

Hedging Basket Credit Derivatives with CDS

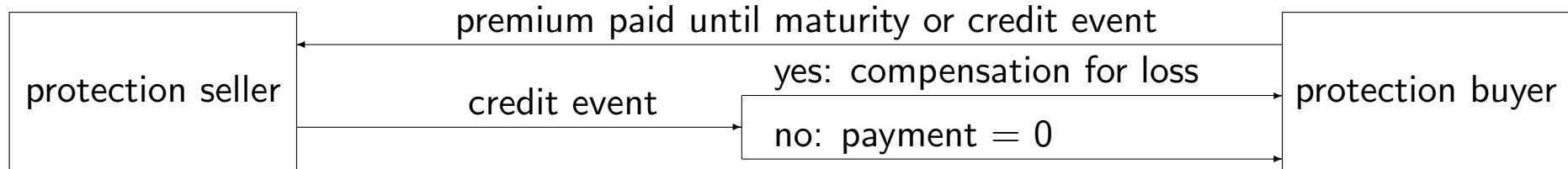
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1 Introduction

1.1 Credit Basket Derivatives

Credit Default Swaps are the primary securities in the credit market. A credit default swap (CDS) offers protection against default of an underlying entity. An insurance fee (spread) s is paid to the protection seller in return for the payment of $(1 - R)$ at the random time τ of default before maturity.



The market quotes fair CDS spreads $s(0, T)$ for "all" maturities T .

Given the spread curve $s(0, T), T > 0$, and an assumption about the (deterministic) recovery rate R one can back out the market implied (risk neutral) distribution of the default time and vice versa

$$s(0, T), T > 0 \quad \leftrightarrow \quad F(t) = \mathbf{P}(\tau < t), t \geq 0.$$

Prices (spreads) of CDS are not calculated based on pricing model - the market "makes" them by supply and demand. **CDS** play the role of **underlings in the credit market**.

Given n default risky entities with random times of default τ_1, \dots, τ_n .

Basket Credit Derivative: The payoffs are functions P of all default times

$$P = P(\tau_1, \dots, \tau_n).$$

Examples are Basket Default Swaps and CDO's.

A ***k*th-to-default swap** is exactly like a plain default swap but the event to protection against is the *k*th default of the *n* underlying names. The fair insurance fee is called the *k*th-to-default spread $s^{k\text{th}}$. Most popular are **first-to-default swaps**, FTD.

Pricing a basket credit derivative (mark-to-market, fair spread) requires a **model for the joint behavior of the default times** τ_1, \dots, τ_n . For pricing purposes what matters is solely the joint distribution

$$\mathbf{P}(\tau_1 < t_1, \tau_2 < t_2, \dots, \tau_n < t_n),$$

where the marginal distributions $F_i(t) = \mathbf{P}(\tau_i < t)$, $i = 1, \dots, n$, are given by the market.

