generalised CR+ framework …

- Consider $\{X_k\}$ as $N(0,\Sigma)$ distrib. as before, but set

$$\gamma_k = \frac{1}{p_k} N\left(\frac{N^{-1}(p_k) - R_k X_k}{\sqrt{1-R_k^2}}\right)$$

- p_k: Average sector PD
- R_k^2: Level of systematic risk of sector k
- Marginals "scaled-Vasicek" distributed with $E(\gamma_k) = 1$ and values in $[0, 1/p_k]$
- **Back-and-forth translation** possible from

$$\mathbf{\Sigma}, \left\{R_k^2\right\} \quad \leftrightarrow \quad \boldsymbol{\sigma}$$

MVM: Hybrid model with Monte-Carlo integration

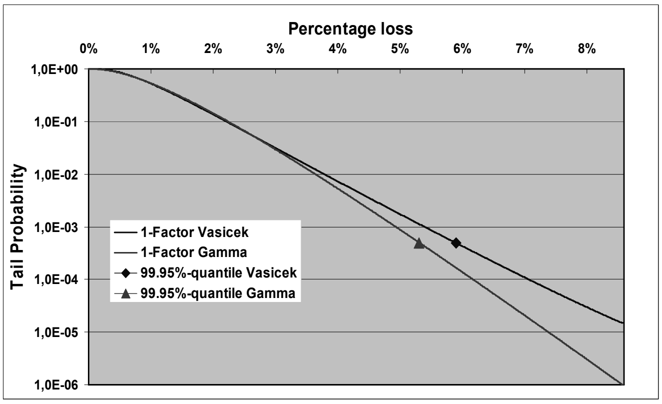
- No closed form for MGF of $\{\gamma_k\}$ and its derivatives available
- Perform necessary integrations over $\{\gamma_k\}$ by (Quasi-)Monte-Carlo

$$G(z) = E^{\gamma}\left(e^{\vec{\gamma}\cdot\vec{P}(z)}\right) \approx \frac{1}{M}\sum_{m=1}^{M} e^{\vec{\gamma}^{(m)}\cdot\vec{P}(z)}$$

$$G_k(z) \approx \frac{1}{M}\sum_{m=1}^{M} \gamma_k^{(m)}\; e^{\vec{\gamma}^{(m)}\cdot\vec{P}(z)}$$

- In each MC step one **realisation of loss distributions** $G^{(m)}(z)$ and $G_k^{(m)}(z)$
- Use nested evaluation for the exponential in each MC step

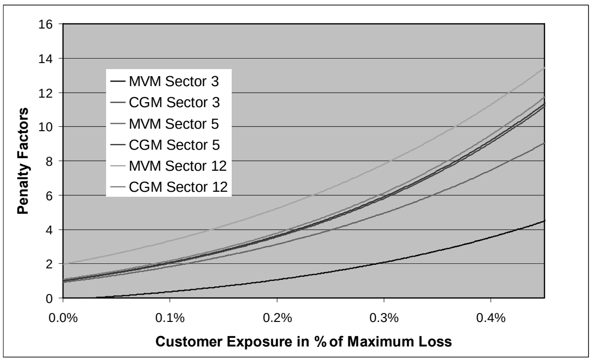
Impact of PD distribution on loss distribution



German Middle Market portfolio, 7000 customers, average PD =1.21%
1-factor generalised CR+ models with factor std. dev. σ = 0.6
$\Rightarrow$ 1st and 2nd moment of both distributions are equal

Merton models and the MVM

Richer covariance structure leads to differentiated penalty factors



13-factor version of Compound Gamma (CGM) and Multivariate Vasicek Model (MVM)
Same Middle Market portfolio as before
Stylised correlation assumptions:
- **Flat** inter-sector correlation **20% except**
- Sector 3 **independent**
- **90%** correlation between sector 12 and sector 13

Advantages of the MVM

- **Flexible parameterisation** in terms of asset correlations as well as PD covariances or a mixture of both

- More **complex covariance** structure σ attainable than with most analytically tractable factor distributions

- Sensible PD distribution even for **high levels of systematic risk**

- **Tail risk** behaviour comparable to Merton models

- Output for **exact CVaR contributions** (penalty factors) calculated in one go with the loss distribution