Modeling Default Dependence with Threshold Models

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Contents

1 Introduction: Modelling Default Dependence

Any problem involving more than one default risky security/entity calls for mastering the problem of default dependence:

- quantifying portfolio credit risk
- multi-credit products, basket credit derivatives, CDO's . . .
- counter-party default risk
- . . .

Modelling default involves two tasks: modelling the (random) *default time* τ and the (random) *default magnitude* $(1 - R)$. The *loss* L up to time T is then

$$L = (1 - R)\mathbf{1}_{\{\tau < T\}}.$$

Default dependence = (?) default correlation = correlation between what?

- *event correlation:* correlation between $\{\tau_1 < T\}$ and $\{\tau_2 < T\}$, T?;
  this is basically the pairwise joint default probability
- *default time correlation:* $\mathrm{corr}(\tau_1, \tau_2)$
- *loss correlation:* $\mathrm{corr}(L_1, L_2)$

As we know correlation is a poor measure of dependence! Better: think about default dependence in general ⇒ **copulas**

Single name default modelling approaches → multi-name models, default dependence

- **Structural models, firm value models:** Merton (1974), Longstaff & Schwarz (1995), . . . default event is triggered by the value of the firm being below a certain trigger level, **pro:** intuitive, simple, **con:** calibration to given term structure of defaults not straightforward, defaults predictable
  default dependence: just make the underlying firm value (ability to pay) processes correlated!

- **Reduced form models:** Jarrow & Lando & Trunbull (1994), Duffie & Singleton (1999),. . . model the (infinitesimal) likelihood of default, **pro:** analogy to interest rate term structure modeling, easy calibration, **con:** missing intuition
  default dependence: extension to sufficient default dependence harder: Schmidt (1998), Davis & Lo(1999), Duffie & Singleton(1999), Jarrow & Yu(2000)

In this talk: follow the structural approach to model (dependent) defaults. Propose model for dependent defaults which

- is straightforward to calibrate to any given term structure of defaults,
- admits analytic expression for the probability of joint default ⇒ efficient calibration of the model to given dependency information.

Joint work with L. OVERBECK, 2003

2 Formulation of the problem

Random times $\tau_1, \dots, \tau_n$ of default of credits $i = 1, \dots, n$. Distribution functions F_i

$$\mathbf{P}(\tau_i < t) = F_i(t),\ t \geq 0,\ i = 1, \dots, n, \tag{1}$$

given from market information or historical data. F_i continuous, strictly increasing, $F_i(0) = 0$.

Default dependence: conditions on the joint distribution of $(\tau_1, \dots, \tau_n)$. Assume given pairwise joint default probabilities for a given fixed time horizon t_0:

$$p_{ij} = \mathbf{P}(\tau_i < t_0, \tau_j < t_0). \tag{2}$$

Equivalent to specifying the event correlation

$$\rho_{ij}^{E} = \frac{p_{ij} - F_i(t_0)F_j(t_0)}{\sqrt{F_i(t_0)(1 - F_i(t_0))F_j(t_0)(1 - F_j(t_0))}}.$$

Problem: Given distribution functions F_i and joint default probabilities p_{ij} (or, equivalently, event correlations) we are looking for stochastic processes (Y_t^i) called *ability-to-pay processes* and barriers $K_i(t)$ such that the default time τ_i for credit i can be modelled as the first hitting time of the barrier $K_i(t)$ by the process (Y_t^i):

$$\tau_i = \inf\{t \geq 0 : Y_t^i \leq K_i(t)\}. \tag{3}$$

In other words, the hitting time τ_i defined by (3) satisfies equations (1) and (2).

