“Numerical Analysis of the Extended Black-Scholes Model”

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Overview of the talk

- Advantages and drawbacks of existing financial market models.
- Motivation for a new class of models specialized on multi-asset markets.
- Presentation of the multidimensional extension of the Black-Scholes model.
- Model calibration vs. statistical parameter estimation: which one to use ?
- Parameter estimation methods for the model using historical data.
- Option pricing techniques: multidimensional Euler schemes and Monte Carlo.
- Numerical results: single-asset vanilla options, implied volatility surfaces, European options on baskets, exotic multi-asset options.
- Practical advantages of pricing and hedging in this model compared to the standard multidimensional Black-Scholes model.
- Work in progress/future perspectives

Equity financial market models: theory and practice I

- Observations from the market:
  - "excess kurtosis" and "fat tails" of asset returns, volatility clustering effects,
  - "smile" and "skew" patterns of implied volatility surfaces,
  - multidimensional interdependency structure between assets prices and their volatilities, e.g. negative correlation between volatility and price.
- Arbitrage free single-asset models:
  - Black-Scholes '73 model, $dS_t = S_t(\mu dt + \sigma dW_t)$,
  - Extensions with jumps (e.g. Merton '76), $dS_t = S_{t-}(\mu dt + \sigma dW_t + \lambda dN_t)$,
  - Extensions with time dependent or local volatility process: $\sigma_t = \sigma(t, S_t)$, where $\sigma(\ ,\ )$ is a deterministic function (e.g. Dupire '94, Derman-Kani '94).
  - Stochastic volatility extensions: $(\sigma_t)_{t\geq 0}$ is an exogenous stochastic process, usually negatively correlated with the asset price process $(S_t)_{t\geq 0}$, e.g. $d\sigma_t = \alpha(t, \sigma_t)dt + \beta(t, \sigma_t)dW_t^\sigma$, where $d < W, W^\sigma >_t = \rho dt$, $\rho < 0$. $\rightarrow$Wiggins '87, Hull-White '87, Stein-Stein '91, Heston '93, Schöbel-Zhu '01,
  - Extensions with stochastic volatilities and jumps, e.g. Scott '97, Bakshi-Cao-Chen '97.

Equity financial market models: theory and practice II

- Arbitrage free single-asset models (pricing by Fourier/FFT methods):
  - Exponential Lévy models, i.e. the log-price $X_t = log(S_t)$ is modelled by a Lévy process with characteristics (μ, σ^2, ν), $\nu =$ Lévy measure of the jumps. Special cases: Brownian motion with drift for $\sigma^2 > 0$ and $\nu = 0$, compound Poisson process for $\sigma^2 = 0$ and $\nu(dx) = \lambda p(dx)$.

    $\rightarrow$parametrisation of the Lévy measure ν to obtain a good fit to market returns: hyperbolic model Eberlein-Keller '95 (hyperbolic distribution), Eberlein-Prause '98 (generalized hyperbolic), Barndorff-Nielsen '97 (normal inverse Gaussian), Madan-Carr-Chang *'98* (variance gamma), Schoutens '01 (Meixner), Carr-Geman-Madan-Yor '02 (CGMY, logistic distribution)
  - Time changed exponential Lévy models: $S_t = exp(L(T_t))$, where L is a Lévy process and T a stochastic time change, e.g. an integrated CIR process.
- Arbitrage free multi-asset models (pricing by multidimensional Monte Carlo):
  - Multidimensional Black-Scholes model,
  - Various d-dimensional versions of the mentioned single-asset models,
  - Extension of BS introduced by Albeverio and Steblovskaya in '02, FinStoch.

Presentation of the Extended Black-Scholes (EBS) model

• In the EBS model, the equity market under the historical probability measure $\mathbb{P}$ is modelled by the following SDE:

$$dS_t = \mathbf{A}_t(S_t)dt + \mathbf{B}_t(S_t)d\tilde{W}_t, \tag{0.1}$$

- where $S_t = (S_t^1, ..., S_t^n)^T$ is the n-dimensional asset price vector,
- $\mathbf{A}_t : \mathbb{R}^n \to \mathbb{R}^n$ is the asset price dependent drift vector,
- $\mathbf{B}_t : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is the asset price dependent volatility mapping and
- $\tilde{W}_t$ is a standard n-dimensional Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0}, \mathbb{P})$.

