# MODELLING LIQUIDITY AND ITS EFFECTS

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## 1. What is liquidity?

*To buy a lot quickly, you have to pay more*

- Effect of a sudden trade should be transient
- Liquidity should be costly
- The more rapidly you try to trade, the worse your terms of trade should be
- No arbitrage from ramping, etc...

Don't want to do

(a) $S_t = F(H_t, Z_t)$ ($H_t$= no. of shares held at $t$)

(i) Free round trips!

(ii) Block trades?

(iii) Many agents?

(iv) Option pricing?

(v) How to value holdings?

(b) Microstructure model of order book?

Too complicated!

*Illiquidity is due to temporary imbalance of supply and demand*

## 2. A simple model for liquidity.

- Let

$$p_n \equiv \log S(t_n) \equiv \log S(n\Delta t), \qquad p_n = p_{n-1} + \xi_n,$$

$(\xi_n)$ IID. In period $I_n = ((n-1)\Delta t, n\Delta t]$, $\lambda_a \Delta t$ shares may be offered for sale, $\lambda_b \Delta t$ sought for purchase, so

$$\Delta H_n + \lambda_b \Delta t f_d(\tilde{p}_n - p_n) = \lambda_a \Delta t f_s(\tilde{p}_n - p_n)$$

where $\Delta H_n$ is additional amount required in period $n$.

- Hence

$$\frac{\Delta H_n}{\Delta t} = (\lambda_a f_s - \lambda_b f_d)(\tilde{p}_n - p_n)$$

so

$$\tilde{p}_n - p_n = \psi\left(\frac{\Delta H_n}{\Delta t}\right)$$

- So if $w_n$ is wealth at time $n\Delta t$, have

$$w_n - w_{n-1} = H_{(n-1)\Delta t}(S_{t_n} - S_{t_{n-1}}) - S_{t_n} h_n (e^{\psi(h_n)} - 1)\Delta t,$$

where $h_n \equiv \Delta H_n / \Delta t$.

- Letting $\Delta t \downarrow 0$, we propose the dynamics

$$\begin{aligned} dH_t &= h_t dt \\ dw_t &= H_t dS_t - h_t f(\varepsilon h_t) S_t dt \end{aligned}$$

where $f(\varepsilon h) = \exp(\psi(h)) - 1$, and $\psi$ is increasing, with $\psi(0) = 0$.

## 3. The Merton problem.

- We have the wealth dynamics

$$dw_t = rw_t dt + H_t(dS_t - rS_t dt) - C_t dt - h_t f(\varepsilon h_t) S_t dt$$

with $dH_t = h_t dt$, $dS_t = S_t(\sigma dW_t + \mu dt)$. Objective is to find

$$V(w, H, S; \varepsilon) \equiv \sup E\left[\int_0^\infty e^{-\rho t} U(C_t) dt \middle| w_0 = w, H_0 = H, S_0 = S\right]$$

- Original (Merton) solution for $U(x) = x^{1-R}/(1-R)$:

$$\begin{aligned} V_M(w, H, S; 0) &= \gamma^{-R} U(w) \\ H_t &= \frac{\pi w_t}{S_t} \equiv \frac{\mu - r}{\sigma^2 R} . \frac{w_t}{S_t} \\ C_t &= \gamma w_t \equiv \frac{\rho + (R-1)(r + \frac{1}{2}\sigma^2 R \pi^2)}{R} . w_t \end{aligned}$$

- *Scaling*: for any $\lambda > 0$,

$$\begin{aligned} V(\lambda w, H, \lambda S; \varepsilon) &= \lambda^{1-R} V(w, H, S; \varepsilon) \\ &= (\lambda S)^{1-R} v(z, H; \varepsilon) \end{aligned}$$

where $z = w/S$, wealth in units of stock.

- *Another scaling !* for any $\lambda > 0$,

$$V(w, \lambda H \lambda^{-1} S; \varepsilon/\lambda) = V(w, H, S; \varepsilon)$$

- so we can scale out the dependence on $\varepsilon$, and take $\varepsilon = 1$.

