# Numerical Analysis of the Extended Black-Scholes Model

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**Abstract**

In this article some numerical results regarding the multidimensional extension of the Black-Scholes model introduced by Albeverio and Steblovskaya in [AlSt] are presented. The focus lies on aspects concerning the use of this model for the practice of financial derivatives. Two parameter estimation methods for the model using historical data from the market and an analysis of the corresponding numerical results are given. Practical advantages of pricing derivatives using this model compared to the original multidimensional Black-Scholes Model are pointed out. In particular the implied volatility surfaces computed in the [AlSt] model are close to those observed on the market.

## 1 Introduction

Over the last three decades the multidimensional Black-Scholes model (see e.g. [HaPl], [Ka]) has been used as a basic and very effective tool for the valuation of derivative instruments in financial markets with several assets. In the last years empirical observations from the market like "excess kurtosis" and "fat tails" of asset returns, "smile" and "skew" patterns of implied volatility surfaces, volatility clustering effects or the multidimensional interdependency structure between assets prices and their covariances hinted to the fact that the classical Black-Scholes framework is too restrictive for an accurate modelling of multidimensional financial markets. Heavy tailed marginals and long range dependence for stock price returns (contradicting the assumptions in the Black-Scholes model) have been observed in empirical studies even in the early papers by Fama [Fama] and Mandelbrot [Mand]. Many approaches have been developed to overcome these difficulties, e.g. by introducing market models with stochastic or level dependent volatility or by using jump-diffusion or Lévy extensions of the Black-Scholes model. Widely used extensions of the Black-Scholes model are e.g. the models introduced by Merton [Merton], Wiggins [Wiggins], Hull and White [HuWh], Dupire [Dupire], Stein and Stein [Stein], Heston [Heston], Scott [Scott], Madan, Carr and Chang [MaCaCh] or Carr, Geman, Madan and Yor [CGMY]. For a survey of stochastic

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volatility and jump diffusion models see e.g the articles by Frey [Frey] or Bakshi, Cao and Chen [BaCaCh] and the references therein. We remark that the models mentioned above are one dimensional extensions of the Black-Scholes model and do not take into consideration the multidimensional character of price processes in a multi-asset market model.

An extension of the Black-Scholes model focusing on the interdependency structure of several assets in the financial market was introduced by Albeverio and Steblovskaya in [AlSt] (we will abbreviate it as EBS - the Extended Black-Scholes Model). More financial and mathematical details concerning the EBS model, e.g. sufficient conditions for the existence of an equivalent martingale measure (absence of arbitrage) and completeness as well as methods for pricing contingent claims on several assets are given in [AlSt].

In the present paper we focus on aspects concerning the numerical analysis of the EBS model. Our primal goal consists in finding effective methods for the estimation of the model parameters and identifying the main characteristics of the model for the practical use in financial institutions, e.g. for pricing and hedging multi asset financial derivatives. To do so we will consider the multidimensional Black-Scholes model as a benchmark and investigate the advantages and drawbacks of the EBS model relative to it. From the point of view of a practitioner we are also interested in comparing the numerical results in the EBS model to market data to determine wether the model is consistent with the empirical observations from the market mentioned before (e.g "smile" and "skew" patterns of implied volatility surfaces).

The paper is organized as follows. In section 2 we give a short description of the EBS model. In section 3 we discuss the problem of parameter estimation and describe two estimation methods to solve it. In section 4 we present a numerical method for pricing and hedging various derivatives on several assets. The method is based on the Euler discretization scheme for multidimensional diffusions and on the Monte Carlo technique. Concrete numerical results concerning prices of derivatives, implied volatility surfaces and the interdependency structure of financial assets as well as a comparison with market data are presented. In section 5 we present some concluding remarks concerning the use of the EBS model for the practice of financial derivatives.

## 2 Description of the model

In the EBS model, the equity market under the historical probability measure $\mathbb{P}$ is modelled by the following SDE:

$$dS_t \;=\; \mathbf{A}_t(S_t)dt + \mathbf{B}_t(S_t)d\tilde{W}_t, \tag{1}$$

where

- $S_t = (S_t^1, ..., S_t^n)^T$ is the n-dimensional asset price vector,
- $\mathbf{A}_t : \mathbb{R}^n \to \mathbb{R}^n$ is the asset price dependent drift vector,
- $\mathbf{B}_t : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is the asset price dependent volatility operator, and
- $\tilde{W}_t$ is a standard n-dimensional Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0}, \mathbb{P})$.

It was shown in [AlSt] that under some additional assumptions concerning the operators $\mathbf{A}_t$ and $\mathbf{B}_t$, $t \geq 0$, there exists a unique risk neutral measure $\mathbb{P}^*$ equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0})$ such that the dynamics of the n-dimensional stock price vector $S_t$ under $\mathbb{P}^*$ is given by

$$dS_t \;=\; r_t S_t dt + \mathbf{B}_t(S_t)dW_t, \tag{2}$$

where

- $(r_t)_{t\geq 0}$ is a given short interest rate process,
- $W_t$ is a standard n-dimensional Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0}), \mathbb{P}^*)$.

Since we are interested in numerical results in this model we assume from now on that the interest rate is constant and non-negative and that the volatility operator $\mathbf{B}_t$ is time independent and given by the linear operator $\mathbf{B} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, i.e. for any $x \in \mathbb{R}^n$, $\mathbf{B}(x)$ is a real $n \times n$ matrix with respect to some orthonormal base of $\mathbb{R}^n$.

We will assume that the conditions of Proposition 1 of [AlSt] guaranteeing the completeness of the EBS model are fulfilled, and therefore one can price derivatives in this model using the standard method of taking the expectation of the discounted derivative under the risk neutral probability measure $\mathbf{P}^*$.

**Remark.** One of the sufficient conditions for the completeness of the EBS model can be formulated in the considered case as follows: for each $k = 1, ..., n$, the vectors $\mathbf{B}(e_k)e_j$ $(j = 1, ..., n)$ are linearly independent.

There are in general no closed formulas for prices of derivatives in this model, hence the derivatives prices have to be evaluated using numerical techniques. Before starting with the numerical computation of options, we first have to address the problem of parameter estimation in this model.

# 3 Parameter estimation

In order to use the EBS model in practice, e.g. to compute concrete prices of derivatives and their hedging parameters, one first needs to estimate the model parameters using historical data available from the market. In the case of the EBS model we have to estimate the linear diffusion coefficient **B** in some given orthonormal base. The estimation of the drift coefficient is not required for the pricing derivatives since in a complete market setting we can use the risk-neutral dynamics (2) and the corresponding arbitrage free evaluation techniques. In this section we will first apply the parameter estimation technique described in [Ge-CaJa] to the EBS model. The method is time consuming and can be used only for small values of n. Then we present an empirical method for estimating the diffusion coefficient **B** which has the advantage that it reduces to solving a linear system of equations and can therefore be applied also for higher dimensions.

## 3.1 A method of Genon-Catalot and Jacod

### 3.1.1 Description of the method

In [Ge-CaJa], a method for estimating the diffusion coefficient $a(\theta, t, X_t)$ depending on some parameter $\theta \in \Theta \subseteq \mathbb{R}^k$ of a multidimensional diffusion process

$$dX_t = b(t, X_t)dt + a(\theta, t, X_t)dW_t \tag{3}$$

observed at p discrete time points $\{t_0 < t_1 < .. < t_p\}$ is described. We denote by $\hat{\theta}_p$ the estimator minimizing over all $\theta \in \Theta$ the following contrast function

$$U(\theta) = \frac{1}{p}\sum_{i=1}^{p}[\log\det c(\theta, t_i) + \Delta X_i^T c(\theta, t_i)^{-1}\Delta X_i], \tag{4}$$

where

$$c(\theta, t_i) = a(\theta, t_i, X_{t_i})a(\theta, t_i, X_{t_i})^T \tag{5}$$

is assumed to be invertible for all $i \in \{0, 1, .., p\}$ and

$$\Delta X_i = \frac{1}{\sqrt{t_i - t_{i-1}}}(X_{t_i} - X_{t_{i-1}}), i \in \{1, .., p\}. \tag{6}$$

Under some additional hypothesis, it was shown in [Ge-CaJa] that $\hat{\theta}_p$ is a good estimator for $\theta$ in the sense that that $\hat{\theta}_p \to \theta$ in $P^\theta$ measure as $p \to \infty$. Moreover the law of $\sqrt{p}(\hat{\theta}_p - \theta)$ under $P^\theta$ converges weakly to a mixed normal distribution as $p \to \infty$.