• Under some additional assumptions concerning the operators $\mathbf{A}_t$ and $\mathbf{B}_t$, $t \geq 0$, there exists a unique risk neutral measure $\mathbb{P}^*$ equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0})$ such that the dynamics of the n-dimensional stock price vector S_t under $\mathbb{P}^*$ is given by

$$dS_t = r_t S_t dt + \mathbf{B}_t(S_t)dW_t, \tag{0.2}$$

- where $(r_t)_{t\geq 0}$ is a given short interest rate process and
- W_t is a standard n-dimensional Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0}), \mathbb{P}^*)$.

The EBS model with constant linear coefficients

• In the general case of the EBS model, the volatility $\mathbf{B}_t$ is a time-dependent non-linear mapping of the stock price vector S_t. Since we are interested in numerical results in the EBS model, we assume that the general non-linear mapping $\mathbf{B}_t(\, . \,)$ is differentiable and replace it by its first order linear approximation (i.e. Fréchet derivative) at some representative level of the share price vector. Hence we assume for the numerical implementation from now on that

- the interest rate is constant and non-negative and that
- the volatility mapping $\mathbf{B}_t$ is time independent and given by the linear operator $\mathbf{B} : \mathbb{R}^n \to \mathbb{R}^{n \times n} \Rightarrow \mathbf{B}(S_t)$ is for every $S_t \in \mathbb{R}^n_+$ a real $n \times n$ matrix.

• The EBS model is viable and complete for an adequate choice of $\mathbf{B}$.

• Therefore, one can price derivatives in the EBS model using the standard method of taking the expectation of the discounted derivative payoff under the risk neutral probability measure $\mathbb{P}^*$.

Some characteristics of the EBS model

- The intuitive advantages of the EBS model compared to the multi-dimensional Black-Scholes model is the fact that asset prices are allowed to interact and can influence each other through the volatility operator $\mathbf{B}$.
- Consequently, the EBS model provides specific characteristics of share prices observed on the market ('fat tails' and excess kurtosis of returns, 'smile' and skew patterns of implied volatility, stochastic correlation between asset prices, etc) which do not appear in the classical multi-dimensional Black-Scholes model.
- The operator $\mathbf{B}$ captures the interdependency structure of the multi-dimensional share price process $(S_t)_{t\geq 0}$ and introduces stochastic volatilities in the model.
- The stochastic volatility processes of each single asset depends only on the other assets and is therefore an endogenous quantity of the model.
- There is no need to introduce an exogenous stochastic volatility process for each asset driven by a separate source of risk (like e.g. in the Heston model).
- The advantage of the EBS model compared to BS becomes evident in practice when considering stock prices which interact and influence each other, e.g. stock prices of companies belonging to the same market segment.

Model calibration vs. statistical parameter estimation

Calibration of parameters technique

- Idea: find model parameters $\theta^* \in \Theta$ so that option prices computed in the model using θ^* coincide with prices of options observed on the market.
- Numerical implementation uses multidimensional optimization techniques of some distance/utility function $U(\theta)$ over the parameter space.
- Distance/utility function U depends on the employed market data, which usually contains imperfections (bid-ask spread, etc).
- Used market data: prices of liquidly traded derivatives (e.g. European vanilla call and put options or the corresponding implied volatilities).
- Option prices should contain relevant information about the assets.
- Calibration is a standard example of an ill-posed inverse problem!
- Non-convexity and local extremes of U lead to instability of the algorithm.
- Complexity is exponential in the dimension of the parameter space Θ.
- Works very well in practice only for single-asset models (5 to 6 params.)!
- New information on the market $\Rightarrow$ re-calibration of the model is required.
- The model parameters change over time and have to be hedged (vega in BS)!

Statistical estimation of parameters I

• Idea: find a 'good' estimator $\hat{\theta}$ for the model parameter $\theta \in \Theta$ using historical data on asset prices, i.e. data under the historical probability measure $\mathbb{P}^{\theta}$.