Assume $0 < \pi < 1$

## 4. The HJB equation.

- After some calculations,

$$\sup_{c,h}\Big\{ U(c) - \tilde{\rho} v + \frac{1}{2}\sigma^2 (H-z)^2 v_{zz} - \alpha(z-H) v_z$$
$$-(hf(\varepsilon h)+c)v_z + h v_H \Big\} = 0.$$

$[\tilde{\rho} = R\gamma + \frac{1}{2}\sigma^2 R(1-R)(1-\pi)^2,\ \alpha = \sigma^2 R(\pi - 1).]$

Maximum over $c, h$:

$$\begin{aligned} c &= v_z^{-1/R}, \\ v_H &= v_z \bar{f}'(\varepsilon h) \qquad\qquad (\bar{f}(t) = t f(t)) \end{aligned}$$

leading to

$$0 = \tilde{U}(v_z) - \tilde{\rho} v + \frac{1}{2}\sigma^2 (H-z)^2 v_{zz} - \alpha(z-H) v_z + \frac{v_z}{\varepsilon}\Phi\left(\frac{v_H}{v_z}\right)$$

where $\Phi(a) \equiv \sup_t \{at - tf(t)\}$.

Heuristic: Consider

$$\inf E\left[\int_0^T \{(W_t - X_t)^2 + \varepsilon \dot{X}_t^2\}\, dt\right]$$

solved by $\dot{X}_t = \tanh((T-t)/\sqrt{\varepsilon})\Delta_t/\sqrt{\varepsilon}$, $\Delta_t \equiv W_t - X_t$, so optimally controlled $\Delta$ is like an OU process with variance approximately $2\sqrt{\varepsilon}$. Hence loss is $O(\sqrt{\varepsilon})$, and can look for an expansion in this scale ...

## 5. Pricing a European option.

- Suppose (wlog) $r = 0$. Bounded European option with expiry $T$ has Black-Scholes price $p(t, s)$ at time $t$ when (discounted) stock price is $s$:

$$p_t + \frac{1}{2}\sigma^2 s^2 p_{ss} = 0.$$

If $x_t$ is cash account at time $t$, then

$$dx_t = -s_t(h_t + h_t f(\varepsilon h_t)dt.$$

- *Utility-indifference pricing:* CARA agent seeks

$$V(t, s, H) \equiv \inf E\Big[e^{-\gamma(x_T - p(T, s_T))}\Big| s_t = s, H_t = H, x_t = 0\Big]$$

If there were no liquidity costs, then

$$\begin{aligned} V(t, s, H) &= \exp(\gamma p(t, s) - \gamma s H - \frac{\mu^2}{2\sigma^2}(T - t)) \\ &\equiv \exp(\gamma f_0(t, s, H)) \end{aligned}$$

say. Thus with liquidity costs, we expect

$$V(t, s, H) = \exp(\gamma f_0(t, s, H) + \gamma\varphi(t, s, H))$$

for some small function $\varphi$.

HJB equation again:

$$0 = \inf\Big[V_t + (\mu - r)sV_s + \frac{1}{2}\sigma^2 s^2 V_{ss} + \gamma s h(1 + f(\varepsilon h))V + hV_H\Big]$$

leading to

$$\varphi_t + \frac{1}{2}\sigma^2 s^2 \gamma \Big\{ p_s + \varphi_s - H + \frac{\mu - r}{\gamma\sigma^2 s} \Big\}^2 + \frac{1}{2}\sigma^2 s^2 \varphi_{ss}$$
$$= s\Phi\Big(\frac{-\varphi_H}{s}\Big) \geq 0$$

Once again, can look for expansion.

