

Valuation of American-Asian Options with the Longstaff-Schwartz Algorithm

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Agenda

- Introduction
 - American-Asian Options with Moving Time Window
- Longstaff-Schwartz / Least-Squares Monte Carlo (LSM) Algorithm
- Implementation
- Numerical Results
- Conclusion and Outlook

Agenda

- **Introduction**
 - Markovian
 - American-Asian Options with Time Window Fixed at Origin
 - Non-Markovian
 - American-Asian Options with Rolling Time Window
- Longstaff-Schwartz / Least-Squares (LS) Algorithm
- Implementation
- Numerical Results
- Conclusion and Outlook

Introduction

Consider an option with payoff:

$$V(S,A,T)=\max(S-A/T,0)$$

where A is the sum of S taken over the time interval $[0,t]$

$$A = \int_0^t S(\tau) d\tau$$

In this case, the process $(S(t),A(t))$ is Markov and can be described as a pde with two base variables S,A :

$$\frac{\sigma^2 S^2}{2} V_{SS} + rSV_S + \frac{1}{t}(S - A)V_A - rV + V_t = 0$$

Introduction

Consider an option with payoff:

$$V(S,A,T)=\max(S-A/t_0,0)$$

where A is the moving sum taken over the time interval $[t-t_0,t]$

($t_0 > 0$ is fixed):

$$A = \int_{t-t_0}^t S(\tau) d\tau$$

European version of this option is an Asian tail.

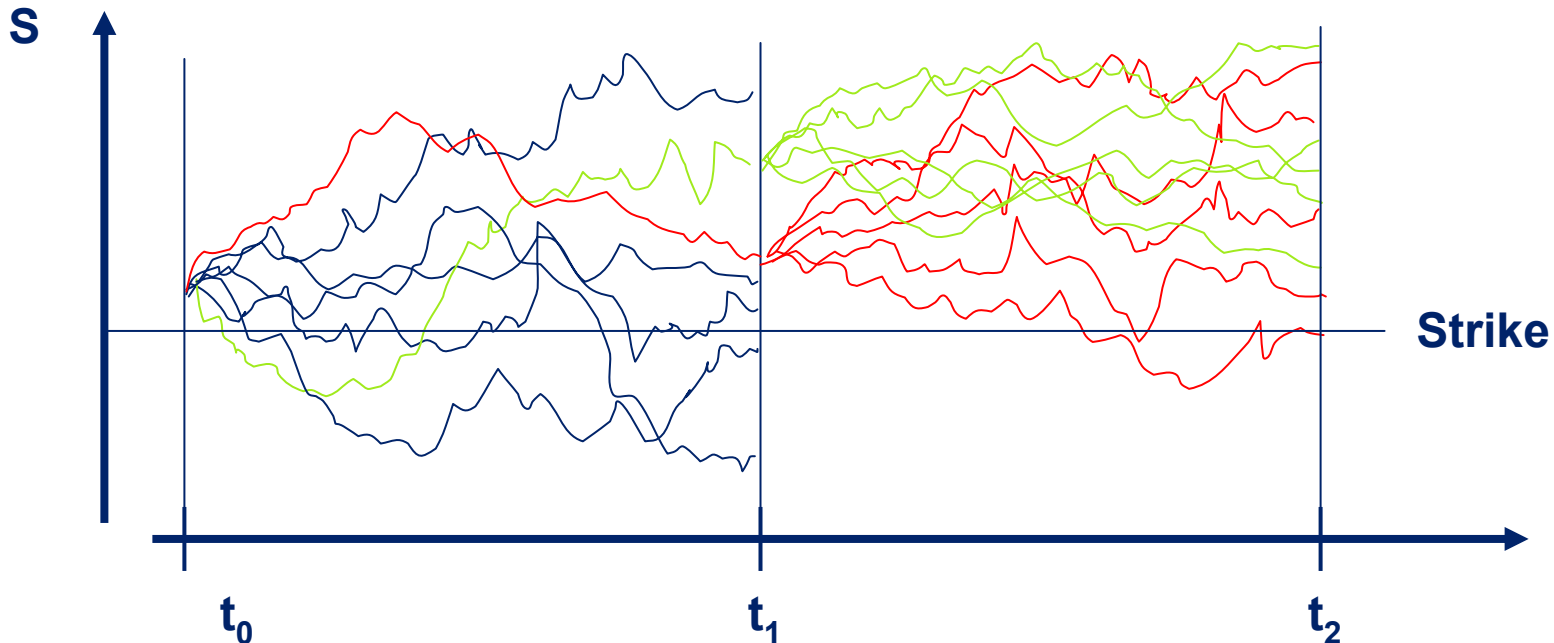
Valuation is straightforward:

First solve for $T-t_0 < t < T$, then for $t < T-t_0$

Introduction

- Difficult to value American version
 - Non-Markovian process
 - cannot be formulated as a finite dimensional pde/complementarity problem
 - need to remember every price $S(\tau)$ for $t-t_0 \leq \tau \leq t$ in order to implement the moving window
 - cannot be formulated as a tree/finite difference problem
 - since it is American, plain Monte Carlo simulation is not possible
 - Monte Carlo on Monte Carlo is very slow

Introduction



Early exercise difficult to implement using Monte Carlo simulation

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Longstaff-Schwartz for American (and Bermudan) Options

F.A. Longstaff, E.S. Schwartz (UCLA):

Valuing American Options by Simulation: A Simple Least-Squares Approach, Review of Financial Studies vol. 14 (2001)

Idea:

Approximation of conditional continuation values with linear regression

Algorithm:

- 1. discrete** timesteps
- 2. Monte Carlo** simulation of the underlying during the **lifetime** of the option
- 3. early exercise** backwards in time:
at each opportunity comparison between value for **immediate payoff** and **continuation value** – determined by linear regression
- 4. discounting** cashflows (one per path at the most) and averaging over the paths → **expected payoff**

Longstaff-Schwartz for American Options

Path	t_0	t_1	t_2	t_3
1	1.00	0.78	0.97	0.79
2	1.00	0.77	0.70	0.72
3	1.00	0.97	0.86	1.63
4	1.00	1.22	1.61	0.79
5	1.00	1.28	1.03	1.27
6	1.00	1.31	1.43	1.39
7	1.00	1.03	1.16	0.61
8	1.00	0.82	0.62	1.26

Basis functions:

$b_1(x)=1; b_2(x)=x; b_3(x)=x^2$

$t_2: 1.61 \rightarrow 0.00$
$1.03 \rightarrow 0.27$
$1.43 \rightarrow 0.39$
$1.16 \rightarrow 0.00$

$Cont(x) = -1.42x^2 + 3.60x - 2.05$

$t_1: 1.22 \rightarrow 0.61$
$1.28 \rightarrow 0.27$
$1.31 \rightarrow 0.43$
$1.03 \rightarrow 0.00$

$Cont(x) = -19.36x^2 + 46.38x - 27.23$

Spot = 1.00
Strike = 1.00

Option Value:

$$\frac{0.63 + 0.61 + 0.27 + 0.43 + 0.03 + 0.26}{8}$$

Path	t_1	t_2	t_3
1			
2			
3			0.63
4		0.61	
5			0.27
6		0.43	
7	0.03		
8			0.26

Path	Payoff t_1	Cont t_1	Payoff t_2	Cont t_2
1				
2				
3				
4	0.22	0.54	0.61	0.07
5	0.28	0.43	0.03	0.16
6	0.31	0.33	0.43	0.20
7	0.03	0.00	0.16	0.22
8				

Least Squares Monte Carlo simulation

- Successfully applied to Markovian processes
 - American put options
 - American-Bermudan-Asian option
 - Cancelable index-amortizing swaps
 - American option with a jump diffusion process
 - $dS=(r+\lambda)Sdt+\sigma SdZ-Sdq$ (dq : Poisson process)
 - Deferred American swaption
 - etc.