• Since $\hat{\theta}$ is estimated under the historical measure $\mathbb{P}^{\theta}$ but the pricing of options is performed under the risk-neutral/martingale measure $\mathbb{P}^*$, we have to restrict our estimation only to quantities which remain invariant under an equivalent change of measure from $\mathbb{P}^{\theta}$ to $\mathbb{P}^*$!

• From Girsanov's theorem for semimartingales, we have an explicit formula for the change of the local semimartingale characteristics of our price process:

$$\begin{array}{ccc} (A, C, \mu) & \rightarrow & (\tilde{A}, C, \tilde{\mu}) \\ \mathbb{P}^{\theta} & \rightarrow & \mathbb{P}^*. \end{array}$$

• Only the Brownian covariation matrix C of the continuous martingale part of $(S_t)_{t \geq 0}$ remains unchanged, the drift A and the jump measure μ change.

• Hence, only the quadratic variation ('volatilities') and covariation ('correlation matrix') structure of $(S_t)_{t \geq 0}$ are invariant under a Girsanov transform!

Statistical estimation of parameters II

• As a consequence, the 'jump' parameters estimated using historical market data (i.e. jump intensity, distribution of the jumps, etc) are valid only under the historical measure $\mathbb{P}$ and cannot be used for pricing and hedging derivatives under the risk neutral measure $\mathbb{P}^*$!

• Therefore, models based on jumps (e.g. exponential Lévy models like GH, NIG, CGMY, Meixner, with 4 to 6 params.) have to be calibrated, since a statistical estimation of risk-free model parameters (i.e. under $\mathbb{P}^*$) is not possible.

• In the case of the EBS model, the parameters are given by the linear diffusion coefficient $\mathbf{B} \in \mathbb{R}^{n\times n\times n}$ $\Rightarrow$ a calibration of the model is impossible for $n \geq 4$.

• $\mathbf{B}$ can be estimated by statistical methods using historical market data.

• Disadvantages: $\mathbf{B}$ is high-dimensional since it contains the first-order dependence structure between assets and generates at the same time the endogenous stochastic volatility process for each of the assets !

• Advantages: $\mathbf{B}$ has to be estimated only once for a given set of assets.

• No pricing and hedging errors due to weekly re-calibration of the model !

• No calibration problems (convergence, bid-ask spread, liquidity of options).

Statistical estimation of diffusion coefficients I (Genon-Catalot and Jacod '93)

- A method for estimating the diffusion coefficient $a(\theta, t, X_t)$ depending on some parameter $\theta \in \Theta \subseteq \mathbb{R}^k$ of a multidimensional diffusion process

$$dX_t = b(t, X_t)dt + a(\theta, t, X_t)dW_t$$

observed at p discrete time points $\{t_0 < t_1 < .. < t_p\}$ is given.

- Let $\hat{\theta}_p$ be the contrast estimator minimizing over $\theta \in \Theta$ the contrast function

$$U(\theta) = \frac{1}{p}\sum_{i=1}^{p}[\log\det c(\theta, t_i) + \Delta X_i^T c(\theta, t_i)^{-1}\Delta X_i],$$

where $c(\theta, t_i) := a(\theta, t_i, X_{t_i})a(\theta, t_i, X_{t_i})^T$ is assumed to be invertible for all $i \in \{0, 1, .., p\}$ and $\Delta X_i := \frac{1}{\sqrt{t_i - t_{i-1}}}(X_{t_i} - X_{t_{i-1}}), i \in \{1, .., p\}$.

- Under some weak hypothesis, it was shown in [GeCa-Ja] that $\hat{\theta}_p$ is a 'good' estimator for θ in the sense that $\hat{\theta}_p \to \theta$ in P^θ measure as $p \to \infty$.

- Moreover the law of $\sqrt{p}(\hat{\theta}_p - \theta)$ under P^θ converges weakly to a mixed normal distribution as $p \to \infty$ (asymptotic mixed normality).