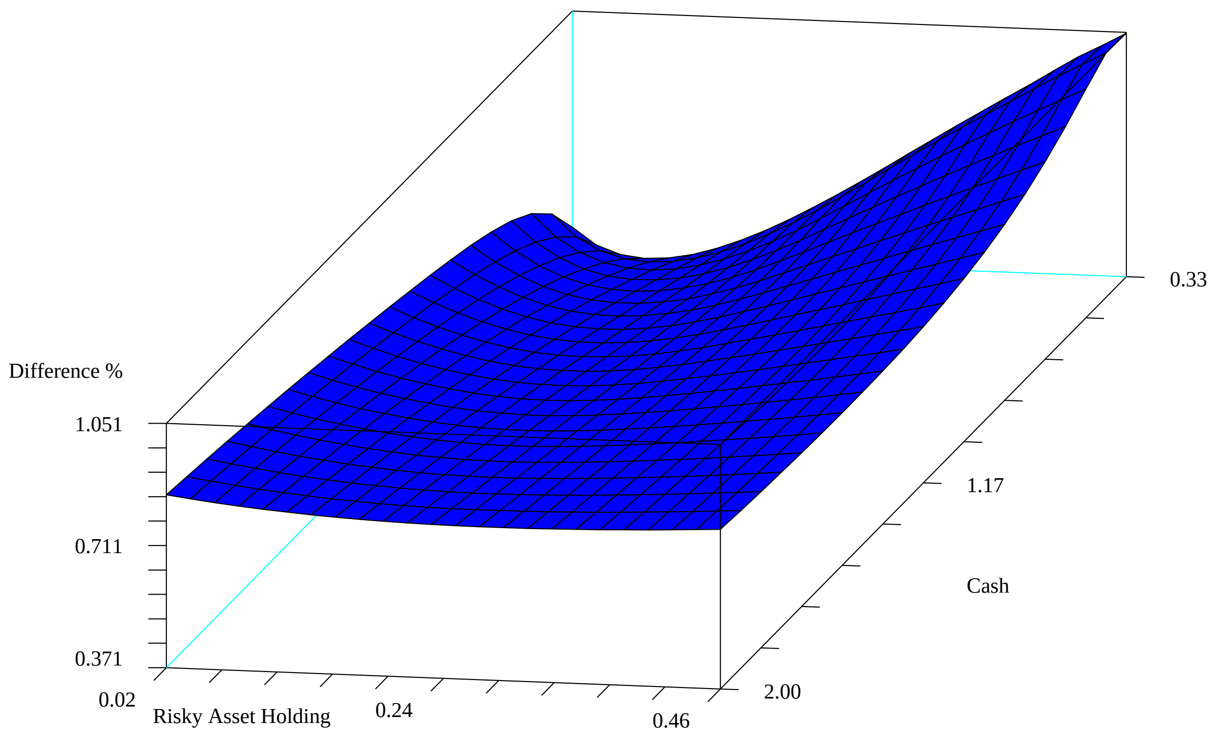


Figure 1: (Merton Soln-V)/Merton Soln n=21, m=24

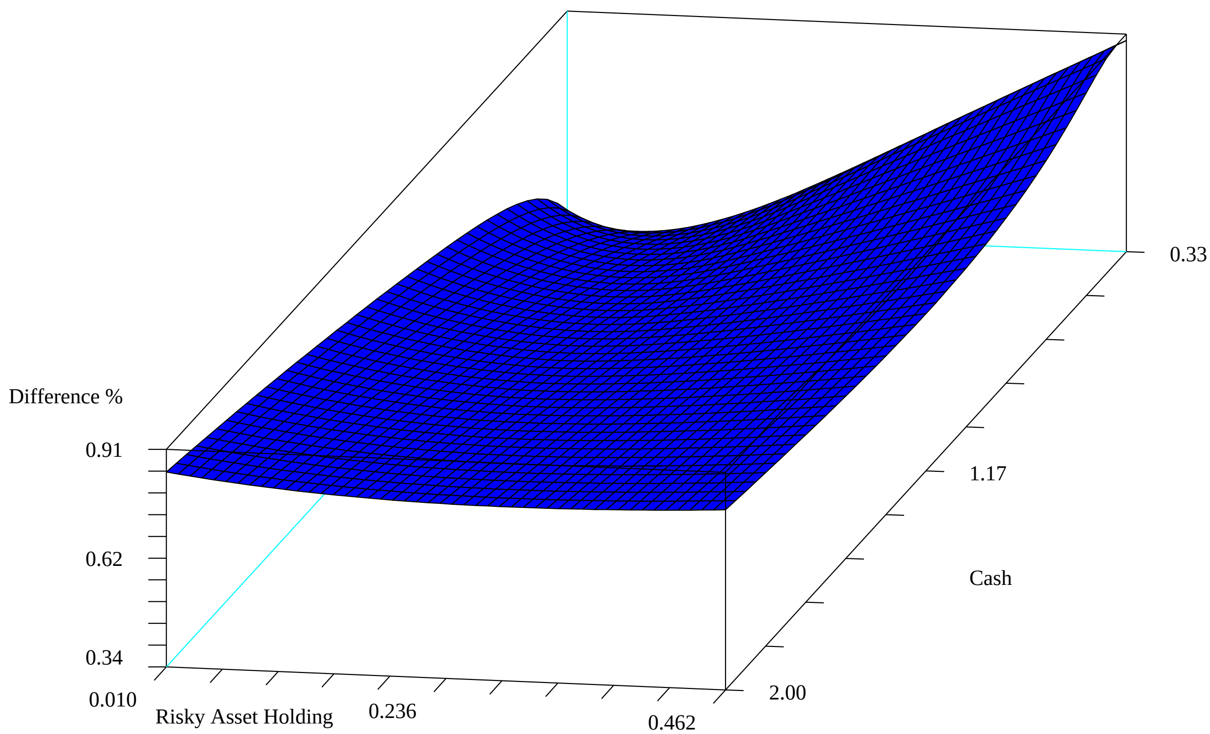