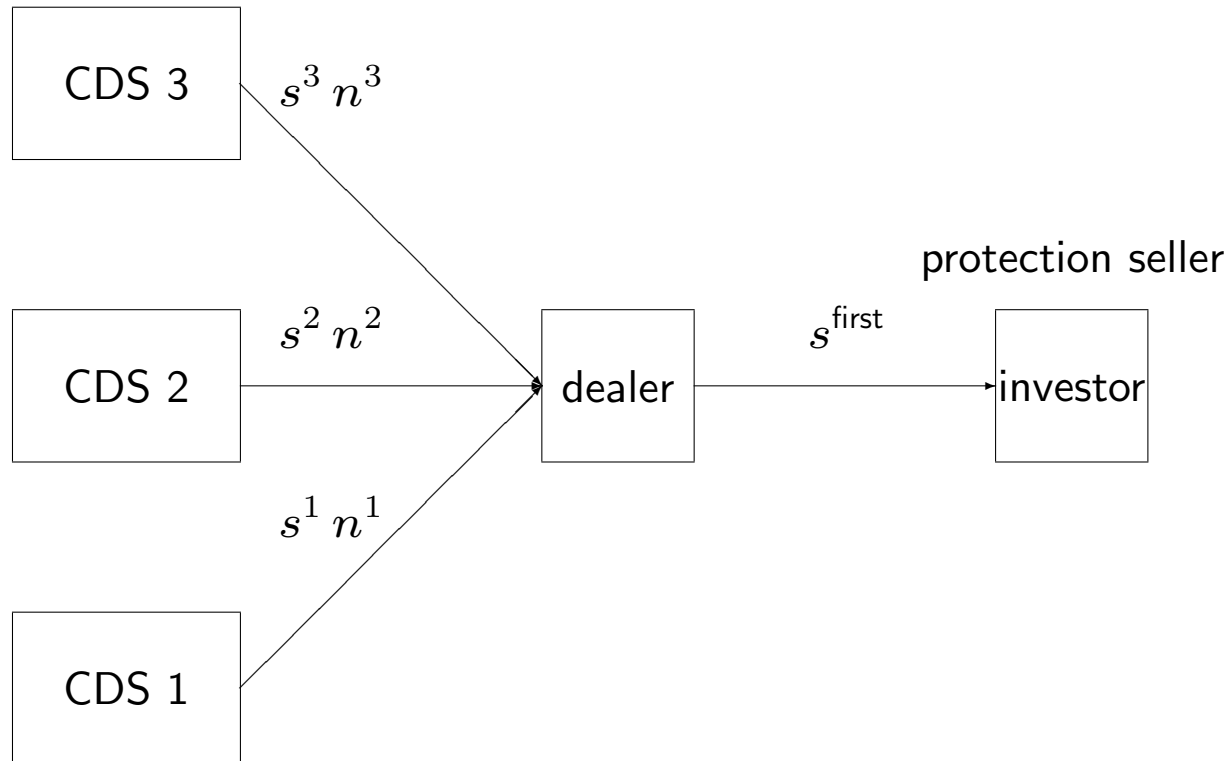
- Copula approaches: $\mathbf{P}(\tau_1 < t_1, \dots, \tau_n < t_n) = C(F_1(t_1), \dots, F_n(t_n))$
 - Monte Carlo implementation
 - quasi analytic implementation in case of dimension reduction (factor models)
- Structural models, e.g. HULL & WHITE 2001, OVERBECK & SCHMIDT 2003

1.2 Intuitive Hedging of Baskets

Hedging a basket credit derivative intuitively involves hedging of two types of risk, **spread risk** of the individual names in the basket and **event risk**, the risk of actual default.

Consider as an example a first-to-default swap where we have bought protection. The hedging strategy would be to sell protection on the individual names in the basket via single name credit default swaps (CDS) on each credit $i = 1, \dots, n$ with some notional amount $n^i \leq 1$ and market spread $s^i(0, T)$.

Hedging Credit Baskets with CDS



Hedging Credit Baskets with CDS

- immunize basket product against changes in spreads of the underlying names (spread risk) - spread hedge

$$\frac{\partial \text{Price}(\text{Basket})}{\partial s^i} = n_i \frac{\partial \text{Price}(\text{CDS}_i)}{\partial s^i}.$$

- generate enough spread income from the hedge to be able to pay the spread on the basket
- should the credit event occur and the basket trade terminates, be able to fulfill contract obligations (default risk) and unwind the outstanding default swap hedges at the then prevailing market conditions.

SCHMIDT & WARD (2002), SCHÖNBUCHER & SCHUBERT (2001) investigate how the **default implied spread widening (spread shock)** is related to the copula of the joint distribution of the default times.

2 Setup and notation

We work on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$ with right continuous filtration (flow of information) $(\mathcal{F}_t)_{t \geq 0}$. Consider positive random variables τ_1, \dots, τ_n on this probability space. We interpret τ_i as the random time of default of credit i and suppose that τ_1, \dots, τ_n are $(\mathcal{F}_t)_{t \geq 0}$ stopping times but we do not make any particular assumption on the way how the default times τ_1, \dots, τ_n are modelled.

To simplify the exposition and to put a clear focus on the problem of correlated defaults we **assume that riskless interest rates are zero** in our model.

Also assume that all recovery rates $R_i(t)$ are deterministic and constant.