General idea: Start with correlated Wiener processes (W_t^i) and define appropriate processes (Y_t^i) by suitable transformations G^i:

$$Y_t^i = G^i(W_t^i, t).$$

3 The Hull and White approach

Hull and White (2001) model: time grind $0 = t_0 < t_1 < t_2 \ldots, \delta_k = t_k - t_{k-1}$.

$$\tau_i = \inf\{t_k : W^i_{t_k} < K_i(t_k),\ k = 1, \ldots\}. \tag{4}$$

Barriers $K_i(t_k)$ are calibrated successively to match the default distribution function F_i for the time points $t_1, t_2, \ldots$:

$$\begin{aligned} K_i(t_1) &= \sqrt{\delta_1}\mathrm{N}^{(-1)}(F_i(t_1)) \\ F_i(t_k) - F_i(t_{k-1}) &= \int_{K_i(t_{k-1})}^{\infty} f_i(t_{k-1}, u)\ \mathrm{N}\left(\frac{K_i(t_k) - u}{\sqrt{\delta_k}}\right) \mathrm{d}u. \end{aligned}$$

$f_i(t_k, x)$ is the density of $W^i_{t_k}$ given that $W^i_{t_j} > K_i(t_j)$ for all $j < k$:

$$\begin{aligned} f_i(t_1, x) &= \frac{1}{\sqrt{2\pi\delta_1}} \exp\left(-\frac{x^2}{2\delta_1}\right) \\ f_i(t_k, x) &= \int_{K_i(t_{k-1})}^{\infty} f_i(t_{k-1}, u) \frac{1}{\sqrt{2\pi\delta_k}} \exp\left(-\frac{(x-u)^2}{2\delta_k}\right) \mathrm{d}u. \end{aligned}$$

Numerical procedures; evaluate the integrals recursively.

Calibrate the correlations ρ_{ij} of the Wiener processes W^i, W^j: simulate the default times, estimate the joint default probabilities from the samples, and calibrate over the results to match given p_{ij}.

Continuous time limit

$$\tau_i = \inf\{t \geq 0 : W_t^i < K_i(t)\}.$$

If $K_i(t)$ absolutely continuous, $K_i(t) = K_i(0) + \int_0^t \mu_s^i \mathrm{d}s$, default time τ_i is the first hitting time of the constant barrier $K_i(0)$ for a Wiener process with drift:

$$\begin{aligned} Y_t^i &= W_t^i - \int_0^t \mu_s^i \mathrm{d}s \\ \tau_i &= \inf\{t \geq 0 : Y_t^i < K_i(0)\}. \end{aligned}$$

Interpretation of drift: *default trend* – the higher μ_s^i the higher is the increase in the likelihood of default.

4 Time changed Wiener process

Alternative solution approach to our problem:
Wiener processes (W_t^i), strictly increasing (deterministic) time transformations (T_t^i):

$$T^i|[0,\infty) \to [0,\infty),\ T_0^i = 0.$$

Define

$$\begin{aligned} Y_t^i &= W^i_{T_s^i} && (5) \\ \tau_i &= \inf\{s \geq 0 : Y_s^i < K_i\}. && (6) \end{aligned}$$

4.1 Calibrating the term structure of defaults

Proposition 1. *Let F_i be a continuous distribution function, strictly increasing on $[0, \infty)$ with $F_i(0) = 0$. If the time transformation (T_t^i) is given by*

$$T_t^i = \left[\frac{K_i}{\mathrm{N}^{(-1)} \left(\frac{F_i(t)}{2} \right)} \right]^2, \ t \geq 0 \tag{7}$$

then the default time τ_i defined by (6) admits the distribution function F_i, i.e., condition (1) is satisfied.

Proof.

$$\mathbf{P}(\min_{s\leq t} W_s < K_i) = 2\,\mathrm{N}(K_i/\sqrt{t}),$$

for τ_i defined by (6)

$$\begin{aligned} \mathbf{P}(\tau_i < t) &= \mathbf{P}(\min_{s\leq t} W^i_{T_s} < K_i) \\ &= \mathbf{P}(\min_{s\leq T^i_t} W^i_s < K_i) \\ &= 2\,\mathrm{N}\left(K_i/\sqrt{T^i_t}\right). \end{aligned}$$

Interpretation of time transformation (T^i_t): the higher the increase of the function T^i_t, the higher is the speed at which the ability-to-pay process $Y^i_t = W^i_{T^i_t}$ passes along the Wiener path thereby increasing the likelihood of default.

