

Optimal and near-optimal advection–diffusion finite-difference schemes III. Black–Scholes equation

BY RONALD SMITH

Mathematical Sciences, Loughborough University, Loughborough LE11 3TU, UK

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Optimal and near-optimal compact finite-difference schemes are presented and tested for the numerical solution of an extended Black–Scholes equation:

$$\partial_t c - rc + rs\partial_s c + \frac{1}{2}\sigma^2 s^2 \partial_s^2 c = -q - s\partial_s f - s\partial_s(s\partial_s g).$$

Here $c(s, t)$ is the expected value of the right to buy or sell an asset at some future date, s is the asset price, $r(t)$ is the rate of increase available from alternative riskless investments, and $\sigma(t)$ is the asset volatility. The terms on the right-hand side allow the applicability to be extended beyond the basic European options model. The compactness of the numerical scheme keeps any computer programming elementary. The required computational resources can be as small as 0.001 of conventional schemes.

Keywords: Black–Scholes equation; computational finance; option pricing

1. Introduction

Black & Scholes (1973) and Merton (1973) gave derivations of a model equation

$$\partial_t c - rc + rs\partial_s c + \frac{1}{2}\sigma^2 s^2 \partial_s^2 c = 0, \tag{1.1 a}$$

which has had such financial impact that Robert Merton and Myron Scholes shared the 1997 Nobel prize for economics (Fischer Black having died in 1995). At the future time T when the option is exercised, the final call value is

$$c = \max(s - L, 0) \quad \text{at } t = T, \tag{1.1 b}$$

where s is the actual share price and L is the strike price. The stochastic evolution of the asset value and of its expectation $c(s, t)$ is backwards in time to the known share price s at the earlier time t^0 of agreeing to purchase the option. The state of the market at intermediate times t is characterized by the rate of return $r(t)$ for riskless investments, or a scenario of alternative rates of returns. Different assets can be characterized as having different price volatilities $\sigma(t)$ and different strike prices L , or scenarios of alternative volatilities and strike prices.

Departures from the simplest European type of option can be modelled by adding further terms (see Howison *et al.* 1994; Rogers & Talay 1997). An extended Black–Scholes model with right-hand-side source, gradient and curvature terms is

$$\partial_t c - rc + rs\partial_s c + \frac{1}{2}\sigma^2 s^2 \partial_s^2 c = -q - s\partial_s f - s\partial_s(s\partial_s g). \tag{1.2}$$

A sign change and a logarithmic change of dependent variable,

$$x = \log(s/s_0) - \int_{t_0}^t R(t') dt', \quad s = s_0 \exp\left(x + \int_{t_0}^t R(t') dt'\right), \quad (1.3 a)$$

converts this extended Black–Scholes model (1.2) to a time-reversed advection–diffusion equation with forcing:

$$-\partial_t c + rc + \left(\frac{1}{2}\sigma^2 - r + R\right)\partial_x c - \frac{1}{2}\sigma^2\partial_x^2 c = q + \partial_x f + \partial_x^2 g. \quad (1.3 b)$$

The mean drift of the expected values $c(s, t)$ in the original financial s -variable can be absorbed into stretching of the x -coordinate if the grid inflation $R(t)$ is chosen to remove the $\partial_x c$ advective term from equation (1.3 b):

$$R_{\text{opt}} = r(t) - \frac{1}{2}\sigma^2(t). \quad (1.3 c)$$

If there is a scenario of different $r(t)$ and $\sigma^2(t)$ to be explored, then it might be more convenient to use a single choice for $R(t)$ for the full scenario.

Numerical methods developed for the advection–diffusion equation (1.3 b) can be adapted to apply to Black–Scholes models (1.2). If the chosen computational grid s_j^0 at the current time happens to be uniformly spaced in the financial variables (say at \$5 intervals) it is non-uniform in x_j . In Part II (Smith 2000) it was shown that for advection–diffusion equations, the computational simplicity combined with exceptional accuracy of optimal compact numerical methods is not restricted to uniform grids. Although the expected values $c(s, t)$ are required in the financial variables s , the computations will be optimized with respect to a non-moving x_j grid.

2. Compact finite-difference scheme

Time is indexed forwards from the present time t^0 by superscripts n . Subscripts j index the present price grid s_j^0 . The numerical approximations at the grid points to the expected values are denoted C_j^n .