Statistical estimation of diffusion coefficients II (Genon-Catalot and Jacod '93)

• In the case of the EBS model, the parameter θ is given the linear volatility operator $\mathbf{B}$ and the coefficients of the multidimensional diffusion equation for the stock price vector S_t are given by $b(t, S_t) = rS_t$, $a(\mathbf{B}, t, S_t) = \mathbf{B}(S_t)$.

• Therefore, the contrast function is given in the case of the EBS model by

$$U(\mathbf{B}) = \frac{1}{p} \sum_{i=1}^{p} [\log\det(\mathbf{B}(S_{t_i})\mathbf{B}(S_{t_i})^T) + \Delta S_i^T (\mathbf{B}(S_{t_i})\mathbf{B}(S_{t_i})^T)^{-1} \Delta S_i],$$

where ΔS_i is defined for $i \in \{1, .., p\}$ by

$$\Delta S_i = \frac{1}{\sqrt{t_i - t_{i-1}}} (S_{t_i} - S_{t_{i-1}})$$

and we have to assume that $\mathbf{B}(S_{t_i})\mathbf{B}(S_{t_i})^T$ is invertible for all $i \in \{0, 1, .., p\}$.

• Therefore, given the set of historical share prices $\{S_{t_1}, .., S_{t_p}\}$ sampled at discrete time points $\{t_1, .., t_p\}$, the minimizer $\hat{\mathbf{B}}_p$ of the contrast function is a 'good' estimator for the diffusion coefficient $\mathbf{B}$ in the sense explained before.

• The method requires a numerical optimization in dimension $n \times n \times n$!

An empirical estimation method for the parameter $\mathbf{B}$ I

• Intuitive motivation: since the volatility matrix $\mathbf{B}(S_t)$ is dependent of the level of the share price vector S_t, we divide the historical data into time intervals $[T_k, T_{k+1}]$ corresponding to different levels of the asset price vector S_t.

• On each of these intervals $[T_k, T_{k+1}]$, the share prices should not vary "too much", so that we can approximate $\mathbf{B}(S_t)$ using the historical correlation matrix $\hat{C}_k$ and historical volatilities $(\hat{\sigma}_k^1, .., \hat{\sigma}_k^n)$ of the share price on $[T_k, T_{k+1}]$.

• The estimated Black-Scholes historical volatility matrix $\hat{\Sigma}_k$ on $[T_k, T_{k+1}]$ is given by $\hat{\Sigma}_k = Diag(\hat{\sigma}_k)\hat{C}_k^{\frac{1}{2}}$ (here we used a special form of $\hat{\Sigma}_k$).

• The estimated diffusion coefficient on $[T_k, T_{k+1}]$ is given by $Diag(S_{T_k})\hat{\Sigma}_k$ and is a 'good' estimator for $\mathbf{B}(S_{T_k})$, i.e. we set $\hat{\mathbf{B}}(S_{T_k}) = Diag(S_{T_k})\hat{\Sigma}_k$.

• Since $\hat{\mathbf{B}}$ is a linear operator, the latter equation is a linear system of $n \times n$ equations with $n \times n \times n$ unknowns given by $\hat{\mathbf{B}} = (\mathbf{b}_{i,j}^l)_{i,j,l \in \{1,...,n\}}$.

• To compute $\hat{\mathbf{B}}$ we need n systems of linear equations of this type, hence we choose n disjoint time intervals $[T_0, T_1]$, ..., $[T_{n-1}, T_n]$ to obtain a (non-singular) linear system with $n \times n \times n$ equations and unknowns characterizing $\hat{\mathbf{B}}$.

An empirical estimation method for the parameter B II

- By choosing suitable time intervals, one can ensure that the resulting system of linear equations has full rank and therefore a unique solution $\hat{\mathbf{B}}$.
- Hence, in order to estimate the diffusion coefficient $\hat{\mathbf{B}}$ using the method described in this section, one needs to solve an $n \times n \times n$-dimensional system of linear equations.
- For this purpose, the usually employed techniques are the Gauss elimination method for low dimensions or iterative methods for higher dimensions.
- As a consequence, the empirical parameter estimation method has a big computational advantage compared to the parameter estimation method of [GeCaJa], where an $n \times n \times n$-dimensional numerical optimization of the contrast function has to be performed.
- In practice, a good choice for the time intervals $[T_0, T_1], .., [T_{n-1}, T_n]$ used for the estimation of $\hat{\mathbf{B}}$ is a partition of a recent time interval (6 to 12 months) containing high frequency historical prices of the considered assets. The partition should be chosen such that asset price do not have major changes within an interval, otherwise the estimation technique becomes inaccurate.