In the case of the EBS model the parameter $\theta$ can be identified with the linear volatility operator $\mathbf{B}$ and the coefficients of the multidimensional diffusion equation for the stock price vector $S_t$ are given by

$$b(t, S_t) = rS_t,$$

$$a(\mathbf{B}, t, S_t) = \mathbf{B}(S_t).$$

Hence the contrast function is given in the case of the EBS model by

$$U(\mathbf{B}) = \frac{1}{p}\sum_{i=1}^{p}[\log\det(\mathbf{B}(S_{t_i})\mathbf{B}(S_{t_i})^T) + \Delta S_i^T(\mathbf{B}(S_{t_i})\mathbf{B}(S_{t_i})^T)^{-1}\Delta S_i], \tag{7}$$

where we have to assume that $\mathbf{B}(S_{t_i})\mathbf{B}(S_{t_i})^T$ is invertible for all $i \in \{0, 1, .., p\}$ and $\Delta S_i$ is defined by

$$\Delta S_i = \frac{1}{\sqrt{t_i - t_{i-1}}}(S_{t_i} - S_{t_{i-1}}), i \in \{1, .., p\}. \tag{8}$$

Therefore, given the set of historical share prices $\{S_{t_1}, .., S_{t_p}\}$ sampled at the time points $\{t_1, .., t_p\}$, the minimizer $\hat{\mathbf{B}}_p$ of the contrast function (7) is a 'good' estimator for the diffusion coefficient $\mathbf{B}$ in the sense explained before.

#### 3.1.2 A Note on the Numerical implementation

In order to estimate the diffusion coefficient $\hat{\mathbf{B}}_p$ using the method described in [Ge-CaJa], we need to find the minimum over all $\hat{\mathbf{B}}_p \in \mathbb{R}^{n\times n\times n}$ of the contrast function $U(\mathbf{B})$ given by (7). Since in this case closed expressions for the minimum of expression (7) are not available, it has to be computed by standard multidimensional numerical optimization methods like e.g. the gradient method. We remark that we have to deal here with an optimization problem in dimension $n \times n \times n$, so it is very time consuming to solve it in practice for high values of $n$.

### 3.2 An empirical method of estimation of a diffusion coefficient in the linear case

#### 3.2.1 Description of the method

The intuitive motivation of the method presented in this section is the observation that on historical time intervals where the share prices do not vary "too much" (e.g. on small time intervals) the stochastic process that we consider (the solution to the linear SDE with multiplicative noise (2)) can be approximated by the multidimensional geometric Brownian motion, i.e. by the standard multidimensional Black-Scholes model.

To explain the idea in more details, we assume that at time T we have historical prices from the market on a recent time interval $[0, T]$. We fix a short (in the best case infinitesimal) time interval $[T_k, T_{k+1}] \subset [0, T]$,$T_k < T_{k+1}$, such that the historical share prices $(S_s)_{s\in[T_k,T_{k+1}]}$ have only minor changes around the average stock price $\overline{S}_k = \frac{1}{T_{k+1}-T_k}\int_{T_k}^{T_{k+1}} S_s ds$ on that time interval. We can estimate the Black-Scholes historical mean volatility matrix $\hat{\Sigma}_k$ on $[T_k, T_{k+1}]$ using the following standard technique:

- We estimate the historical Black-Scholes volatilities $\hat{\sigma}_k^i$ for each of the stock prices $S^i$ on the time interval $[T_k, T_{k+1}]$, $i = 1, ...n$.
- We estimate the historical correlation matrix $\hat{C}_k = (\hat{\rho}_k^{i,j})$, i.e. the correlation coefficients $\hat{\rho}_k^{i,j}$ between $S^i$ and $S^j$ on the same time interval.
- The multidimensional Black-Scholes mean volatility matrix for the time interval $[T_k, T_{k+1}]$ is then given by[4] $\hat{\Sigma}_k = Diag(\hat{\sigma}_k)\hat{C}_k^{\frac{1}{2}}$.

Once the Black-Scholes historical volatility has been estimated, according to the statistical theory of diffusion processes a good estimation of the diffusion coefficient of the stock prices for the time interval $[T_k, T_{k+1}]$ is given by $Diag(\overline{S}_k)\hat{\Sigma}_k$.

Therefore we have the following corresponding approximation for the dynamics of the share prices vector $S = (S^1, ..., S^n)$ on the time interval $[T_k, T_{k+1}]$:

$$dS_t \simeq rS_t dt + Diag(\overline{S}_k)\hat{\Sigma}_k dW_t \tag{9}$$

On the other hand, by using the relations between diffusion coefficients and solutions of SDEs, one can see that the solution of the stock price equation (2) for the EBS model is approximated on the interval $[T_k, T_{k+1}]$ by the solution of the equation

$$dS'_t = rS'_t dt + \mathbf{B}(\overline{S}_k)dW_t \tag{10}$$

Thus we can look at $\mathbf{B}(\overline{S}_k)$ as an approximation of the diffusion coefficient of the stock prices process (2) on $[T_k, T_{k+1}]$.

[4]By Diag(x) we denote the diagonal matrix having $(x_1, x_2, ..., x_n)$ on the diagonal, by $\hat{C}_k^{\frac{1}{2}}$ we denote the positive self-adjoint square root of $\hat{C}_k$.

A comparison of the estimated diffusion coefficients using the two methods leads to an empirical estimator $\hat{\mathbf{B}}$ of the linear volatility operator in the EBS model, satisfying the condition:

$$\hat{\mathbf{B}}(\overline{S}_k) = Diag(\overline{S}_k)\hat{\Sigma}_k. \tag{11}$$

Since $\hat{\mathbf{B}}$ is a linear operator, equation (11) is a $n \times n$ dimensional system of linear equations having $n \times n \times n$ unknowns given by $\hat{\mathbf{B}} = (\mathbf{b}^l_{i,j})_{i,j,l \in \{1,\ldots,n\}}$.

Hence, in order to compute $\hat{\mathbf{B}}$ we need $n$ systems of linear equations of type (11), which leads to the following method. We choose $n$ disjoint representative time intervals $[T_0, T_1], \ldots,$ $[T_{n-1}, T_n]$ from the historical time interval $[0, T]$ such that the share prices do not have major changes (like 'crashes' or other 'extremal events') on these intervals. For each interval $[T_k, T_{k+1}]$ we compute the historical volatility matrix $\hat{\Sigma}_k$ and the average prices $\overline{S}_k$ to obtain $n$ equations of type (11) involving $\hat{\mathbf{B}}$. The linear system of equations for $\hat{\mathbf{B}}$

$$\hat{\mathbf{B}}(\overline{S}_k) = Diag(\overline{S}_k)\hat{\Sigma}_k, k = 0, \ldots n-1 \tag{12}$$

obtained in this way has $n \times n \times n$ equations and the same number of unknowns. In order to solve this system of equations we have to assume that it has full rank. This can generically be achieved by choosing suitable time intervals $[T_k, T_{k+1}]$, $k \in \{0, .., n-1\}$ when building up the system of equations (12). If this linear system has full rank, it can be solved using standard numerical techniques to obtain an estimator $\hat{\mathbf{B}}$ for the linear volatility operator $\mathbf{B}$ of equation (2). The method has been tested with market data and the corresponding results are presented in the next section.

**Remark**: In the multi-dimensional Black-Scholes model, although one obtains a unique estimator $\hat{C}$ for the covariance matrix of the stock prices, the volatility matrix $\hat{\Sigma}$ is not uniquely determined since any matrix $\hat{\Sigma} \in \mathbb{R}^{n \times n}$ satisfying $\hat{\Sigma}\hat{\Sigma}^T = \hat{C}$ can be taken as a Black-Scholes volatility matrix. Depending on the purpose, one has to specify a special form for $\hat{\Sigma}$ (usually one takes the positive self-adjoint or the Cholesky square root of $\hat{C}$) to ensure the uniqueness of $\hat{\Sigma}$. We would like to point out that in the EBS model the situation is similar, hence we choose a special form for the volatility operator $\hat{\mathbf{B}}$ which is implicitly given by equation (11). Since the estimators $\hat{\mathbf{B}}$ obtained using the method of [Ge-CaJa] described in the last section and using the method of this section do not have in general the same special form, we do not expect them to be equal when using the same historical data. This is not a real inconvenience in practice, since both methods approximate the same asset price dynamics given by the SDE (2).