Compact numerical schemes use the minimum number of successive mesh points and of time steps (i.e. 3 by 2). The accuracy with which the compact finite-difference relationship between the expected values can model the partial-differential equation is limited to errors of fourth order in the step length and of second order in the time-step (Spotz & Carey 2000).

It needs to be explained how the results in Parts I and II (Smith 1999, 2000) improves upon that limit on accuracy, to errors at the grid points of fifth order in the step length and to all orders in time. More impressively, Saul'ev (1958) pointed out that for $r = 0$, $R = \frac{1}{2}\sigma^2$, the error at grid points can be as small as seventh order if the step lengths are precisely tuned. The matching of the finite-difference approximation to the diffusion equation within a 3 by 2 computational module involves the combination of the six grid-point values. It is the inappropriateness of the combination and not the errors in the individual values that dominates the equation mismatch. For example, the time-stepping in equation (2.3) does not conform with a Galerkin approximation to the partial-differential equation unless the parameters Θ , S and K' are all zero.

In Part II (Smith 2000) it was noted that to achieve the simplest expressions for the optimal coefficients, it was appropriate to use the x -centroid as the point of

reference for evaluating compact finite-difference approximations to c , $\partial_x c$ and $\partial_x^2 c$. Accordingly, we introduce the difference operators D , D_x and D_{xx} :

$$D[C^n] = \frac{1}{3}(C_{j-1}^n + C_j^n + C_{j+1}^n) - \frac{2b_j}{3(x_{j+1} - x_{j-1})} \left(\frac{C_{j+1}^n - C_j^n}{(x_{j+1} - x_j)} - \frac{C_j^n - C_{j-1}^n}{(x_j - x_{j-1})} \right), \tag{2.1 a}$$

$$D_x[C^n] = \frac{C_{j+1}^n - C_{j-1}^n}{(x_{j+1} - x_{j-1})} - \frac{(x_{j-1} - 2x_j + x_{j+1})}{3(x_{j+1} - x_{j-1})} \left(\frac{C_{j+1}^n - C_j^n}{(x_{j+1} - x_j)} - \frac{C_j^n - C_{j-1}^n}{(x_j - x_{j-1})} \right), \tag{2.1 b}$$

$$D_{xx}[C^n] = \frac{2}{(x_{j+1} - x_{j-1})} \left(\frac{C_{j+1}^n - C_j^n}{(x_{j+1} - x_j)} - \frac{C_j^n - C_{j-1}^n}{(x_j - x_{j-1})} \right), \tag{2.1 c}$$

where b_j denotes a local effective mean-square spacing for the x_j grid:

$$b_j = \frac{1}{3}((x_j - x_{j-1})^2 + (x_j - x_{j-1})(x_{j+1} - x_j) + (x_{j+1} - x_j)^2). \tag{2.1 d}$$

Any inflationary stretching $R(t)$ does not effect any of the D -operators.

To account for possible time-dependence of $R(t)$, $r(t)$ or $\sigma^2(t)$ within a single time-step of duration $\Delta t = t^{n+1} - t^n$ we will use the hat notation:

$$\left. \begin{aligned} \hat{R} &= \frac{1}{\Delta t} \int_0^{\Delta t} R(t^n + \tau) d\tau, & \hat{r} &= \frac{1}{\Delta t} \int_0^{\Delta t} r(t^n + \tau) d\tau, \\ \hat{\sigma}^2 &= \frac{1}{\Delta t} \int_0^{\Delta t} \sigma^2(t^n + \tau) d\tau. \end{aligned} \right\} \tag{2.2}$$

The D and hat notation allows us to represent neatly a class of optimal or near-optimal compact numerical schemes with adjustable parameters Θ , S and K' :