Pricing and hedging of multi-asset derivatives I

- Vanilla options on the underlying S^i with time to maturity T and strike K:

$$Call = e^{-rT}\mathbb{E}^*[(S^i_T - K)^+],\ \ Put = e^{-rT}\mathbb{E}^*[(K - S^i_T)^+].$$

- European vanilla options on baskets with time to maturity T, basket weights $(\lambda_1, ..\lambda_n)$ and strike K:

$$BasketCall = e^{-rT}\mathbb{E}^*[(\sum_{i=1}^{n} \lambda^i S^i_T - K)^+],$$

$$BasketPut = e^{-rT}\mathbb{E}^*[(K - \sum_{i=1}^{n} \lambda^i S^i_T)^+].$$

- General multi-asset European option with time to maturity T and payoff function $f : \mathbb{R}^n \rightarrow \mathbb{R}^+$:

$$EuropeanOption(f) = e^{-rT}\mathbb{E}^*[f(S_T)].$$

Pricing and hedging of multi-asset derivatives II

• Barrier multi-asset European options with time to maturity T , payoff function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^+$ and boundary D (here τ_D denotes the first hitting time of the boundary D by the share price process $(S_t)_{t\geq 0}$)

$$KnockInOption = e^{-rT}\mathbb{E}^*[f(S_T)\mathbf{1}_{\{\tau_D \leq T\}}]$$

$$KnockOutOption = e^{-rT}\mathbb{E}^*[f(S_T)\mathbf{1}_{\{\tau_D \geq T\}}].$$

• Asian options with time to maturity T and payoff $f : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^+$:

$$AsianOption(f) = e^{-rT}\mathbb{E}^*[f(S_T, \int_0^T S_u du)]$$

• Lookback options with time to maturity T and payoff $f : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^+$:

$$LookbackOption(f) = e^{-rT}\mathbb{E}^*[f(S_T, \max_{0\leq u\leq T} S_u)]$$

• General case: $\mathbb{E}^*[\Phi(S_t, t \in [0,T])]$ for some suitable $\mathbb{P}^*$-integrable functional $\Phi : C[0,T] \longrightarrow \mathbb{R}$ depending on the path $(S_t)_{t\in[0,T]}$ of the asset price process.

Pricing and hedging of multi-asset derivatives III

• If one can compute prices of a derivative numerically, then it is also possible to compute its hedging parameters and therefore the hedging portfolio for that derivative (we are in the complete market setting of the EBS model).

• The hedging parameters can be evaluated using the difference quotient approximation technique, widely used for the calculation of the 'Greeks' of financial derivatives which do not have cosed formulas: one approximate the delta with respect to the i-th asset and the cross-gamma of a derivative by

$$\Delta_i(S_0) = \frac{\partial Option(S_0)}{\partial S^i} \simeq \frac{Option(S_0 + \epsilon S_0^i \mathbf{e}_i) - Option(S_0)}{\epsilon S_0^i},$$

$$\Gamma_{ij}(S_0) = \frac{\partial^2 Option(S_0)}{\partial S^i \partial S^j} = \frac{\partial \Delta_i(S_0)}{\partial S^j} \simeq \frac{\Delta_i(S_0 + \epsilon S_0^j \mathbf{e}_j) - \Delta_i(S_0)}{\epsilon S_0^j}$$

where ϵ is usually set to 1% and $\mathbf{e}_i$ denotes the i-th unit basis vector in $\mathbb{R}^n$.

Numerical methods: Euler schemes and Monte Carlo I

• Since closed formulas for the prices of options in the EBS model do not exist even in the simplest case of plain vanilla options, we use the Euler discretization scheme for SDEs and the Monte Carlo method to compute the expectations mentioned above.