The **prices of the primary securities**, the credit default swaps, are assumed to be $(\mathbf{P}, \mathcal{F}_t)$ -**martingales** ensuring that there is no arbitrage between the primary securities.

The payoff of a *contingent claim* which is paid at time T is described by an \mathcal{F}_T -measurable random variable X . For an integrable claim X we define its value $V_t(X)$ at time $t \leq T$ by

$$V_t(X) = \mathbb{E}(X | \mathcal{F}_t). \quad (1)$$

3 The CDS spread

Investigate a CDS on one of our credits $i = 1, \dots, n$ and write τ for the random time of default. A CDS entered into at time $t = 0$ with maturity T and spread s is a contingent claim with the following payoff at time T

$$-s \cdot (\tau \wedge T) + (1 - R)\mathbf{1}_{\{\tau \leq T\}}.$$

We use the following notation:

$$B(t, T) = \begin{cases} \mathbb{E}(\tau \wedge T | \mathcal{F}_t) - \tau \wedge t & : \text{ for } t \leq T \\ 0 & : \text{ otherwise} \end{cases} \quad (2)$$

$$Q(t, T) = \mathbb{E}(\mathbf{1}_{\{\tau > T\}} | \mathcal{F}_t) \quad \text{defaultable zero bond.} \quad (3)$$

$(B(t, T))_{t \geq 0}$ is the *risky present value of a basis point*, it gives the value at time t of one unit paid for the length of time to default after t and up to T .

We have the relation

$$B(t, T) = \int_t^T Q(t, u) du, \quad \frac{\partial}{\partial T} B(t, T) = Q(t, T). \quad (4)$$

Definition 1. *The fair spread $s(t, T)$ for a CDS entered into at time t and with maturity T is defined as*

$$s(t, T) = \begin{cases} \frac{(1-R)(1-Q(t, T))}{B(t, T)} & : \text{ on } \{\tau > t\}, t < T \\ 0 & : \text{ otherwise .} \end{cases} \quad (5)$$

In practice at each point t in time the CDS spread curve $(s(t, T))_{T>t}$ is the primary market information. The next proposition shows how the spread curve "drives" the other quantities.

Proposition 1. *Let $t > 0$ be fixed and all statements are \mathbf{P} -a.s. on $\{\tau > t\}$.*

(i) The risky present value of a basis point $B(t, T)$ satisfies the following ordinary differential equation

$$\frac{\partial}{\partial T} B(t, T) + \frac{s(t, T)}{1 - R} B(t, T) = 1, T > t, \quad (6)$$

with initial condition $B(t, t) = 0$.

(ii) Suppose that the CDS spread curve $(s(t, T))_{T>t}$ is an integrable function in $T > t$ \mathbf{P} -a.s. on $\{\tau > t\}$. Then the corresponding term structure of risky zero bonds $(Q(t, T))_{T>t}$ can be inverted from the CDS spread curve $(s(t, T))_{T>t}$ and $Q(t, T)$ is given by

$$Q(t, T) = 1 - \frac{s(t, T)}{1 - R} \int_t^T \exp \left(- \int_v^T \frac{s(t, u)}{1 - R} du \right) dv. \quad (7)$$

Moreover, for the risky present value of a basis point we have the relation

$$B(t, T) = \int_t^T \exp \left(- \int_v^T \frac{s(t, u)}{1 - R} du \right) dv. \quad (8)$$

Proof: Assertion (i) is an immediate consequence of (5) and (4).

If $(s(t, T))_{T>t}$ is integrable, the solution to (6) is standard and given by (8). The assertion for $Q(t, T)$ then follows in view of $Q(t, T) = \frac{\partial}{\partial T} B(t, T)$. \diamond

4 CDS strategies

We investigate simple trading strategies in CDS which generate new securities that can be used as primary hedging instruments.

Let $0 = t_0 < t_1 < t_2 < \dots < t_N = T$ be a partition P of time and consider the following strategy. At time t_0 we enter into a fair CDS with maturity T , at time t_1 this CDS is unwound at the then prevailing market value and we enter into a new fair CDS starting at time t_1 with maturity T and so on. Denote by $V(t, u, T)$ the value at time $t \geq u$ of a CDS whose spread is $s(u, T)$, i.e., it was fair at time u .