$$\begin{aligned} & (1 + \frac{1}{2}\hat{r}\Delta t + \Theta(\hat{r}\Delta t)^2) \\ & \times \left\{ \frac{D[C^n]}{\Delta t} + (\frac{1}{2}\hat{\sigma}^2 + \hat{R} - \hat{r})(\frac{1}{2} - S)D_x[C^n] \right. \\ & \quad \left. - (\frac{1}{4}\hat{\sigma}^2 + \frac{1}{2}S\Delta t(\frac{1}{2}\hat{\sigma}^2 + \hat{R} - \hat{r})^2 - K')D_{xx}[C^n] \right\} \\ & + (1 - \frac{1}{2}\hat{r}\Delta t + \Theta(\hat{r}\Delta t)^2) \\ & \times \left\{ -\frac{D[C^{n+1}]}{\Delta t} + (\frac{1}{2}\hat{\sigma}^2 + \hat{R} - \hat{r})(\frac{1}{2} + S)D_x[C^{n+1}] \right. \\ & \quad \left. - (\frac{1}{4}\hat{\sigma}^2 + \frac{1}{2}S\Delta t(\frac{1}{2}\hat{\sigma}^2 + \hat{R} - \hat{r})^2 + K')D_{xx}[C^{n+1}] \right\} \\ & = (1 + \frac{1}{2}\hat{r}\Delta t + \Theta(\hat{r}\Delta t)^2) \{ \frac{1}{2}(D[q^n] + D_x[f^n] + D_{xx}[g^n]) \\ & \quad - \frac{1}{2}S(\frac{1}{2}\hat{\sigma}^2 + \hat{R} - \hat{r})\Delta t(D_x[q^n] + D_{xx}[f^n]) + \frac{1}{2}K'\Delta t D_{xx}[q^n] \} \\ & \quad + (1 - \frac{1}{2}\hat{r}\Delta t + \Theta(\hat{r}\Delta t)^2) \{ \frac{1}{2}(D[q^{n+1}] + D_x[f^{n+1}] + D_{xx}[g^{n+1}]) \\ & \quad - \frac{1}{2}S(\frac{1}{2}\hat{\sigma}^2 + \hat{R} - \hat{r})\Delta t(D_x[q^{n+1}] + D_{xx}[f^{n+1}]) + \frac{1}{2}K'\Delta t D_{xx}[q^{n+1}] \}. \end{aligned} \tag{2.3}$$

Since the x_j grid is not moving, the grid-asymmetry measure Γ_j^n used in Part II (Smith 2000) remains constant, and the U' terms in the numerical scheme of Part II are absent. The time-reversed character of the Black-Scholes equation exchanges superscripts n and $n + 1$ by comparison with Part II. Several terms become zero if the optimal selection (1.3 c) for $R(t)$ can be used.

3. Parameters Θ , S and K'

In the unforced (European option) case, varying Θ has only an order $(\hat{r}\Delta t)^3$ effect. However, the high accuracy acts as an encouragement for us to use large time-steps, months rather than days. Accordingly, Θ should be chosen optimally as in equation (4.1 *a*) of Part II (Smith 2000):

$$\Theta = \left(\frac{1}{1 - \exp(-\hat{r}\Delta t)} - \frac{1}{\hat{r}\Delta t} - \frac{1}{2} \right) \frac{1}{r\Delta t} \approx \frac{1}{12} - \frac{(\hat{r}\Delta t)^2}{720} + \frac{(\hat{r}\Delta t)^4}{30\,240}. \tag{3.1}$$

Likewise, in the unforced (European option) case, varying K' has only a third-order effect in terms of the x step length. As derived in Part II (Smith 2000), the error reduces to fourth order with the selection

$$K' = \frac{1}{12}((\frac{1}{2}\hat{\sigma}^2 + \hat{R} - \hat{r})^2\Delta t) + \frac{b_j}{6\Delta t} - \frac{1}{2}S\hat{\sigma}^2. \tag{3.2}$$

If the optimal choice (1.3 *c*) for $R(t)$ can be made exactly, then it is only via this selection for K' that the finite-difference scheme (2.3) retains any S -dependence.

With this selection (3.2) for K' , varying S modifies the fourth-order error. In Part I (Smith 1999) it is shown that the trivial choice $S = 0$ has the nice feature that on a uniform x grid the stability requirement is the Courant condition

$$(\frac{1}{2}\hat{\sigma}^2 + \hat{R} - \hat{r})^2\Delta t^2 < b_j, \tag{3.3}$$

i.e. the time-step is short enough that in one time-step the price drift of the solution is less than a grid spacing.