- First application to a non-Markovian process

LSM for American-Asian Options

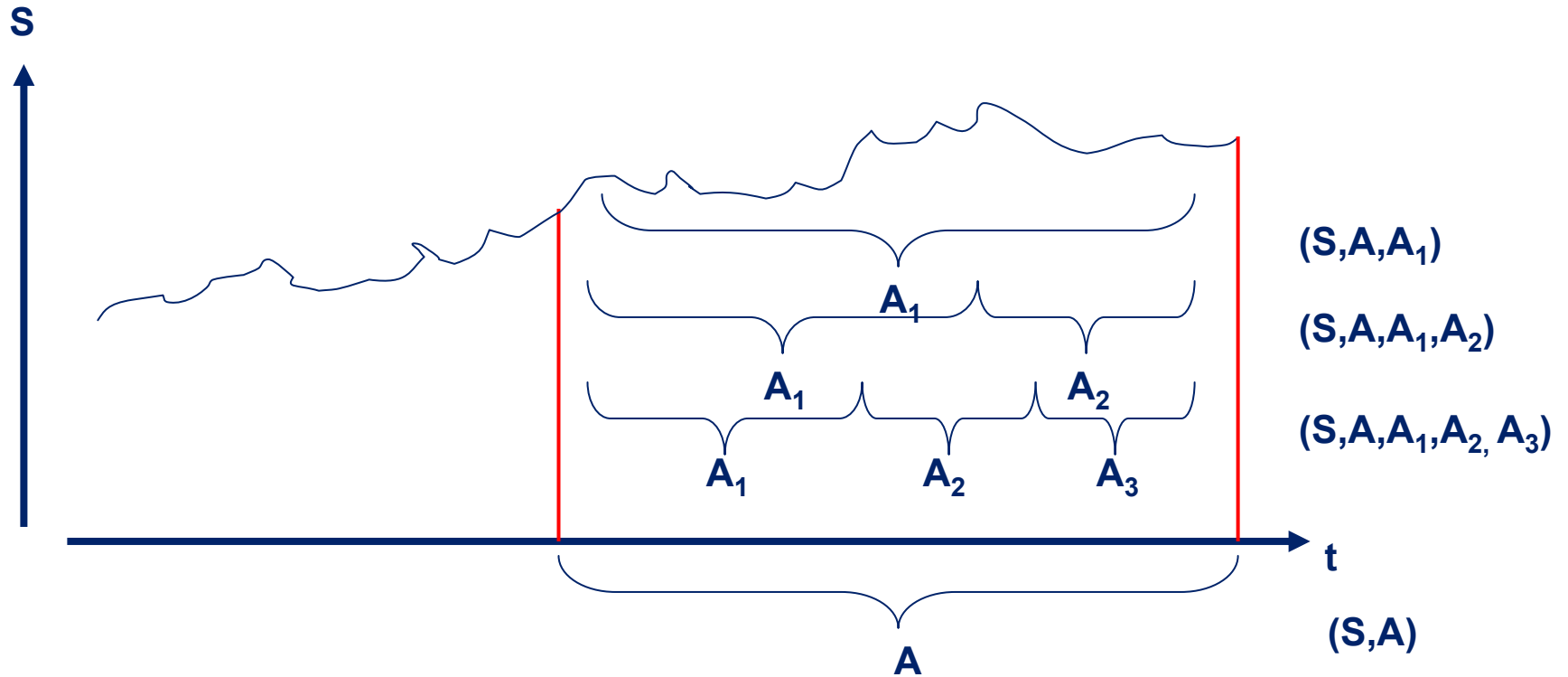
additional dimension in the linear regression: A