The value process $D^P(t, T)$ of this strategy is obviously

$$D^P(t, T) = \sum_{j=1}^N V(t_j \wedge t, t_{j-1} \wedge t, T).$$

and, after some simple algebra, it can also be written in integral form

$$\begin{aligned}
 D^P(t, T) &= - \int_0^t \sum_{j=1}^N s(t_{j-1}, T) \mathbf{1}_{(t_{j-1}, t_j]}(u) \mathbf{1}_{\{\tau > u\}} du \\
 &\quad - \int_{(0, t]} \sum_{j=1}^N s(t_{j-1}, T) \mathbf{1}_{(t_{j-1}, t_j]}(u) dB(u, T) - (1 - R)(Q(t, T) - Q(0, T)).
 \end{aligned}$$

Passing to the limit for $\Delta t_j = t_j - t_{j-1} \rightarrow 0$ we define for $t \leq T$

$$C(t, T) = - \int_0^t s(u-, T) du - \int_{(0, t]} s(u-, T) dB(u, T) - (1 - R)(Q(t, T) - Q(0, T)). \tag{9}$$

We interpret $C(t, T)$ as the price at time t of a strategy in credit default swaps which consists in continuously resettling into a fair credit default swap with maturity T .

Proposition 2. *If the continuous martingale part of the semi martingale $(s(t, T))$ up to time τ vanishes, then*

$$C(t, T) = - \int_0^t s(u, T) du + \int_0^t 1_{\{\tau > u\}} B(u, T) ds(u, T) + (1 - R) 1_{\{\tau \leq t\}}. \quad (10)$$

Proof: Ito's formula . . .

Equation (10) has an intuitive interpretation. The first term quantifies the accrued premiums from the CDS positions up to time t . The second integral $\int_0^t B(u, T) ds(u, T)$ expresses the cumulative costs of resettling the CDS positions to be fair: over the "time interval" du the mark-to-market value of our CDS position from the beginning of this interval is just the change in fair spread

$ds(u, T)$ times the present value of a basis point $B(u, T)$ at the end of the interval and for the remaining time to maturity T .

5 Hedging baskets

Consider n credits with default times τ_1, \dots, τ_n . Assume from now on that

$$\mathbf{P}(\tau_i = \tau_j) = 0, \forall i \neq j. \quad (11)$$

The default times τ_1, \dots, τ_n can then be uniquely ordered and we denote by $\tau_{[k]}$ the time of the k th default, i.e., $\tau_{[k]} \in \{\tau_1, \dots, \tau_n\}$ and

$$\tau_{[1]} < \tau_{[2]} < \dots < \tau_{[n]}.$$

A k th to default swap (basket CDS) with maturity T and premium s is like a credit default swap where the event to protect is the occurrence of the k th default before maturity T . It is a credit basket derivative with payoff

$$V_T^{k\text{th}} = -s \cdot (\tau_{[k]} \wedge T) + \sum_{i=1}^n P_i(\tau_i) \mathbf{1}_{\{\tau_{[k]} \leq T, \tau_{[k]} = \tau_i\}}, \quad (12)$$

where $P_i(\tau_i)$ is an insurance premium paid if credit i is the k th defaulting, in practice usually, $P_i(\tau_i) = 1 - R_i$, where R_i is the recovery rate for credit i .

In the following we investigate the problem of hedging a basket credit derivative with primary securities such as credit default swaps $V^i(t, T)$ on credit i or strategies $D^{i,P}(t, T)$, $C^i(t, T)$. The superscript i indicates that the respective security refers to credit i .