As derived in Part II (Smith 2000), the unforced error at the grid points can be reduced to fifth order, in terms of the x step length, with the selection

$$S_{\text{opt}} = \frac{2\hat{\sigma}^2\{b_j - (\hat{\sigma}^2 + 2\hat{R} - 2\hat{r})^2\Delta t^2\}}{\Delta t\{12\hat{\sigma}^4 - (\hat{\sigma}^2 + 2\hat{R} - 2\hat{r})^2b_j + \frac{1}{4}(\hat{\sigma}^2 + 2\hat{R} - 2\hat{r})^4\Delta t^2\}}. \tag{3.4 a}$$

To achieve that extra accuracy, the stability range is reduced:

$$(\frac{1}{2}\hat{\sigma}^2 + \hat{R} - \hat{r})^2\Delta t^2 < \frac{1}{2}b_j \tag{3.4 b}$$

and, to avoid S_{opt} becoming singular, the coarseness of the grid is restricted:

$$b_j < \frac{12\hat{\sigma}^4}{(\hat{\sigma}^2 + 2\hat{R} - 2\hat{r})^2} + \frac{1}{4}(\hat{\sigma}^2 + 2\hat{R} - 2\hat{r})^2\Delta t^2. \tag{3.4 c}$$

If the optimal choice (1.3 *c*) can be made of the inflationary stretching $R(t)$, then the inequalities (3.4 *b*) and (3.4 *c*) become trivial and the value of S_{opt} becomes

$$S_{\text{opt}} = \frac{b_j}{6\Delta t\hat{\sigma}^2}. \tag{3.4 d}$$

Hence, as deduced by Crandall (1955) in his optimal improvement upon the ($r = 0$, $R = \sigma^2/2$) Crank & Nicolson (1947) scheme,

$$K' = \frac{b_j}{12\Delta t}. \tag{3.4 e}$$

In Part I (Smith 1999) it was noted that, in accordance with the work of Saul’ev (1958), the error can be as small as seventh order if, in addition to the S_{opt} selection, the mean-square grid spacing is tuned:

$$b_j = \sqrt{5}\hat{\sigma}^2\Delta t. \tag{3.5 a}$$

This tuning of the grid spacing can only be achieved when $\hat{\sigma}^2$ does not change between time-steps. The fractional price spacing should be approximately

$$\Delta s_j^0/s_j^0 = 5^{1/4}(\hat{\sigma}^2\Delta t)^{1/2}. \tag{3.5 b}$$

If the optimal choice (1.3 c) for $R_{\text{opt}}(t)$ is also made, then S_{opt} has the fixed value

$$S_{\text{opt}} = \frac{1}{6}\sqrt{5} \approx 0.372\,678. \tag{3.5 c}$$

After N equal time-steps marching backwards from the expiry date for the option, a diffusive estimate for the square of the width of the high-curvature corner of the Black–Scholes solution is $N\hat{\sigma}^2\Delta t$. For the mean-square grid spacing (3.5 a) to be finer than this minimal resolution, N must be greater or equal to 3.

4. Solution of three-point-compact finite-difference schemes

The final call value (1.1 b) initiates the evaluation of the numerical solution at the final time level N . To extend the evaluation backwards from a time level $n + 1$ to n , we first transfer the C_j^{n+1} terms in equation (2.3) to the right-hand side. Symbolically the three-point relationship between the C_j^n terms then takes the form

$$-\alpha_j C_{j-1}^n + \beta_j C_j^n - \gamma_j C_{j+1}^n = \delta_j^n, \tag{4.1}$$

where $\alpha_j, \beta_j, \gamma_j$ and δ_j^n have known values. It is only for time-dependent $r(t), \sigma(t)$ and $R(t)$ that the coefficients α_j, β_j and γ_j will vary with n .

Many numerical analysis texts explain efficient Gaussian elimination methods of solving tridiagonal systems (see, for example, Richtmyer & Morton 1967, § 8.5; Mitchell & Griffiths 1980, § 2.5). The solution C_j^n is assumed to take the form of a given end value C_J^n at the highest computational extremity s_J^n of the share price

$$C_J^n = s_J^n - L \exp\left(-\int_{t^0}^{t^n} r(t') dt'\right), \tag{4.2 a}$$

followed by a right-to-left sweep to smaller share prices

$$C_j^n = \epsilon_j C_{j+1}^n + \zeta_j^n \quad \text{for } j < J, \tag{4.2 b}$$

for some appropriate coefficients ϵ_j, ζ_j^n . If we use this sweep formula (4.2 b) to eliminate C_{j-1}^n from the three-point relationship (4.1), then we can solve for C_j^n :

$$C_j^n = \frac{\gamma_j}{\beta_j - \alpha_j \epsilon_{j-1}} C_{j+1}^n + \frac{\delta_j^n + \alpha_j \zeta_{j-1}^n}{\beta_j - \alpha_j \epsilon_{j-1}}. \tag{4.2 c}$$