Figure 2: (Merton Soln-V)/Merton Soln n=41, m=48

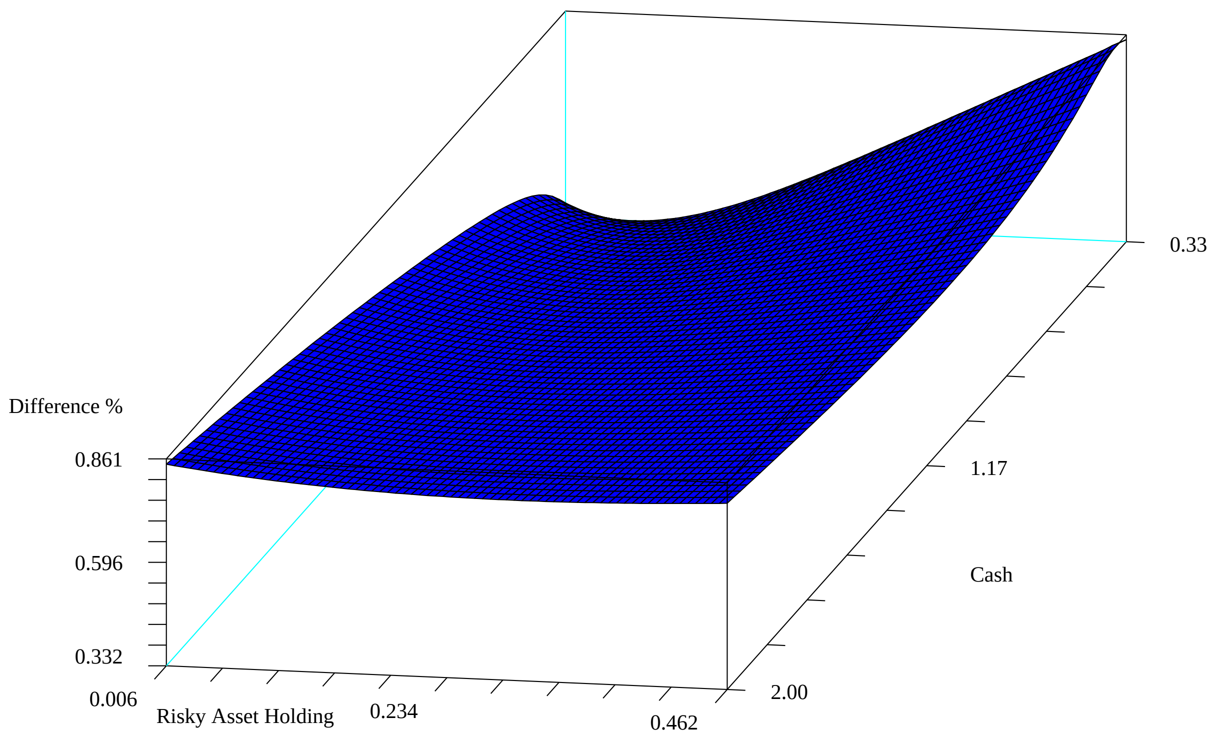


Figure 3: (Merton Soln-V)/Merton Soln n=61, m=72

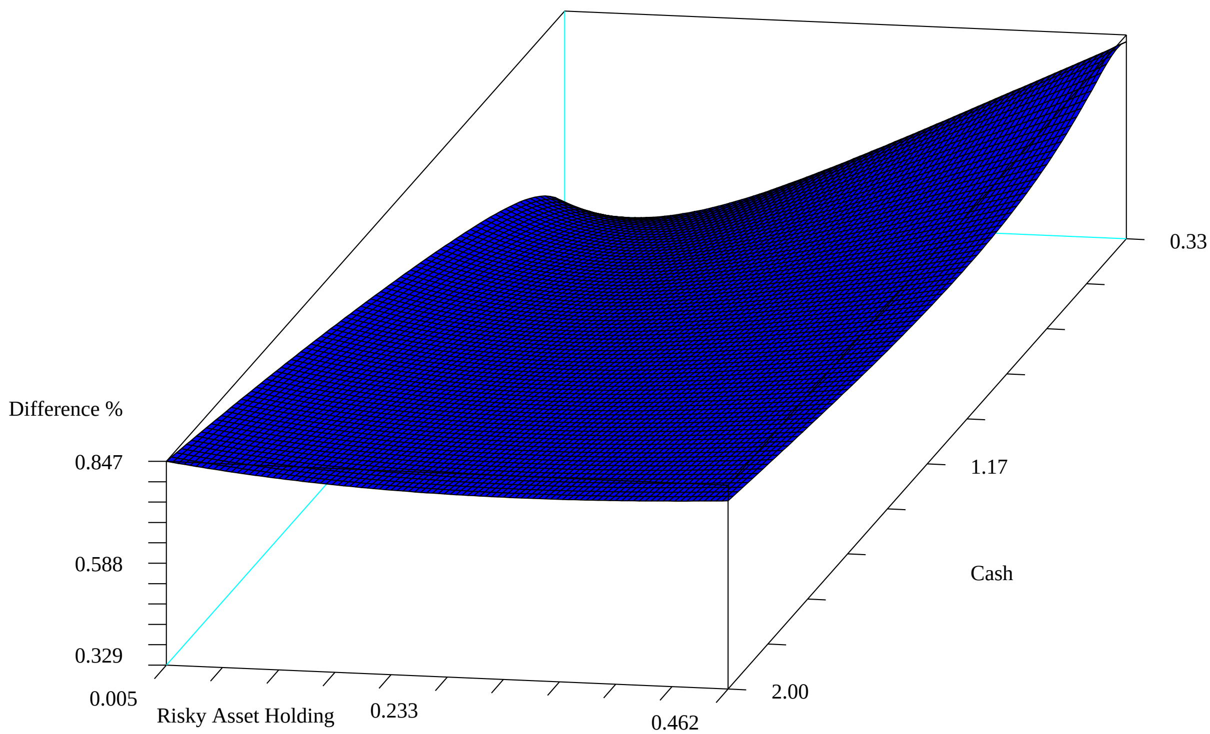