Definition 2. *The basket credit derivative with payoff $V_T = f(T, \tau_1, \dots, \tau_n)$ at time T is called hedgeable in the hedge instruments $H^i(\cdot, S) \in \{V^i(\cdot, S), D^{i,P}(\cdot, S), C^i(\cdot, S)\}$ with $S \in M_i$ and M_i a finite set of maturities, if*

$$V_T = K + \sum_{i=1}^n \sum_{S \in M_i} \int_0^T n^{i,S}(u) dH^i(u, S),$$

with some constant K and predictable integrands $n^{i,S}$ such that the integrals are well-defined. The integrands $\{n^{i,S}, S \in M_i, i = 1, \dots, n\}$ are called a hedging strategy in the hedge securities $\{H^i(\cdot, S), S \in M_i, i = 1, \dots, n\}$.

Remark: The strategy $\{n^{i,S}, S \in M_i, i = 1, \dots, n\}$ can be extended to a self-financing strategy in the securities $\{H^i(\cdot, S), S \in M_i, i = 1, \dots, n\} \cup \{\beta\}$ putting a respective amount n^β into the risk free security (savings account) $\beta_t = 1$:

$$n^\beta(t) = \sum_{i=1}^n \sum_{S \in M_i} n^{i,S}(t) H^i(t, S) - \left(K + \sum_{i=1}^n \sum_{S \in M_i} \int_0^t n^{i,S}(u) dH^i(u, S) \right), t \leq T.$$

5.1 The pure jump case

Denote by N^i the jump process associated with the default time τ_i

$$N_t^i = 1_{\{\tau_i \leq t\}}, t \geq 0.$$

In this section we assume that the underlying filtration (\mathcal{F}_t) is

$$\mathcal{F}_t = \mathcal{F}_t^{N^1, \dots, N^n}, t \geq 0, \quad (13)$$

i.e., the filtration is generated by the pure jump processes N^1, \dots, N^n . In other words, defaults are the only observable information in the market.

It well-known that in the pure jump case (\mathcal{F}_t) -adapted processes possess a very simple and explicit form which will be the key for our further analysis of hedging and pricing basket derivatives.

Denote by z_k the random variable indicating the identity of the k th default:

$$z_k = \sum_{i=1}^n i \mathbf{1}_{\{\tau_{[k]} = \tau_i\}}, k = 1, \dots, n. \quad (14)$$

Lemma 1. *Let (X_t) be right continuous (\mathcal{F}_t) -adapted, then X_t admits a representation*

$$\begin{aligned} X_t = & f_0(t)1_{[0,\tau_{[1]})}(t) + f_1(\tau_{[1]}, z_1, t)1_{[\tau_{[1]},\tau_{[2]})}(t) + \dots & (15) \\ & + f_{n-1}(\tau_{[1]}, \dots, \tau_{[n-1]}, z_1, \dots, z_{n-1}, t)1_{[\tau_{[n-1]},\tau_{[n]})}(t) \\ & + f_n(\tau_{[1]}, \dots, \tau_{[n]}, z_1, \dots, z_n, t)1_{[\tau_{[n]},\infty)}(t), \end{aligned}$$

with deterministic functions $f_k(t_1, \dots, t_k, i_1, \dots, i_k, t)$.

In view of (15) the CDS spreads $s^i(t, T)$ can be written in the form

$$s^i(t, T) = \begin{cases} a^i(t, T) & : t < \tau_{[1]} \\ a^i(t, T) + b^i(\tau_{[1]}, i_1, t, T) & : \tau_{[1]} \leq t < \tau_{[2]}, \tau_{[1]} = \tau_{i_1}, i_1 \neq i \\ a^i(t, T) + b^i(\tau_{[1]}, i_1, t, T) + b^i(\tau_{[1]}, \tau_{[2]}, i_1, i_2, t, T) & : \tau_{[2]} \leq t < \tau_{[3]}, \tau_{[1]} = \tau_{i_1}, \\ & \tau_{[2]} = \tau_{i_2}, i_1 \neq i, i_2 \neq i \\ \dots & : \dots \end{cases} \quad (16)$$

The function $a^i(t, T)$ is the deterministic base CDS spread up and until the time of first default, the function $b^i(\tau_{[1]}, i_1, t, T)$ is the spread widening relative to the base spread $a^i(t, T)$ which is caused by the occurrence of the time of first default and the first default being credit i_1 etc.