• We generate discretized paths $\widetilde{S}^{(L)}$ sampled at L discrete time points $\{t_0 = 0, t_1, .., t_L = T\}$ using the standard n-dimensional Euler discretization scheme.

• $\widetilde{S}^{(L)}$ defined recursively by $\widetilde{S}^{(L)}_0 = S_0$ and

$$\widetilde{S}^{(L)}_{t_{j+1}} = \widetilde{S}^{(L)}_{t_j}[1 + r(t_{j+1} - t_j)] + \mathbf{B}(\widetilde{S}^{(L)}_{t_j})\sqrt{t_{j+1} - t_j}G_j,$$

where $\{G_j\}_{j=0}^{L-1}$ are independent n-dim. standard normal random variables.

• Known result on Euler schemes: the accuracy is of order $O(\frac{1}{\sqrt{L}})$, i.e.

$$\mathbb{E}^*[\Phi(\widetilde{S}^{(L)}_t, t \in [0, T])] = \mathbb{E}^*[\Phi(S_t, t \in [0, T])] + O(\frac{1}{\sqrt{L}}).$$

Numerical methods: Euler schemes and Monte Carlo II

• The Monte Carlo method consists in approximating the expectation

$$\mathbb{E}[\Phi(\widetilde{S}_t^{(L)}, t \in [0,T])]$$

by the sum

$$\frac{1}{M}\sum_{i=i}^{M} \Phi(\widetilde{S}^{(L)}(i)_t, t \in [0,T]),$$

where $\{(\widetilde{S}^{(L)}(1), \dots, \widetilde{S}^{(L)}(M)\} \subset C[0,T]$ is a sample of M independent realizations of discretized paths of financial asset prices using the Euler method.

• The strong law of large numbers implies the almost sure pointwise convergence of the Monte Carlo for $M \to \infty$ and the central limit theorem implies that the approximation error is of order $O(\frac{1}{\sqrt{M}})$. Therefore:

$$\mathbb{E}[\Phi(S_t, t \in [0,T])] = \frac{1}{M}\sum_{i=i}^{M} \Phi(\widetilde{S}^{(L)}(i)_t, t \in [0,T]) + O(\frac{1}{\sqrt{M}}) + O(\frac{1}{\sqrt{L}}).$$

Numerical results for the parameter estimation

• Although the numerical implementation of the model works for all dimensions, we restrict the presentation to dimension two since these results already reveal the main properties of the EBS model and are easy to present.

• The stock price dynamics is given in this case by

$$d\begin{pmatrix} S_t^1 \\ S_t^2 \end{pmatrix} = r\begin{pmatrix} S_t^1 \\ S_t^2 \end{pmatrix} dt + \mathbf{B^1}\begin{pmatrix} S_t^1 \\ S_t^2 \end{pmatrix} dW_t^1 + \mathbf{B^2}\begin{pmatrix} S_t^1 \\ S_t^2 \end{pmatrix} dW_t^2,$$

where $\mathbf{B^1} = \mathbf{B}(\mathbf{e_1})$ and $\mathbf{B^2} = \mathbf{B}(\mathbf{e_2})$ characterize the linear operator $\mathbf{B}$.

• The numerical estimation results for S^1 =IBM and S^2 =DELL using data from January to April 2002 for the two estimation techniques described before:

$$GeCa, Jacod: \quad \hat{\mathbf{B}}^1 = \begin{pmatrix} 0.3366 & 0.0550 \\ -0.0561 & -0.4712 \end{pmatrix}, \hat{\mathbf{B}}^2 = \begin{pmatrix} 0.1772 & 0.0829 \\ -0.3086 & 0.0431 \end{pmatrix}$$

$$Empirical: \quad \hat{\mathbf{B}}^1 = \begin{pmatrix} 0.2445 & 0.2774 \\ 0.4155 & -1.3032 \end{pmatrix}, \hat{\mathbf{B}}^2 = \begin{pmatrix} -1.3611 & 6.2997 \\ 0.4044 & -1.1423 \end{pmatrix}$$

Figure 0.1. European Call prices on IBM computed with parameters estimated by GeCa-Jacod and Empirical.