### 3.2.2 A Note on the Numerical Implementation

In order to estimate the diffusion coefficient $\hat{\mathbf{B}}$ using the method described in this section, we only need to solve a $n \times n \times n$-dimensional system of linear equations. For this purpose one can use standard techniques, e.g. the Gauss elimination method for low dimensions or iterative methods for higher dimensions. As a consequence, the empirical parameter estimation method described in section 3.1 has a big computational advantage compared to the parameter estimation method described in section 3.2, when one had to perform an $n \times n \times n$-dimensional optimization of the contrast function to obtain the estimator $\hat{\mathbf{B}}$.

In practice, a good choice for the time intervals $[T_0, T_1], .., [T_{n-1}, T_n]$ used for the estimation of $\hat{\mathbf{B}}$ is a partition of a recent time interval containing high frequency historical prices of the considered assets. The partition should be chosen such that asset price do not have major changes within an interval.

# 4 Numerical results

In this section we discuss the numerical implementation of the EBS model as well as the pricing of various derivatives on single assets and on baskets, e.g. European vanilla options, European barrier calls and puts, Asian and lookback options. We analyze the main characteristics of the model in practice as well as its advantages compared with the classical multi-dimensional Black-Scholes model.

## 4.1 Pricing Options using the Monte Carlo method

It is well known that working in a complete market under the risk neutral measure $\mathbb{P}^*$, the price of an option can be computed as the expectation of the discounted payoff of that options, see [AlSt] for more details. In the following we will give some examples of price formulas of widely used financial derivatives, under the assumption that we are situated at time 0 and that the interest rate is constant end equal to $r$:

1. Plain vanilla options on the underlying $S^i$ with time to maturity T and strike K:

$$Call = e^{-rT}\mathbb{E}[(S^i_T - K)^+], Put = e^{-rT}\mathbb{E}[(K - S^i_T)^+] \tag{13}$$

2. European call and put options on baskets of assets with time to maturity T, basket weights $(\lambda_1, ..\lambda_n)$ and strike K:

$$BasketCall = e^{-rT}\mathbb{E}[(\sum_{i=1}^{n} \lambda^i S^i_T - K)^+], BasketPut = e^{-rT}\mathbb{E}[(K - \sum_{i=1}^{n} \lambda^i S^i_T)^+] \tag{14}$$

3. General multi-asset European option with time to maturity T and payoff function $f : \mathbb{R}^n \to \mathbb{R}^+$:

$$EuropeanOption(f) = e^{-rT}\mathbb{E}[f(S_T)] \tag{15}$$

4. Barrier multi-asset European options with time to maturity T , payoff function $f : \mathbb{R}^n \to \mathbb{R}^+$ and boundary D (here $\tau_D$ denotes the first hitting time of the boundary D by the share price process $(S_t)_{t\geq 0}$).

$$\begin{aligned} KnockInOption &= e^{-rT}\mathbb{E}[f(S_T)\mathbf{1}_{\{\tau_D \leq T\}}] \\ KnockOutOption &= e^{-rT}\mathbb{E}[f(S_T)\mathbf{1}_{\{\tau_D \geq T\}}] \end{aligned} \tag{16}$$

5. Asian options with time to maturity T and payoff function $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^+$:

$$AsianOption(f) = e^{-rT}\mathbb{E}[f(S_T, \int_0^T S_u du)] \tag{17}$$

6. Lookback options with time to maturity T and payoff function $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^+$:

$$LookbackOption(f) = e^{-rT}\mathbb{E}[f(S_T, \max_{0\leq u\leq T} S_u)] \tag{18}$$

**Remark**: The expectations in formulas (13) to (18) can be written as $\mathbb{E}[\Phi(S_t, t \in [0,T])]$ for some suitable $\mathbb{P}^*$-integrable functional $\Phi : C[0,T] \to \mathbb{R}$ depending on the path $(S_t)_{t\in[0,T]}$ of the asset price process.

Since closed formulas for the prices of options in the EBS model do not exist even in the simplest case of plain vanilla options, we use the Euler discretization scheme for SDEs and the Monte Carlo method to compute the expectations in the formulas (13) to (18).

Continuous paths of stochastic processes cannot be generated on a digital computer, therefore one has to use discretized paths $\widetilde{S}^{(L)}$ sampled at discrete time points $\{t_0 = 0, t_1, .., t_L = T\}$. For simplicity we assume that the sample time points divide the time interval $[0,T]$ into $L$ equidistant subintervals. To generate one path we use the standard $n$-dimensional Euler discretization scheme for the SDE (2) to obtain $\widetilde{S}^{(L)}$ defined recursively by

$$\begin{aligned} \widetilde{S}_0^{(L)} &= S_0 \\ \widetilde{S}_{t_{j+1}}^{(L)} &= \widetilde{S}_{t_j}^{(L)}[1 + r(t_{j+1} - t_j)] + \mathbf{B}(\widetilde{S}_{t_j}^{(L)})\sqrt{t_{j+1} - t_j}G_j, \end{aligned} \tag{19}$$

where $\{G_j\}_{j=0}^{L-1}$ is a sequence of independent $n$-dimensional standard Gaussian random variables. Since $\widetilde{S}^{(L)}$ is defined by the equation (19) only at discrete time points we use the linear interpolation method to define $\widetilde{S}^{(L)}$ between two consecutive discretization points $t_j$ and $t_{j+1}$ by

$$\widetilde{S}_t^{(L)} = \frac{t_{j+1} - t}{t_{j+1} - t_j} \widetilde{S}_{t_j}^{(L)} + \frac{t - t_j}{t_{j+1} - t_j} \widetilde{S}_{t_{j+1}}^{(L)} \tag{20}$$

to obtain continuous sample paths for $\widetilde{S}^{(L)}$. Since $\widetilde{S}^{(L)} \in C[0,T]$, the path dependent payoff functional $\Phi : C[0,T] \to \mathbb{R}$ can now be applied to $\widetilde{S}^{(L)}$.

A well known result on Euler schemes (see e.g. [KlPl]) states that in the special case when the functional $\Phi$ corresponds to general European, barrier, Asian and lookback options the convergence speed and hence the accuracy of the Euler method is of order $O(\frac{1}{\sqrt{L}})$ i.e.

$$\mathbb{E}[\Phi(\widetilde{S}_t^{(L)}, t \in [0,T])] = \mathbb{E}[\Phi(S_t, t \in [0,T])] + O(\frac{1}{\sqrt{L}}) \tag{21}$$

The Monte Carlo method for pricing financial derivatives consists in approximating the expectation

$$\mathbb{E}[\Phi(\widetilde{S}_t^{(L)}, t \in [0,T])] \tag{22}$$

by the sum

$$\frac{1}{M} \sum_{i=i}^{M} \Phi(\widetilde{S}^{(L)}(i)_t, t \in [0,T]), \tag{23}$$

where $\{(\widetilde{S}^{(L)}(1), ..., \widetilde{S}^{(L)}(M)\} \subset C[0,T]$ is a sample of $m$ independent realizations of discretized paths of financial asset prices using the Euler method described in (19). The strong law of large numbers implies the pointwise convergence of (23) to (22) for $M \to \infty$. We also know due to the central limit theorem that the approximation error is of order $O(\frac{1}{\sqrt{M}})$. Therefore we have (for large $M$) the following relation:

$$\frac{1}{M} \sum_{i=i}^{M} \Phi(\widetilde{S}^{(L)}(i)_t, t \in [0,T]) = \mathbb{E}[\Phi(\widetilde{S}_t^{(L)}, t \in [0,T])] + O(\frac{1}{\sqrt{M}}). \tag{24}$$

Putting together equations (21) and (24) we obtain

$$\frac{1}{M} \sum_{i=i}^{M} \Phi(\widetilde{S}^{(L)}(i)_t, t \in [0,T]) = \mathbb{E}[\Phi(S_t, t \in [0,T])] + O(\frac{1}{\sqrt{M}}) + O(\frac{1}{\sqrt{L}}). \tag{25}$$

Consequently, using this numerical method we can approximate the price of financial derivatives of types (13) to (18) up to a discretization error of order $O(\frac{1}{\sqrt{L}})$ and a Monte Carlo error of order $O(\frac{1}{\sqrt{M}})$, where $L$ is the number of discretization points of the considered time interval and $M$ is the number of paths generated using the Monte Carlo method.