If this result (4.2 c) is to be identical to the sweep formula (4.2 b), then ϵ_j, ζ_j^n satisfy the recurrence relations

$$\epsilon_j = \frac{\gamma_j}{\beta_j - \alpha_j \epsilon_{j-1}}, \quad \zeta_j^n = \frac{\delta_j^n + \alpha_j \zeta_{j-1}^n}{\beta_j - \alpha_j \epsilon_{j-1}} \quad \text{for } 1 < j. \tag{4.2 d}$$

If at the lowest computational extremity s_1^n of the share price, the option value is constrained to be zero, then

$$\epsilon_1 = 0, \quad \zeta_1^n = 0. \tag{4.2 e}$$

Time-stepping backwards in time from $n + 1$ to n is achieved by retracing the above derivation. First, ϵ_1, ζ_1^n are initialized with equation (4.2 e). Then a left-to-right j -increasing sweep (4.2 d) is performed to evaluate ϵ_j, ζ_j^n . Next, C_j^n is evaluated from equation (4.2 a) followed by a final right-to-left j -decreasing sweep (4.2 b) to evaluate C_j^n .

5. Numerical tests

It is for the forced case of exotic options that numerical methods become indispensable (Zvan *et al.* 1998). However, Black & Scholes's (1973) exact solution of the unforced equations (1.1 a), (1.1 b):

$$c(s, t) = s[1 - \frac{1}{2} \operatorname{erfc}(d/\sqrt{2})] - L \exp(-\rho)[1 - \frac{1}{2} \operatorname{erfc}((d - \Sigma)/\sqrt{2})], \tag{5.1 a}$$

with

$$\rho = \int_t^T r(t') dt', \quad \Sigma^2 = \int_t^T \sigma^2(t') dt', \quad d = \frac{\log(s/L) + \rho + \frac{1}{2}\Sigma^2}{\Sigma}, \tag{5.1 b}$$

allows absolute rather than comparative testing of alternative numerical schemes.

Representing the abrupt change in the gradient of the call value (1.1 b) is a source of error, particularly when the grid is coarse. The call value at the grid point nearest to the strike price L is interpolated to keep the dependence upon L differentiable as grid points are crossed:

$$C_j^N = 0 \quad \text{for } \frac{1}{2}(s_j^N + s_{j+1}^N) \leq L, \tag{5.2 a}$$

$$\begin{aligned} &= \left(\frac{s_j^N + s_{j+1}^N - 2L}{s_{j+1}^N - s_{j-1}^N} \right)^2 \\ &\quad \times \left(\frac{1}{2}s_j^N - \frac{1}{2}s_{j-1}^N - \frac{(L - \frac{1}{2}s_j^N - \frac{1}{2}s_{j-1}^N)(s_{j-1}^N - 2s_j^N + s_{j+1}^N)}{(s_{j+1}^N - s_{j-1}^N)} \right) \\ &\quad \text{for } \frac{1}{2}(s_{j-1}^N + s_j^N) \leq L \leq \frac{1}{2}(s_j^N + s_{j+1}^N), \end{aligned} \tag{5.2 b}$$

$$= s_j^N - L \quad \text{for } L \leq \frac{1}{2}(s_{j-1}^N + s_j^N). \tag{5.2 c}$$

As a test case with round figures, we measure the time in years and we specify the future date to be after one year, with constant market rates, constant volatility and strike price:

$$r = 0.1, \quad \sigma^2 = 0.01, \quad L = 100. \tag{5.3}$$

For the trivial $S = 0$ non-optimal scheme with zero inflationary stretching $R = 0$, we take the present price grid to be clustered about $s = 90$:

$$[s_1^0, \dots, s_9^0] = [60, 70, 80, 85, 90, 95, 100, 110, 120], \tag{5.4 a}$$

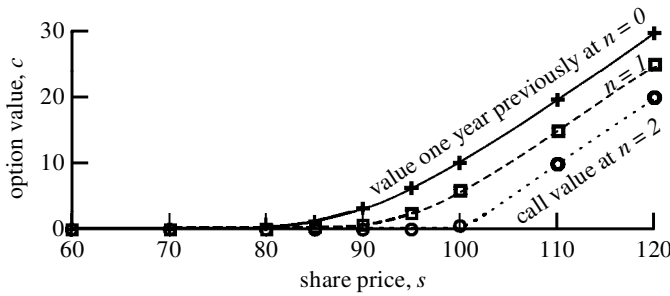


Figure 1. Exact Black-Scholes solution compared with a near-optimal $S = 0$ finite-difference solution on a fixed non-uniform nine-point grid with two 6 month time-steps.