sufficient for fixed-start time window: Markovian process

not sufficient for moving time-window: non-Markovian process

Idea: add further averages: A_1, A_2, \dots

adding further base variables



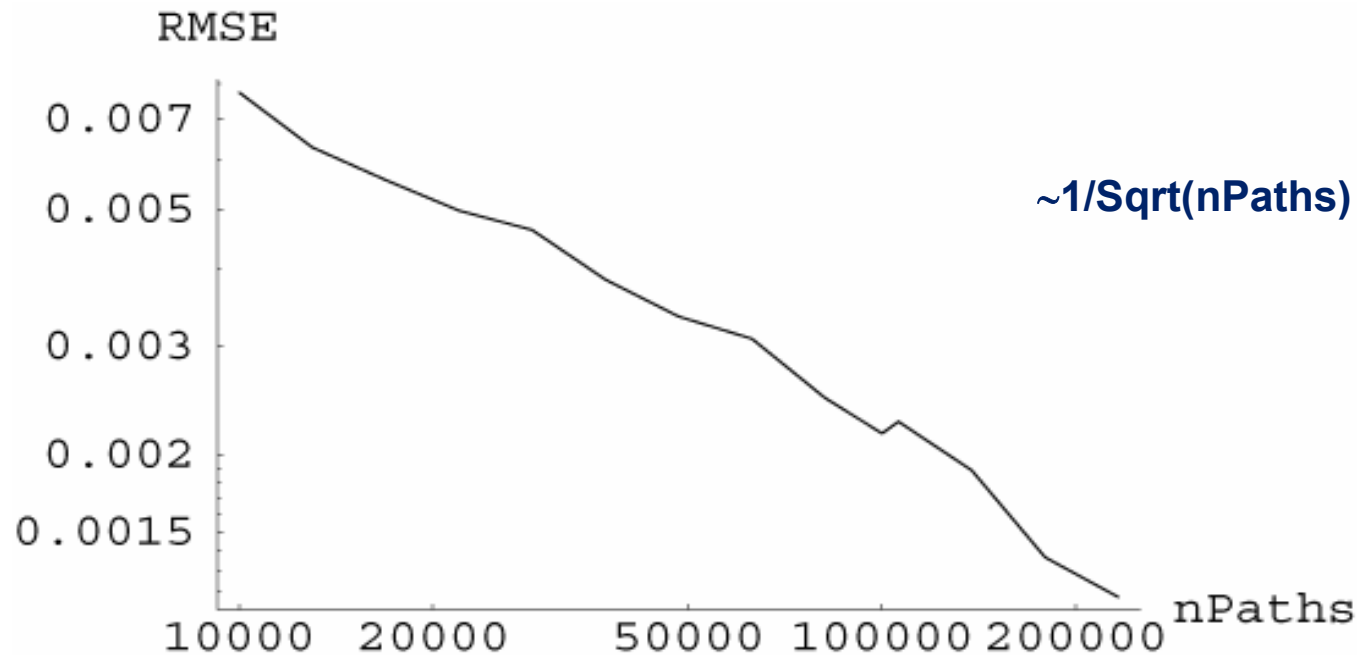
Base Functions used in the Linear Regression

- Laguerre polynomial of order n $L_n(x)$
 - $L_0(x)=1$; $L_1(x)=1-x$; $L_2(x)=1/2 (2-4x+x^2)$;...
 - up to order n
- Number of base functions for $n=3$
 - two base variables $(S,A) \rightarrow 10$ base functions
 - $L_0(S) * L_0(A)$
 - $L_1(S) * L_0(A)$; $L_0(S) * L_1(A)$
 - $L_1(S) * L_1(A)$; $L_2(S) * L_0(A)$; $L_0(S) * L_2(A)$
 - $L_2(S) * L_1(A)$; $L_1(S) * L_2(A)$; $L_3(S) * L_0(A)$; $L_0(S) * L_3(A)$;
 - Three base variables $(S,A,A_1) \rightarrow 20$ base functions
 - d base variables $\rightarrow O(d^3)$ base functions

Implementation

- C# on an Intel-P5 W2K PC
 - execution time approx. 45% slower than C++
 - factor of three faster than Java or Mathematica
 - rapid prototyping
- No information about C# random algorithm, therefore uniform random number generation based on algorithm ran2() in Numerical Recipes
- Box-Muller method for normal deviates
- Solution of Least Squares problem by QR decomposition
- Parameters:
 - $r=5\%$; $\text{vol}=30\%$; $T=2y$; $t_0=40\%*T$; 100 time steps (+40 in the past)