Figure 4: (Merton Soln-V)/Merton Soln n=81, m=96

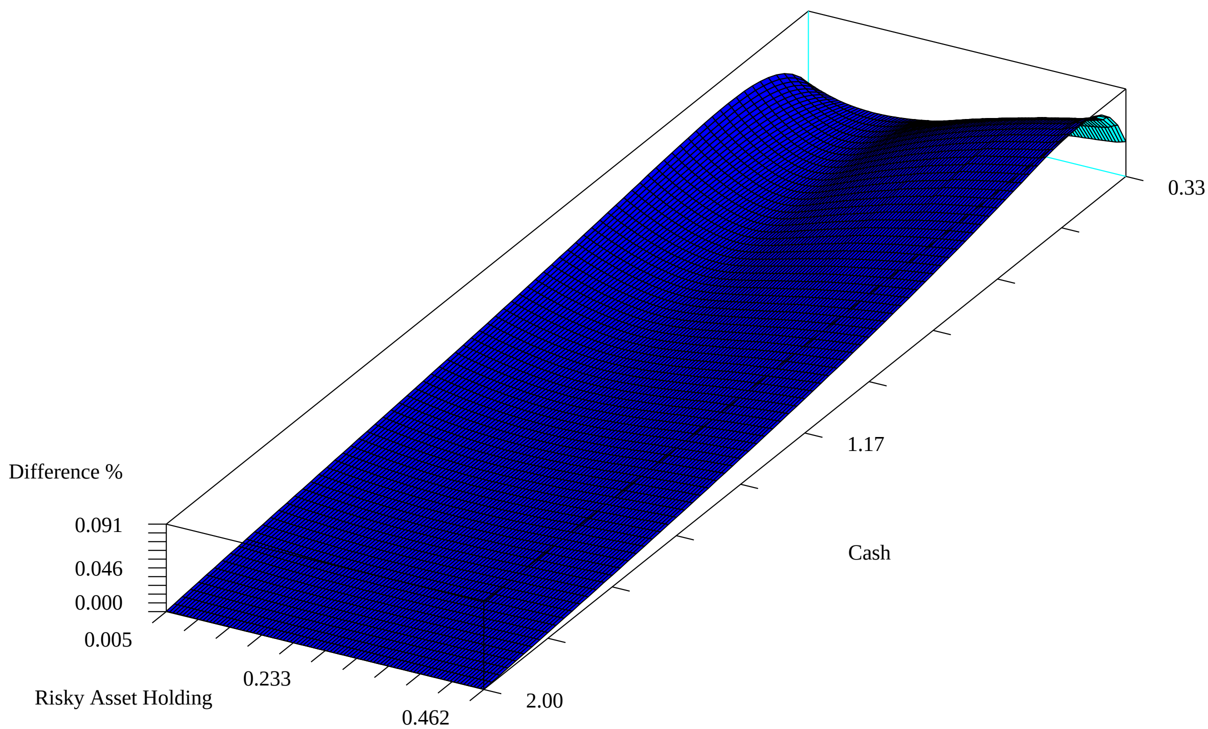


Figure 5: (Power Series-V)/Power Series Soln n=81, m=96