Since $s^i(t, T)$ is a semimartingale until τ_i we can assume that the functions

$$a^i(t, T), b^i(t_1, \dots, t_k, i_1, \dots, i_k, t, T)$$

are of finite variation and right continuous in the variable t .

For the predictable hedging strategy $n(t)$ in any of our hedge instruments we make the following ansatz

$$n(t) = \begin{cases} n_0(t) & : 0 \leq t \leq \tau_{[1]} \\ n_1(\tau_{[1]}, i_1, t) & : \tau_{[1]} < t \leq \tau_{[2]}, \tau_{[1]} = \tau_{i_1}, i_1 \neq i \\ n_2(\tau_{[1]}, \tau_{[2]}, i_1, i_2, t) & : \tau_{[2]} < t \leq \tau_{[3]}, \tau_{[1]} = \tau_{i_1}, \\ & \tau_{[2]} = \tau_{i_2}, i_1 \neq i, i_2 \neq i \\ \dots & : \dots \end{cases} . \quad (17)$$

Now it turns out that the hedging and pricing of a first to default swap can be made very explicit.

Proposition 3. Consider a first-to-default (FTD) swap with maturity T and payoff V_T^{first} ,

$$V_T^{first} = -s \cdot (\tau_{[1]} \wedge T) + \sum_{i=1}^n P_i(\tau_i) 1_{\{\tau_{[1]} \leq T, \tau_{[1]} = \tau_i\}}.$$

Chose as hedge instruments CDS strategies $C^i(t, T)$ (see (9)) for the underlying credits $i = 1, \dots, n$. The FTD swap is hedgeable in the instruments $C^i(t, T)$ with strategies $n^i(t)$ as in (17), i.e.,

$$V_T^{first} = K + \sum_{i=1}^n \int_0^T n^i(s) dC^i(s, T)$$

if and only if the vector function $\mathbf{n}_0(t) = (n_0^1(t), \dots, n_0^n(t))^T$ satisfies the following

system of ordinary integral equations

$$-s t \cdot \mathbf{1} + \mathbf{P}(t) = K \cdot \mathbf{1} + \mathbf{E}(t, T) \cdot \mathbf{n}_0(t) + \int_0^t d\mathbf{F}(u, T) \cdot \mathbf{n}_0(u) \cdot \mathbf{1}, \quad 0 \leq t \leq T \quad (18)$$

$$-s \cdot T = K + \int_0^T d\mathbf{F}(u, T) \cdot \mathbf{n}_0(u), \quad (19)$$

with the notation

$$\begin{aligned} \mathbf{P}(t) &= (P_1(t), \dots, P_n(t))^T \\ \mathbf{1} &= (1, \dots, 1)^T \end{aligned}$$

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$$\mathbf{E}(t, T) = (E_{i,j}(t, T))_{i,j=1,\dots,n}$$

$$E_{i,j}(t, T) = \begin{cases} 1 - R_i & : i = j \\ b^i(t, j, t, T) \int_t^T \exp\left(-\int_v^T \frac{a^i(t, u) + b^i(t, j, t, u)}{1 - R_i} du\right) dv & : i \neq j \end{cases}$$

$$d\mathbf{F}(u, T) = (-a^1(u, T)du + B^1(u, T)da^1(u, T), \dots, \\ -a^n(u, T)du + B^n(u, T)da^n(u, T))$$

$$B^i(u, T) = \int_t^T \exp\left(-\int_v^T \frac{a^i(t, u)}{1 - R_i} du\right) dv$$

and functions a^i, b^i from (16).

Proof: Using (10) from Proposition 2 the FTD swap is hedgeable in the instruments $C^i(t, T)$ with strategies $n^i(t)$ as in (17) if and only if

$$\begin{aligned}
 & -s \cdot (\tau_{[1]} \wedge T) + \sum_{i=1}^n P_i(\tau_i) \mathbf{1}_{\{\tau_{[1]} \leq T, \tau_{[1]} = \tau_i\}} \\
 & = K + \sum_{i=1}^n \left[- \int_0^{\tau_{[1]} \wedge T} n_0^i(u) s^i(u, T) du + \int_0^{\tau_{[1]} \wedge T} n_0^i(u) B^i(u, T) ds^i(u, T) \right. \\
 & \quad \left. + n_0^i(\tau_{[1]}) (1 - R_i) \mathbf{1}_{\{\tau_{[1]} = \tau_i \leq T\}} \right].
 \end{aligned}$$