Figure 0.2. Implied volas on IBM computed with parameters estimated by GeCa-Jacod and Empirical.

FIGURE 0.3. Volatility surfaces for IBM computed in the EBS model and observed on the market.

Numerical results on option prices and implied volatilities

- The corresponding computed implied volatilities have a 'smile', are skew and have a term structure.
- The results are typical for implied volatility surface observed on equity markets, like the one computed from vanilla options on IBM in figure.
- The implied volatility surface generated by the model and the one observed on the market for the same date are similar (the absolute implied volatility difference remains uniformly under 4%), although we used a statistical parameter estimation based on historical data and not a calibration technique!
- Contrary to the multi-dimensional Black-Scholes model (where the implied volatility surface is flat), the EBS models provides the 'stylized facts' known from market observations.
- Since the EBS model leads to prices of vanilla type options which are very close to prices observed on the market, we can use this model to prices and hedge more complex derivatives, remaining at the same time consistent with prices of vanilla options on the market.

Figure 0.4. Knockout barrier call and put on IBM-DELL basket

Numerical results on barrier options on baskets

- The numerical example shows prices of knock-out barrier call and put option on a basket composed of 20 shares of IBM and 80 shares of DELL. On 02/28/02 the share of IBM and DELL were 103.8$ respectively 26.11$ so the spot price of the basket is 4164.8$ and about half of the capital is invested in IBM shares and the rest in DELL shares.
- The downside knock-out barriers are at 80$ for IBM and at 20$ for DELL.

Further numerical results on various assets

- In the next figures we present further numerical results obtained in the EBS model using
  - the pairs of assets Johnson&Johnson (JNJ) and Procter&Gamble (PG)
  - and the pair of shares indices DAX and FTSE.
- We used historical data from December 2002 to March 2003 for the estimation of parameters. The vanilla option prices and the implied volatilities are computed relative to 03/03/03 for different strikes and maturities. We included implied volatility surfaces observed on the market on 03/03/03 for comparison.

FIGURE 0.5. Implied volatility surfaces generated in the model and observed on the market for Johnson&Johnson on 03/03/03.

FIGURE 0.6. Implied volatility surfaces generated in the model and observed on the market for DAX on 03/03/03.

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Disadvantages of the multi-dimensional Black-Scholes model

• The major practical drawback of the multi-dimensional Black-Scholes model and of other multi-dimensional models introduced in recent years is that prices of derivatives computed in these models do not correspond to observations on the market.

• After the estimation or calibration of the model parameters, implied volatility surfaces (which are in one-to-one correspondence with prices of vanilla options for different strikes and maturities) generated in these models are either flat or have no 'smile', skew or term structure.

• This implies that for any choice of model parameters, these models are not consistent with information observed on the market.

• The multi-dimensional BS model cannot be used for an accurate pricing and hedging of complex derivatives, since it does not provide realistic results even in the case of plain vanilla options!

Is the EBS model an alternative to BS ?

- The EBS model captures very well the interdependency structure between financial assets in the model parameter $\mathbf{B}$ and provides therefore a realistic dynamics of the n-dimensional share price process.
- From the point of view of the practitioner, one of the main advantages of the EBS model compared to the BS model is the fact that prices of vanilla options in the EBS model are consistent with the ones observed on the market.
- Therefore we suggest to use the EBS model for pricing and hedging multi-asset derivatives as an alternative to the multi-dimensional Black-Scholes model.
- We observe that after the parameter estimation with historical data the non-diagonal elements of $\mathbf{B}^1$ and $\mathbf{B}^2$ are non-zero. The classical Black-Scholes model corresponds to the case when the non-diagonal elements are zero, i.e. the case where the level of other assets is not relevant for the dynamics of the considered share price. The existence of non-trivial non-diagonal elements in the estimated parameters is a sign that the share prices are highly dependent on the level of other assets, justifying *a posteriori* and from a numerical point of view the extension of the Black-Scholes model.