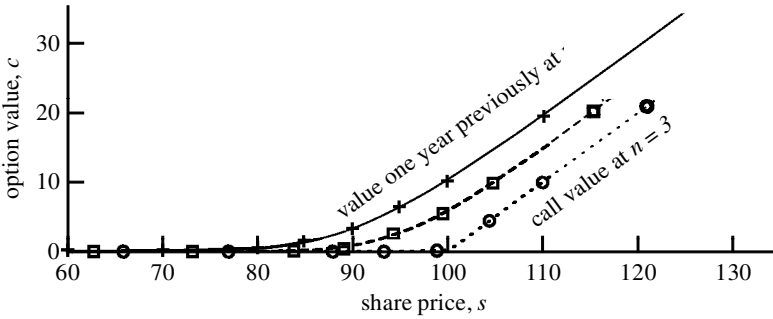


Figure 2. Exact Black-Scholes solution compared with a near-optimal $S = 0$ finite-difference solution on an inflating non-uniform nine-point grid with two 6 month time-steps.

with coarseness marginally sufficient to permit a two-step calculation $\Delta t = \frac{1}{2}$, i.e. 6 months. The computations use simplified end conditions:

$$C_1^n = 0, \quad C_j^n = s_j^n - 100 \exp(-r(1-n)\Delta t). \tag{5.4 b}$$

Figure 1 compares the exact continuous and numerical discrete results evolving backwards from $n = 2$. The purpose of the figures is to illustrate the different styles of grids, the small number of grid points and of time-steps, and to indicate the magnitude of the evolution for the expected value of the share options. The interpolation of the corner value at $s = 100$ for $n = 2$ is the only perceptible error. At $n = 0$ the errors are less than 0.20. The size of these error can be contrasted with the range up to 30 in the same units of the results shown in figure 1.

With the optimal inflationary stretching $R = 0.095$, the trivial $S = 0$ scheme is closer to optimal. Figure 2 compares the exact continuous and numerical discrete results for the same non-uniform coarse grid (5.4 a) and two-step calculation $\Delta t = 0.5$. The inflationary stretching can be seen in the right-to-left shift of the positions of the grid points as n decreases from 2 to 0. At the calculation points there are far smaller changes between time-steps of the option value in figure 2 than in figure 1. At $n = 0$ the errors are less than 0.13.

For the optimal $S = S_{\text{opt}}$ scheme with zero inflationary stretching $R = 0$ and the same non-uniform coarse grid (5.4 a), the stability requirement (3.4 b) requires a three-step calculation $\Delta t = \frac{1}{3}$, i.e. 4 months. Figure 3 compares the exact continuous and numerical discrete results. At $n = 0$ the maximum error is less than 0.09.

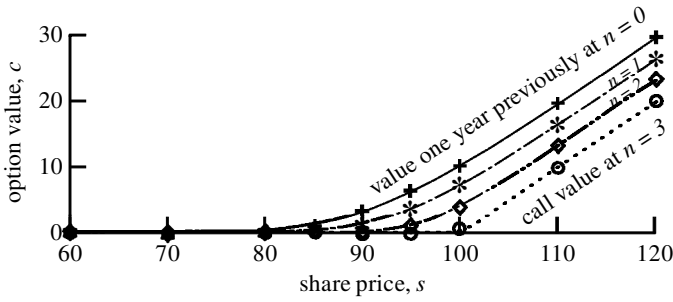


Figure 3. Exact Black–Scholes solution compared with the optimal finite-difference solution on a fixed non-uniform nine-point grid with three 4 month time steps.