On the set $\{\tau_{[1]} = \tau_j = t \leq T\}$ using (8) and (16) this can be written as

$$\begin{aligned}
 & -s \cdot t + P_j(t) \\
 &= K + \sum_{i=1}^n \left[- \int_0^t n_0^i(u) a^i(u, T) du \right. \\
 & \quad \left. + \int_0^t n_0^i(u) \int_u^T \exp \left(- \int_v^T \frac{a^i(t, w)}{1 - R_i} dw \right) dv da^i(u, T) \right] \\
 & \quad + \sum_{i \neq j} b^i(t, j, t, T) \int_t^T \exp \left(- \int_v^T \frac{a^i(t, u) + b^i(t, j, t, u)}{1 - R_i} du \right) dv + n_0^j(t)(1 - R_j).
 \end{aligned}$$

In vector notation this is just equation (18). In the same way, on the set $\{\tau_{[1]} > T\}$ we obtain (19).

Corollary 1. *Suppose that for every s, K equation (18) possesses a unique solution*

$$\mathbf{n}_0^{s,K}(t), t \leq T.$$

(i) *For given spread s the price K of the FTD swap is a solution of the equation*

$$K = -s \cdot T - \int_0^T d\mathbf{F}(u, T) \cdot \mathbf{n}_0^{s,K}(u).$$

(ii) *The fair spread s^{FTD} of the FTD swap is a solution of the equation*

$$s^{FTD} = \frac{-\int_0^T d\mathbf{F}(u, T) \cdot \mathbf{n}_0^{s^{FTD},0}(u)}{T}.$$

5.2 Numerical examples

To illustrate the results we start with a model setup as in (16) assuming for simplicity that all functions $a^i(t, T), b^i(u, j, t, T), \dots$ are constant over time, i.e.

$$\begin{aligned} a^i(t, T) &= a^i \\ b^i(u, j, t, T) &= b^i(j) \\ &\dots \end{aligned}$$

For the premiums $P_i(t)$ of the FTD swap we assume, as is common in practice, that

$$P_i(t) = 1 - R_i.$$

In this case equation (18) simplifies considerably and possesses a unique solution for every s, K , which can be made even explicit. However, we prefer a numerical solution based on a time discretization.

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Consider as an example $n = 5$ names with base spreads $a^1 = 0,80\%$, $a^2 = 0,90\%$, $a^3 = 1,00\%$, $a^4 = 1,10\%$, $a^5 = 1,20\%$ and assume recoveries $R_i = 20\%$ throughout. The following table shows the fair FTD spread s^{FTD} for maturities $T = 1, \dots, 5$ and for equal default implied spread jumps $b^i(j) = 1\%, 5\%, 10\%$.

$b^i(j)/T$	1	2	3	4	5
1%	4,878%	4,764%	4,657%	4,555%	4,459%
5%	4,467%	4,073%	3,765%	3,516%	3,310%
10%	4,074%	3,523%	3,148%	2,872%	2,660%

Now consider the same example as above with maturity $T = 5$ and for equal default implied spread jumps $b^i(j) = 5\%$. Here are the fair prices K of the FTD swap for different given spreads s .

spread s	2,00%	2,50%	3,00%	3,31%	3,50%	4,00%	4,50%
K in basis points	598,16	369,85	141,54	0,00	-86,75	-315,06	-543,37

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The final example takes a rather extreme situation. We consider a $n = 5$ basket with base spreads $a^1 = 1,00\%$, $a^2 = 2,00\%$, $a^3 = 3,00\%$, $a^4 = 4,00\%$, $a^5 = 5,00\%$, recoveries $R_i = 20\%$, maturity $T = 5$ and equal default implied spread jumps $b^i(j) = 10\%$. The fair spread is $s^{\text{FTD}} = 7,992\%$ and the following picture shows the hedges over time for the fair FTD swap.