Results

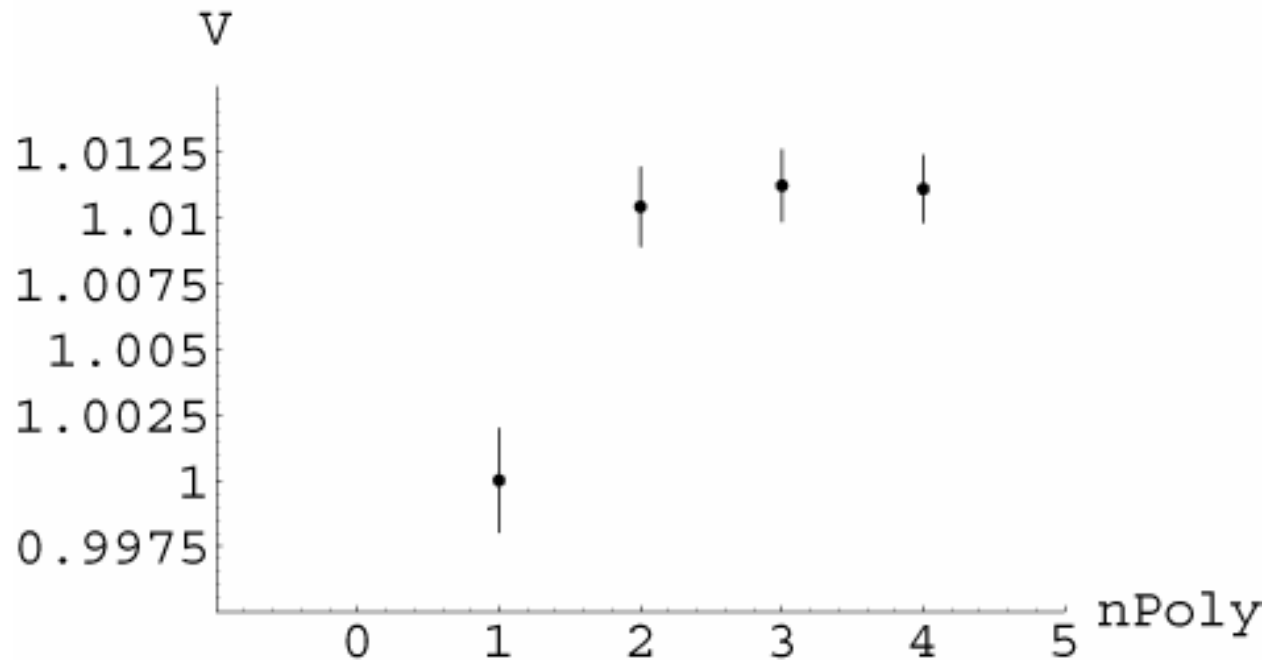


RMSE vs. the number of paths used in the LSMC simulation.

Base variables: (S,A)