Knowing the price of an option enables us to compute its hedging parameters and therefore to hedge the option (we are in the complete market setting of [AlSt]). The hedging parameters can be evaluated using the difference quotient approximation technique, widely used for the calculation of the 'Greeks' of financial derivatives, i.e. one approximate the delta with respect to the i -th asset and the cross-gamma of an option by

$$\Delta_i(S_0) = \frac{\partial Option(S_0)}{\partial S^i} \simeq \frac{Option(S_0 + \epsilon S_0^i \mathbf{e}_i) - Option(S_0)}{\epsilon S_0^i}, \tag{26}$$

$$\Gamma_{ij}(S_0) = \frac{\partial^2 Option(S_0)}{\partial S^i \partial S^j} = \frac{\partial \Delta_i(S_0)}{\partial S^j} \simeq \frac{\Delta_i(S_0 + \epsilon S_0^j \mathbf{e}_j) - \Delta_i(S_0)}{\epsilon S_0^j} \tag{27}$$

where $\epsilon$ is usually set to 1% and $\mathbf{e}_i$ denotes the i-th unit basis vector in $\mathbb{R}^n$.

## 4.2 Estimated volatility operator $\hat{\mathbf{B}}$ for different assets

In order to analyze the properties of the model, we will deal for simplicity with the two dimensional version of the model. The stock price dynamics is given in this case by

$$d\begin{pmatrix} S_t^1 \\ S_t^2 \end{pmatrix} = r\begin{pmatrix} S_t^1 \\ S_t^2 \end{pmatrix} dt + \mathbf{B^1}\begin{pmatrix} S_t^1 \\ S_t^2 \end{pmatrix} dW_t^1 + \mathbf{B^2}\begin{pmatrix} S_t^1 \\ S_t^2 \end{pmatrix} dW_t^2, \tag{28}$$

where $\mathbf{B^1} = \mathbf{B}(\mathbf{e_1})$ and $\mathbf{B^2} = \mathbf{B}(\mathbf{e_2})$ are $2 \times 2$ real valued matrices defining the volatility operator $\mathbf{B} \in \mathbb{R}^{2\times2\times2}$ employed in [AlSt] equation (4). Although the numerical implementation of the model works for all dimensions we restrict our analysis to dimension two since these results reveal the most important characteristics of the model and are easy to present.

The intuitive advantages of the EBS model introduced in [AlSt] compared to the multi-dimensional Black-Scholes model is the fact that asset prices are allowed to interact and can influence each other through the volatility operator $\mathbf{B}$. This advantage becomes evident in practice when considering stock prices which interact and influence each other, e.g. stock prices of companies which belong to the same market segment. In the following we first present some representative numerical results for the pair of assets IBM and DELL, which should reveal the characteristics of the EBS model. For the estimation we used daily historical prices of the companies IBM and DELL from January to April 2002.

Using the contrast estimator method described is section 2 we obtained the following operator (given by two matrices $\mathbf{B}^1$ and $\mathbf{B}^2$ of equation (28)) for the two dimensional price process of IBM and DELL:

$$\hat{\mathbf{B}}^1 = \begin{pmatrix} 0.3366 & 0.0550 \\ -0.0561 & -0.4712 \end{pmatrix}, \tag{29}$$

$$\hat{\mathbf{B}}^2 = \begin{pmatrix} 0.1772 & 0.0829 \\ -0.3086 & 0.0431 \end{pmatrix}.$$

Using the empirical estimation method described is section 3 we obtained the following operator (given by two matrices $\hat{\mathbf{B}}^1$ and $\hat{\mathbf{B}}^2$ of equation (28)) for the two dimensional price process of IBM and DELL:

$$\hat{\mathbf{B}}^1 = \begin{pmatrix} 0.2445 & 0.2774 \\ 0.4155 & -1.3032 \end{pmatrix}, \tag{30}$$

$$\hat{\mathbf{B}}^2 = \begin{pmatrix} -1.3611 & 6.2997 \\ 0.4044 & -1.1423 \end{pmatrix}.$$

As pointed out at the end of section 3, we do not expect the two estimators to be equal, although we expect them to describe equivalent price dynamics for the pair IBM and DELL. This can be verified by comparing implied volatilities (or equivalently prices of European vanilla options) on both underlyings for different strikes and different maturities $t \in [0, T]$given in the next section, since it is known that implied volatility surfaces and distributions of prices of the underlying $S_t$ at different maturities $t \in [0, T]$ are in one-to-one correspondence. This will be done next.

### 4.3 Option Prices and Implied Volatility Surfaces

For the numerical computation of prices of derivatives we used the multi-dimensional Monte Carlo method described in the last section with discretization parameter $L = 100$ and number of Monte Carlo paths $M = 10^6$. It is easy to check (the coefficients being constant) that the sufficient conditions guaranteeing the model completeness described in [AlSt] are satisfied in the considered case.

We start with some results on IBM. Figure (1) shows prices of European call options on IBM computed in the EBS model for different strikes and maturities using the estimators (29) and (30). The corresponding computed implied volatilities can be found in figure (2).

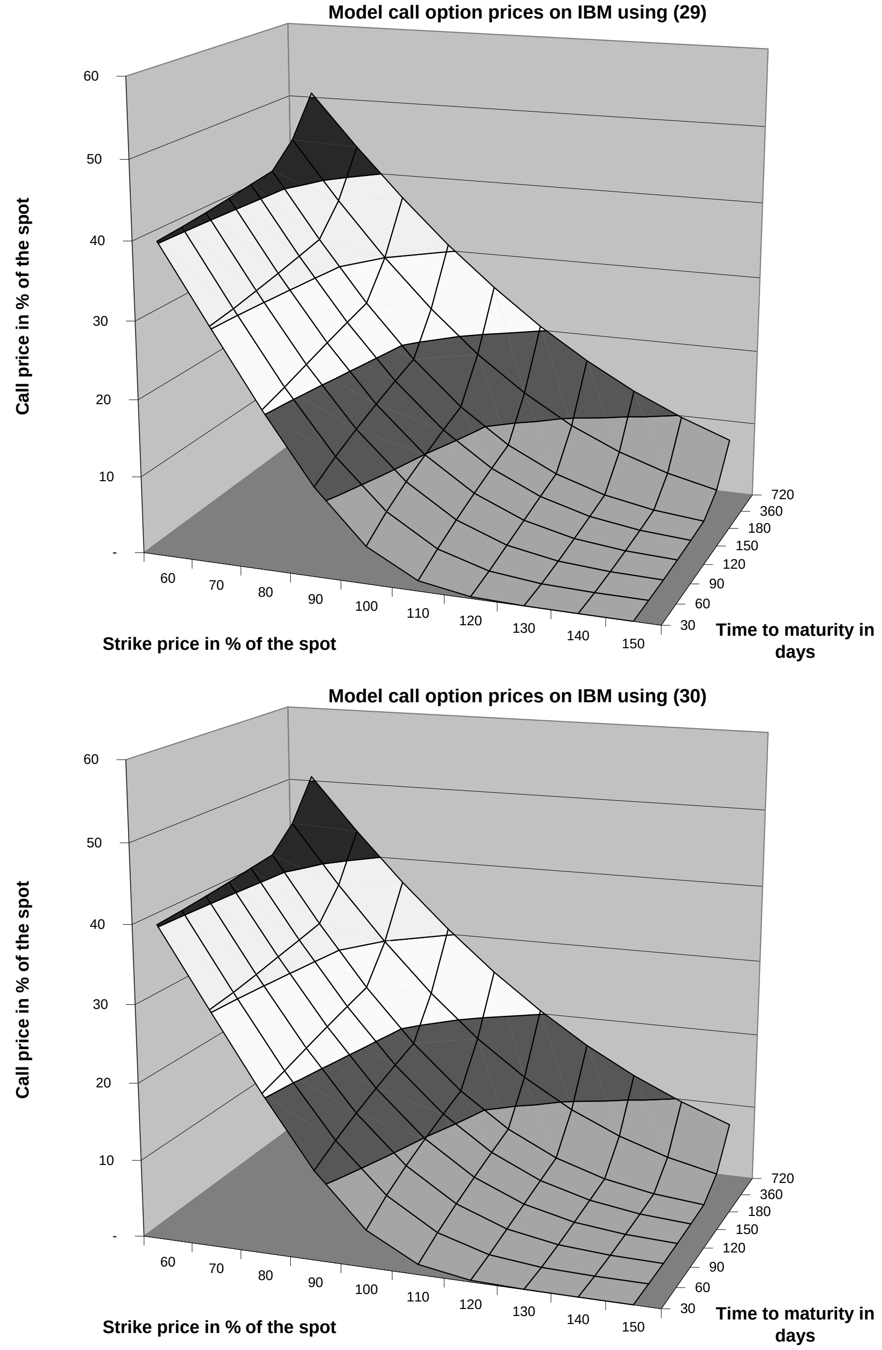



Figure 1: European Call prices on IBM computed with model parameters (29) and (30).