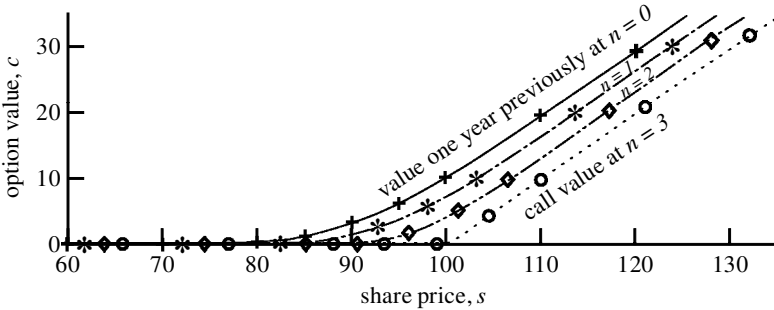


Figure 4. Exact Black–Scholes solution compared with the optimal finite-difference solution on an inflating non-uniform nine-point grid with three 4 month time-steps.

Figure 4 demonstrates the combination of optimal $S = S_{\text{opt}}$ and optimal inflationary stretching $R = 0.095$. As was the case in figure 2, the inflationary stretching eliminates most of the changes in value between time-steps at the computational grid points. The error at $n = 0$ is less than 0.036.

Finally, figure 5 uses the Saul’ev (1958) type of tuned grid spacing, inflationary stretching and the optimal $S = S_{\text{opt}}$ scheme. For the minimal three time-steps the grid points are at a ratio of 1.0863 apart:

$$[s_1^0, \dots, s_5^0, \dots, s_9^0] = [64.6, \dots, 90, \dots, 125.3]. \tag{5.5}$$

The distinctive feature of figure 5 is the geometric spacing of the computational grid. The error reduces yet again to less than 0.023 on a range of valuations up to 38.

To achieve fractional errors of similar magnitude, conventional numerical schemes for solving Black–Scholes models use time-steps of 0.01 years or less and correspondingly close grid spacing (Zvan *et al.* 1998). The number of computations is about a thousand times that of the present schemes.

Although figures 1–5 are concerned with the particular numerical specification (5.3), the size of the errors can be regarded as being representative. For shares with lower volatilities σ and markets with higher interest rates r , the non-inflationary examples of figures 1 and 3 would have to be modified to have price resolution proportional to σ and to keep the time resolution within the inflationary stability bounds (3.3), (3.4 *b*). For the R_{opt} inflation-removed examples of figures 2 and 4, it suffices that the price resolution be proportional to σ . Finally, for the tuned spacing example of figure 5, the grid spacing (3.5 *a*), (3.5 *b*) shares the same inflation and

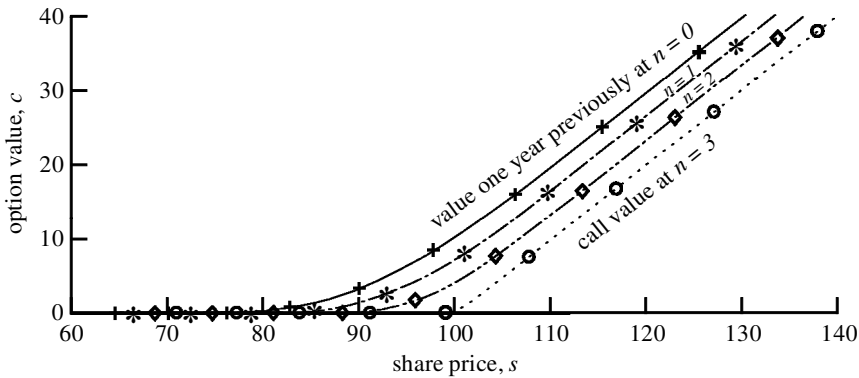


Figure 5. Exact Black–Scholes solution compared with the optimal finite-difference solution on an inflating nine-point grid with tuned spacing and three 4 month time-steps.

self-similarity as the exact Black–Scholes solution (5.1 *a*). For shares with different volatilities σ , the fractional errors would remain unchanged as the tuned spacing changes.

6. Concluding remarks

The near optimal or optimal compact finite-difference schemes live up to their name. The largest of the computations performed in this paper involves only nine grid points and three time steps. To shift the calculations backwards one time-step requires only two sweeps of the nine grid points. Yet, the least accurate, $S = 0$, $R = 0$, of the schemes has errors less than 1% of the largest computed value. The most accurate scheme reduces the errors to less than one sixteenth of a percent. The high order of the numerical schemes implies that there would be marked reductions in error as the number of grid points and of time-steps is increased. The combined simplicity and accuracy would make it feasible to perform the profusion of calculations needed to span scenarios of changing market rates, asset volatilities and strike prices.

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