Max. degree of used Laguerre polynomials: 3

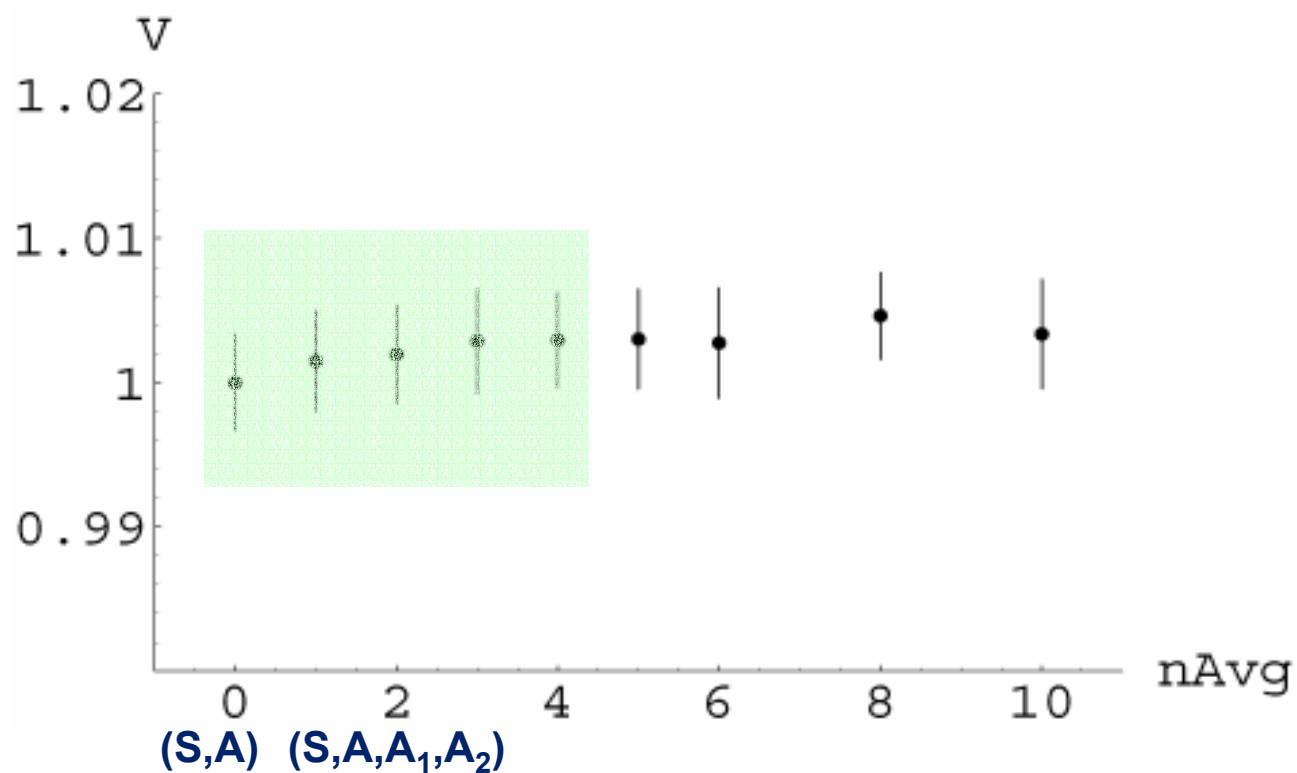
Results



Value of option price as function of the maximum polynomial degree used in the regression (the option value has been renormalised to $V = 1$ for polynomial degree $n = 1$).

The number of paths used in the calculations is $nPaths = 179216$.

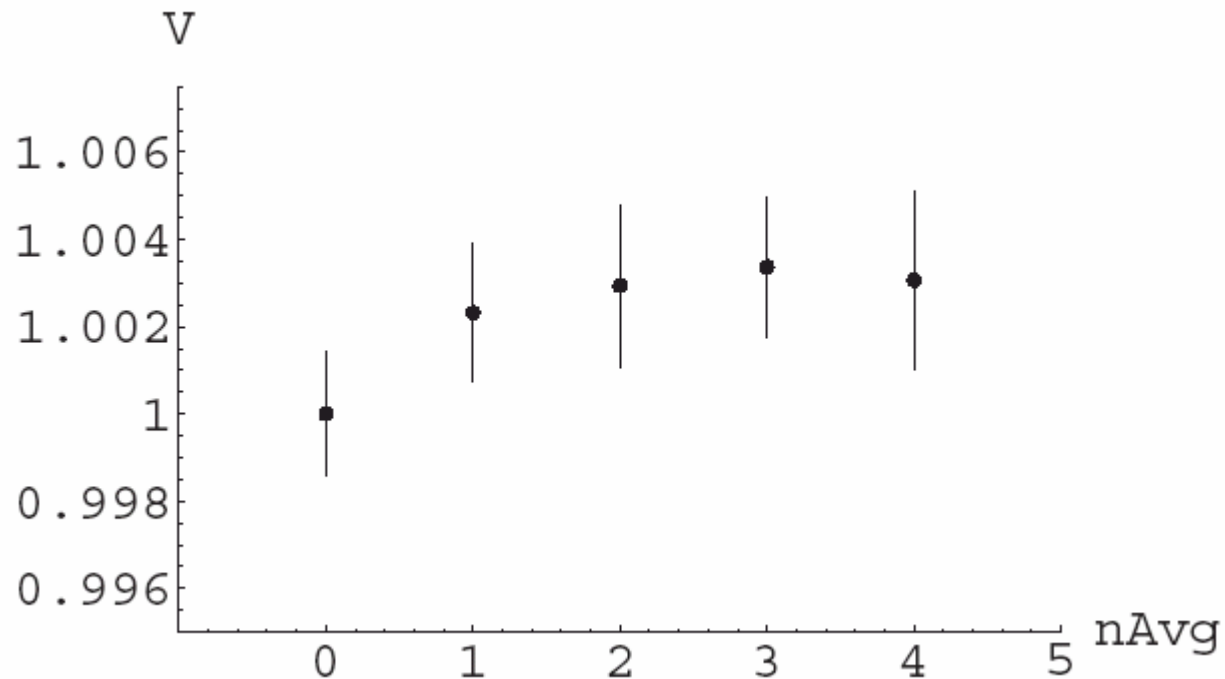
Results



Value of an American-Asian option with constant-width time window versus the number of additional base variables. The maximum polynomial degree used in the regression is $n = 3$.

The data shown here result from runs with $nPaths = 48268$.

Results



Option value versus the number of base variables for polynomial degree $n = 3$ and $nPaths = 179216$.

Summary and Outlook

- American-Asian option with moving time window
 - non-Markovian problem
 - not tractable with standard pricing techniques
 - valuation with Least Squares Monte Carlo simulation by adding further base variables
 - Algorithm can easily be adjusted for other options: geometric average, Bermudan-Asian, etc.
 - considerable computing effort needed

- Techniques for speeding up simulation and improving convergence
 - low discrepancy series: quasi-random numbers
 - has been demonstrated for LSM by Chaudhary (preprint, Math. Dept., UCLA, 2003)
 - parallelization of code

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