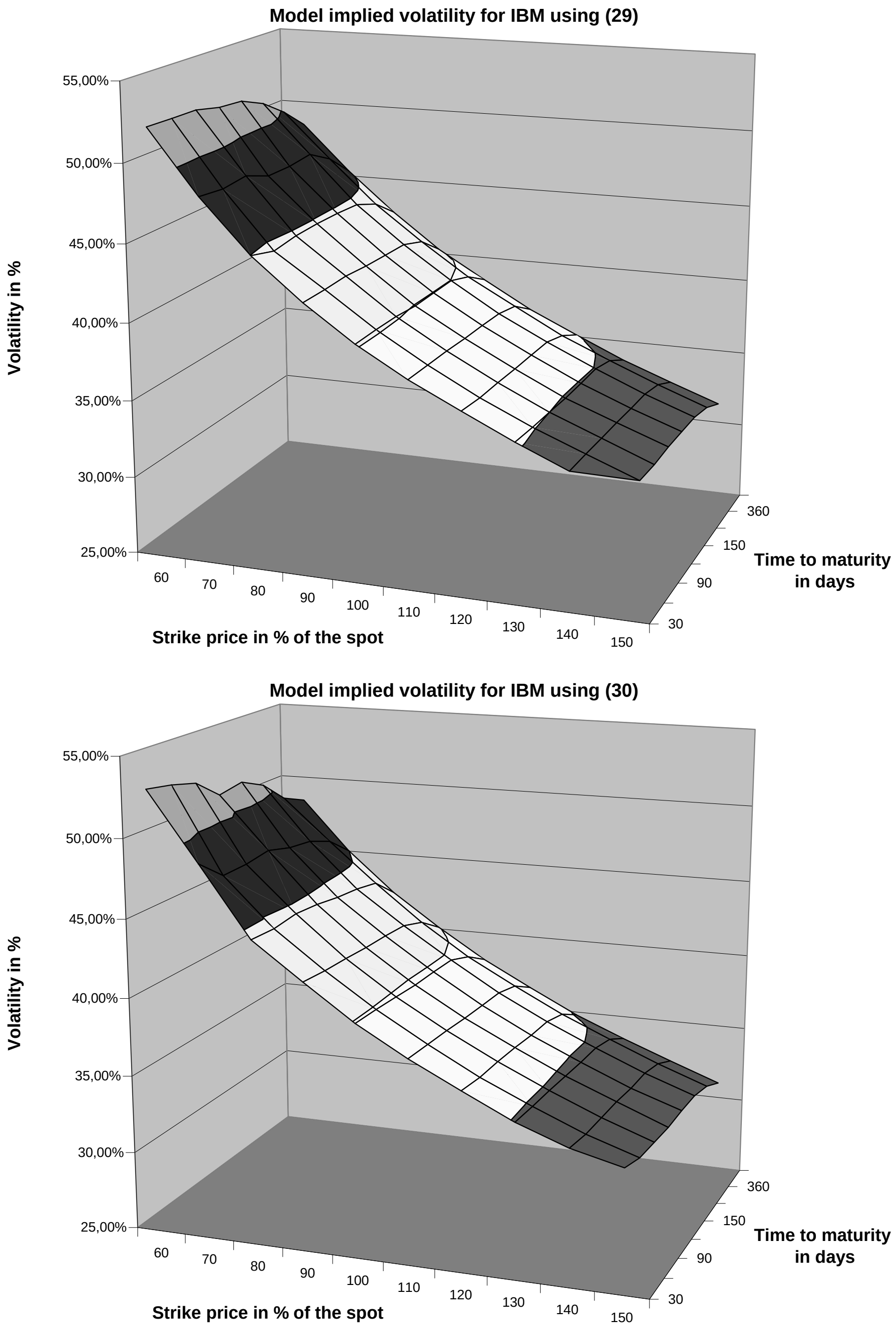



Figure 2: Computed implied volatility surfaces for IBM with model parameters (29) and (30).

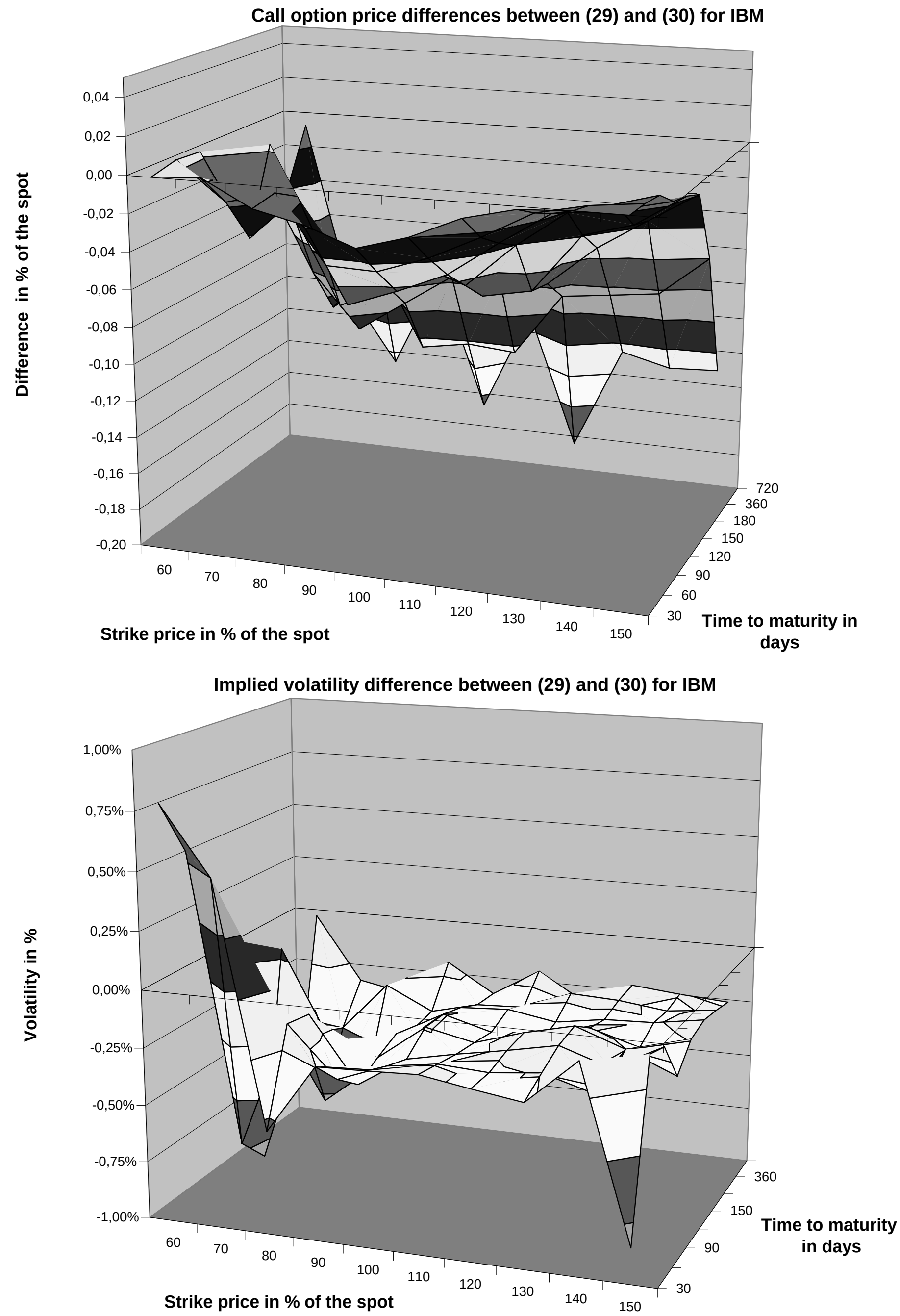



Figure 3: Difference between European call prices and implied volatilities for IBM computed using the model parameters (29) and (30).

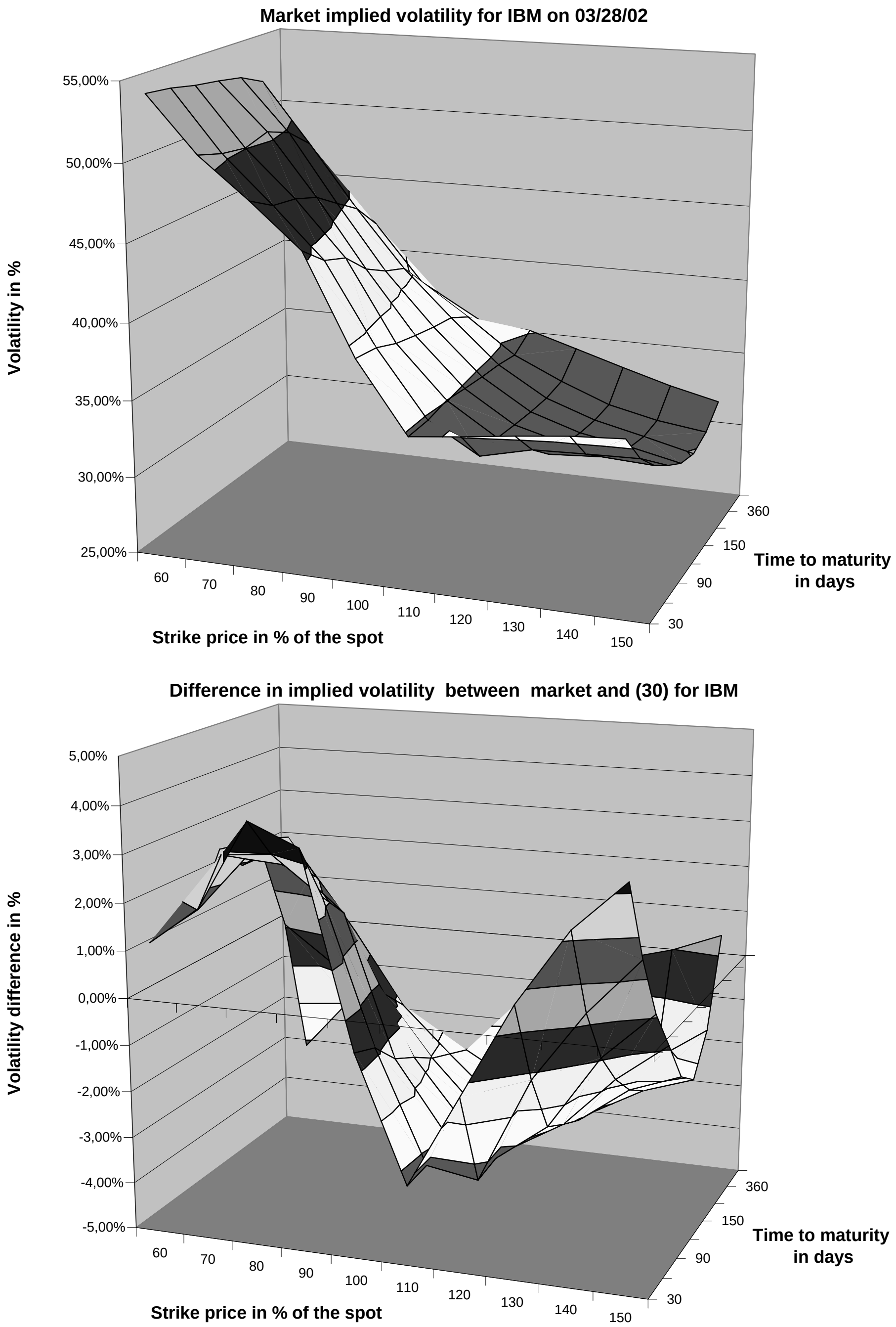



Figure 4: Implied volatility surfaces for IBM observed on the market on 03/28/02 and the difference between market and implied volatilities for IBM computed using (30)

Figure (3) shows the difference between option prices and implied volatilities. The implied volatility surfaces shown in figure (2) have a 'smile', are skew and have a term structure. The results are typical for implied volatility surface observed on equity markets, like the one shown in figure (4). The difference between the model generated implied volatility surface and the one observed on the market on the same date can also be found in (4).

In figure (4) one can also observe that the implied volatility surface computed in the EBS model and the one observed on the market on 03/28/02 are very similar (the absolute implied volatility difference remains uniformly under 4%) , although we used a parameter estimation based on historical data prior to that date. This can be seen as an empirical evidence that the EBS model fits very well the market. To enable the comparison with the multi-dimensional Black-Scholes model, we remark at this point that the implied volatility surface in the latter model is flat, therefore the so called 'stylised facts' curresponding to real world financial markets (e.g. volatility 'smile' and term structure) do not appear at all in the Black-Scholes model.

Since the model leads to prices of vanilla type options which are very close to prices observed on the market, we can use the model to compute prices of more complex derivatives which are consistent with the market. Therefore we can hedge complex derivatives not only using the underlyings and the bank account but also other simple (e.g. vanilla type) options as a trading asset.

In the following we present some numerical results concerning prices of more complex options on IBM and DELL. The first example is a knock-out barrier call option on a basket composed of 20 shares of IBM and 80 shares of DELL. On 02/28/02 the share of IBM and DELL were 103.8$ respectively 26.11$ so the spot price of the basket is 4164.8$ and about half of the capital is invested in IBM shares and the rest in DELL shares. The downside knock-out barriers are at 80$ for IBM and at 20$ for DELL. The numerical results are shown in figure (5).

The second example in figure (6) shows prices of knock-out barrier put option on the same basket for different maturities and strikes, where the downside knock-out barriers are the same as in the previous case.

In the next figures we present further numerical results obtained in the EBS model using the pairs of assets Johnson&Johnson and Procter&Gamble and the pair of shares indices DAX and FTSE. We used historical data from December 2002 to March 2003 for the estimation of parameters. The vanilla option prices and the implied volatilities are computed relative to 03/03/03 for different strikes and maturities. We included implied volatility surfaces observed on the market on 03/03/03 for comparison.

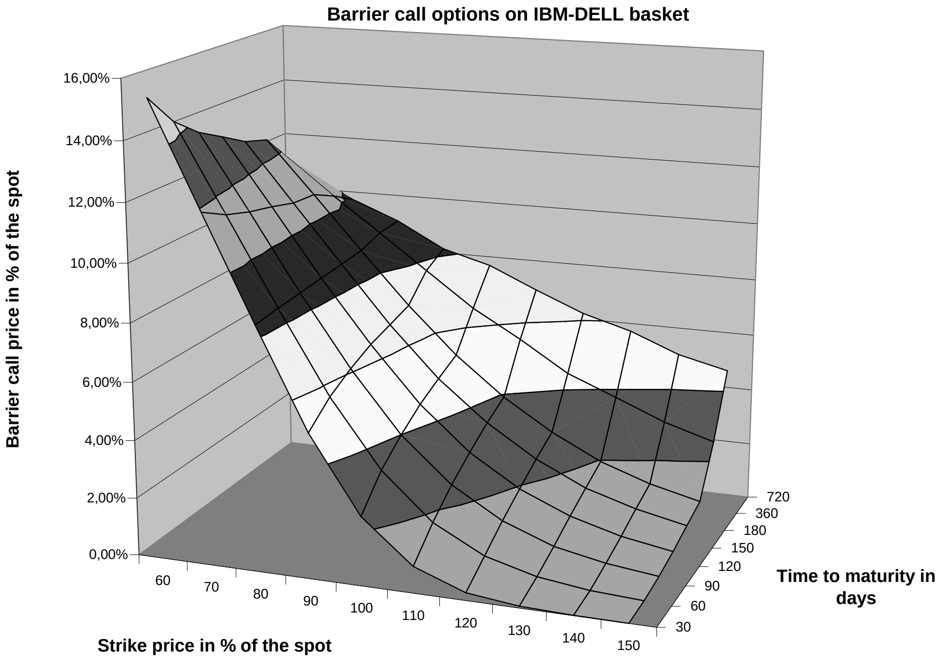



Figure 5: Knockout barrier call on IBM-DELL basket

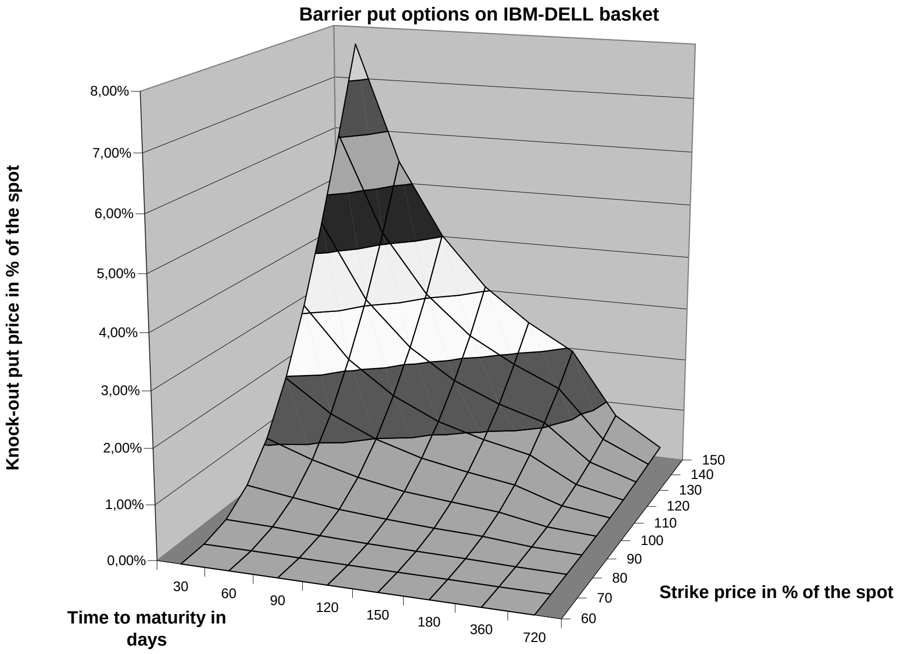
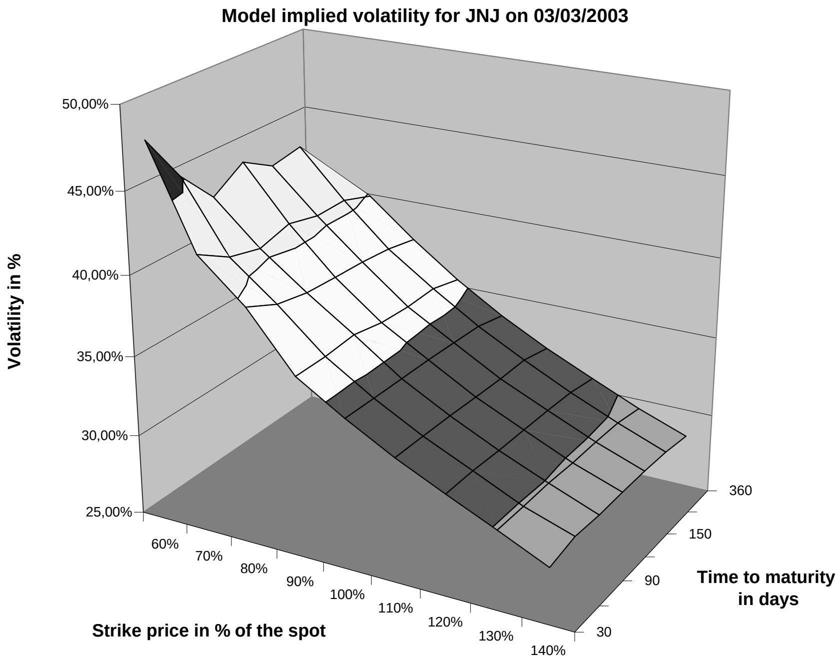



Figure 6: Knockout barrier put on IBM-DELL basket



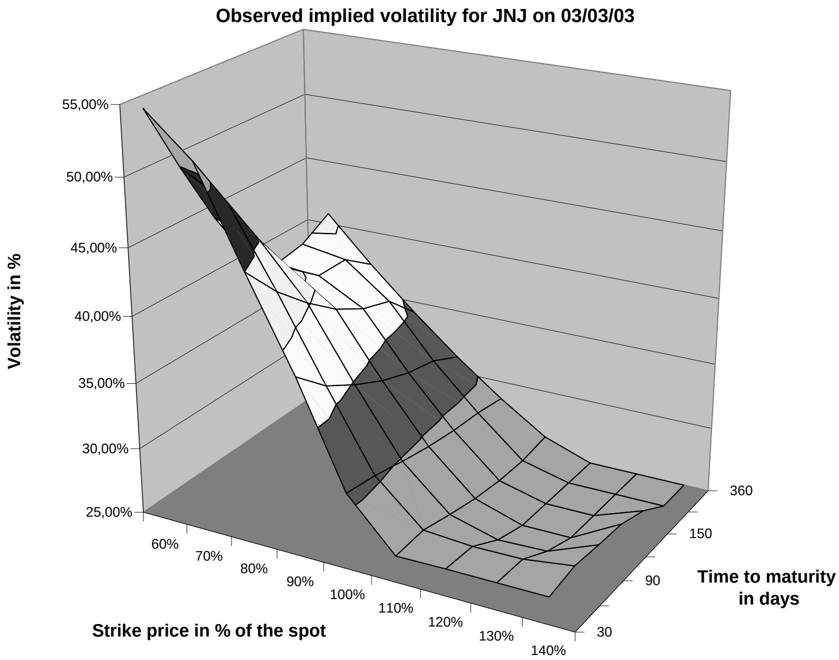




Figure 7: Implied volatility surfaces generated in the model (together with Procter&Gamble) and observed on the market for Johnson&Johnson on 03/03/03

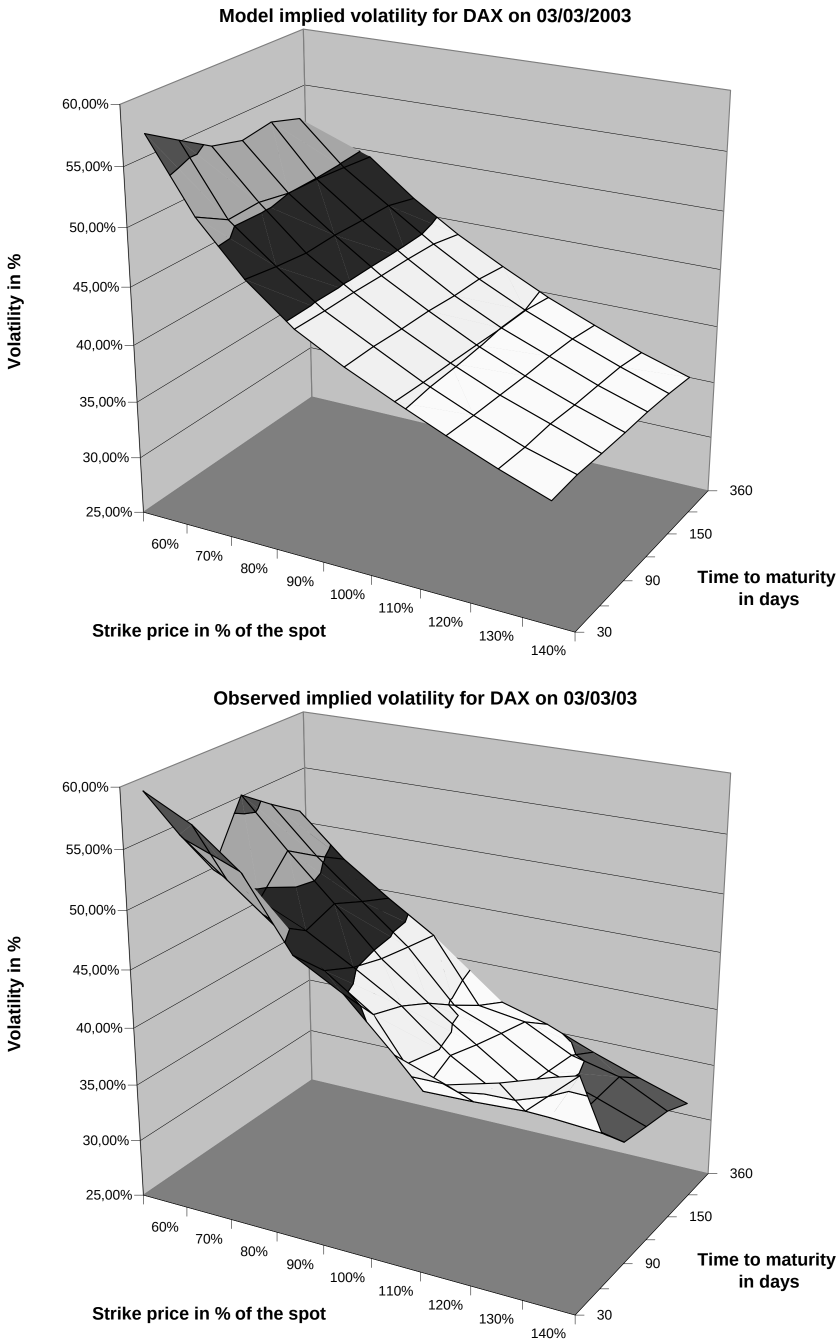
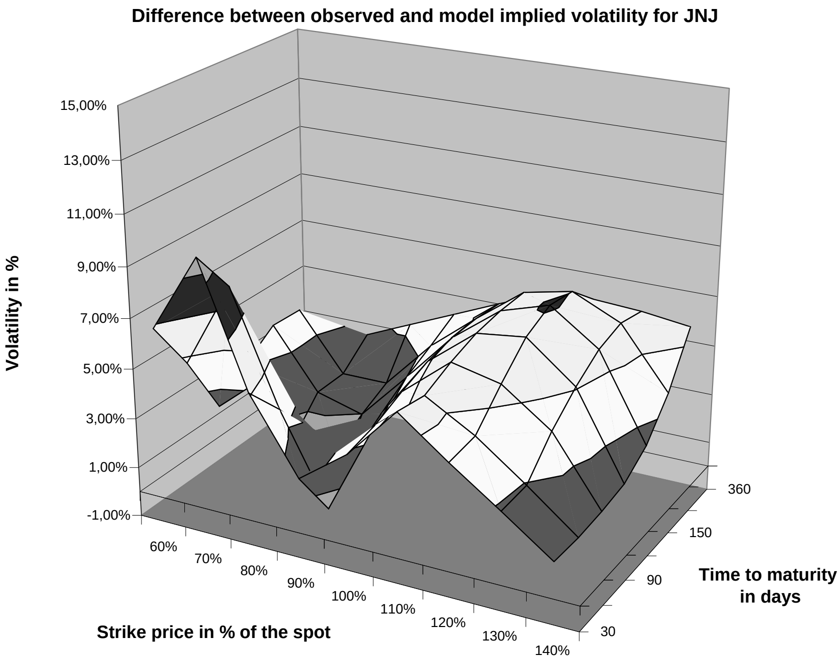



Figure 8: Implied volatility surfaces generated in the model (together with the FTSE index) and observed on the market for DAX on 03/03/03



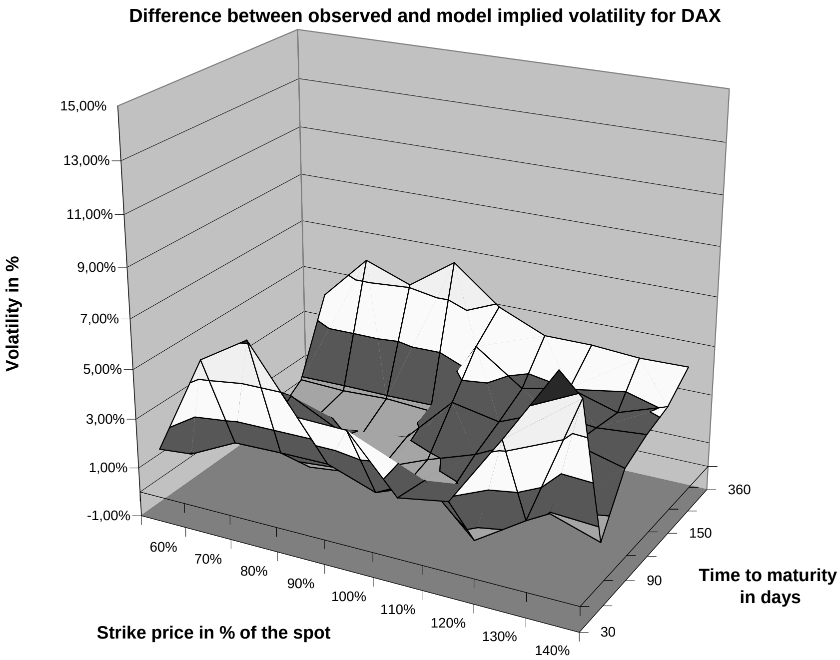




Figure 9: Difference between model generated and observed implied volatilities for Johnson&Johnson and DAX on 03/28/02

# 5 Concluding remarks

The major practical drawback of the multi-dimensional Black-Scholes model and of other multi-dimensional models introduced in recent years is that prices of derivatives computed in these models do not correspond to observations on the market. After the estimation or calibration of the model parameters, implied volatility surfaces (which are in one-to-one correspondence with prices of vanilla options for different strikes and maturities) generated in these models are either flat or have no 'smile', skew or term structure, in contradiction to observations from the market. This implies that for any choice of model parameters, the model can not be consistent with information observed on the market. Therefore these models do not lead to accurate prices and hedging strategies for financial derivatives. In the case of the EBS model we do not have these drawbacks since the numerical results presented in this paper show that prices of derivatives computed in the EBS model show the specific 'stylised facts' and are very close to the ones observed on the markets.

Comparing the implied volatilities diagrams we observe that although the parameters in the EBS model are estimated using historical share prices (and not through calibration with actual prices of derivatives from the market), the model generates implied volatility surfaces similar to the ones observed on the market. From the point of view of the practitioner, one of the main advantages of the EBS model compared to the standard Black-Scholes model is the fact that prices of derivatives in the EBS model are consistent with the ones observed on the market. This characteristic of the model is mainly due to the fact that the dynamics of the share price of one asset is influenced not only by its volatility and its own level but also by the level of other assets. Taking a look at (29) and (30) we observe that after the parameter estimation with historical data the non-diagonal elements of $\mathbf{B}^1$ and $\mathbf{B}^2$ are non-zero. The classical Black-Scholes model corresponds to the case when the non-diagonal elements are zero, i.e. the case where the level of other assets is not relevant for the dynamics of the considered share price. The existence of non-trivial non-diagonal elements in the estimated parameters is a sign that the share prices are highly dependent on the level of other assets, justifying a posteriori and from a numerical point of view the extension of the Black-Scholes model developed in [AlSt]. One minor disadvantage of the EBS model comparing to the Black and Scholes model is that in this model there are no explicit formulas for vanilla options on single assets and that prices of derivatives have to be computed using numerical methods as described at the beginning of this section. This disadvantage of the EBS disappears when pricing multi-asset or more complex derivatives, since these kind of derivatives do not have explicit pricing formulas in the Black-Scholes model and have to be priced anyway by numerical methods.

Since the presented model captures very well the interdependency structure between financial assets in the model parameter **B**, we suggest to use the EBS model for pricing and hedging multi-asset derivatives as an alternative to the multi-dimensional Black-Scholes model. Although we presented in this paper for simplicity only numerical results for a two-dimensional version of the model, the model gives good results also for higher dimensions (e.g. for derivatives depending on 5 to 30 financial assets influencing each other). This is exactly the case where the multi-dimensional Black-Scholes model ceases to give satisfactory results in practice.

When pricing derivatives in the EBS model depending only on one underlying, one important question is how to chose the dimension of the model, i.e. how many additional financial assets which influence the underlying should be taken into account in the model. An empirical answer which gives good results in practice and is easy to implement is that one should take only one additional asset which is strongly correlated with the underlying. In our example (29) we took for IBM the additional asset DELL, another good choice could have been the NASDAQ index.

The best accuracy for pricing and hedging derivatives in the EBS model even in the case of single asset options is of course achieved when one uses a model containing all financial assets traded on the market. In this case the model is "maximal" in the sense that it captures the entire (high dimensional) interdependency structure of the assets available on the market. The disadvantage of such an approach lies in the enormous computational complexity needed for pricing and hedging derivatives. On the other hand such a high dimensional approach might be very interesting when pricing complex multi asset options, e.g. exotic options depending on 30 to 50 assets with individual barrier for each asset.

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