If F_i admits a density $F_i' = f_i$, then

$$T_t^i = \int_0^t (\sigma_s^i)^2 \mathrm{d}s, \tag{8}$$

and

$$W^i_{T_t^i} = \int_0^t \sigma_s^i \mathrm{d}\tilde{W}_s^i. \tag{9}$$

$$\sigma_s^i = \sqrt{- \left[\frac{K_i}{\mathrm{N}^{(-1)} \left(\frac{F_i(s)}{2} \right)} \right]^3 \frac{f_i(s)}{K_i \varphi \left(\mathrm{N}^{(-1)} \left(\frac{F_i(s)}{2} \right) \right)}}.$$

Interpretation of volatility σ_s^i: *default speed*. The higher the default speed, the higher the volatility of the ability-to-pay process and the higher the likelihood of crossing the default threshold K_i.

4.2 Calibrating joint default probabilities

Proposition 2. *Let* (W_t^1) *and* (W_t^2) *be Wiener processes with correlation* ρ*. For time changes* (T_t^i), $i = 1, 2$, *and default times* τ_i, $i = 1, 2$, *defined by (6) we have the following expression for the joint survival probability*

$$\mathbf{P}(\tau_1 > t_0, \tau_2 > t_0) = \tag{10}$$

$$\begin{cases} \frac{2}{\alpha\, T}\mathbf{e}^{\frac{-r_0^2}{2T}} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi\theta_0}{\alpha}\right) \int_{\theta=0}^{\alpha} \int_{r=0}^{\infty} \sin\left(\frac{n\pi\theta}{\alpha}\right) r\mathbf{e}^{\frac{-r^2}{2T}} I_{\frac{n\pi}{\alpha}}\left(\frac{r\, r_0}{T}\right) \\ \qquad \left(-1 + 2\mathrm{N}\left(\frac{r\sin\theta}{\sqrt{\Delta}}\right)\right) \mathrm{d}\theta \mathrm{d}r, & \text{if} \Delta > 0, \\ \frac{2r_0}{\sqrt{2\pi T}}\mathbf{e}^{\frac{-r_0^2}{4T}} \sum_{n=1,3,5,\ldots} \frac{1}{n} \sin\left(\frac{n\pi\theta_0}{\alpha}\right) \\ \qquad \left[I_{\frac{1}{2}(\frac{n\pi}{\alpha}+1)}\left(\frac{r_0^2}{4T}\right) + I_{\frac{1}{2}(\frac{n\pi}{\alpha}-1)}\left(\frac{r_0^2}{4T}\right)\right] & \text{if} \Delta = 0, \end{cases}$$

where

$$\begin{aligned}
T &= \min(T^1_{t_0}, T^2_{t_0}) \\
\Delta &= \max(T^1_{t_0}, T^2_{t_0}) - \min(T^1_{t_0}, T^2_{t_0}) \\
\theta_0 &= \begin{cases} \tan^{(-1)}\left(\frac{K_2\sqrt{1-\rho^2}}{K_1-\rho K_2}\right) & \textit{if } \left(\frac{K_2\sqrt{1-\rho^2}}{K_1-\rho K_2}\right) > 0 \\ \pi + \tan^{(-1)}\left(\frac{K_2\sqrt{1-\rho^2}}{K_1-\rho K_2}\right) & \textit{otherwise} \end{cases} \\
r_0 &= \frac{-K_2}{\sin\theta_0} \\
\alpha &= \begin{cases} \tan^{(-1)}\left(\frac{-\sqrt{1-\rho^2}}{\rho}\right) & \rho < 0 \\ \pi + \tan^{(-1)}\left(\frac{-\sqrt{1-\rho^2}}{\rho}\right) & \rho > 0 \end{cases}
\end{aligned}$$

and I_k denotes the modified Bessel function with order k.

Proof. Heavily based on a result by ZHOU 1997 plus application of Markov property to handle time transformations.

4.3 Calibration Examples

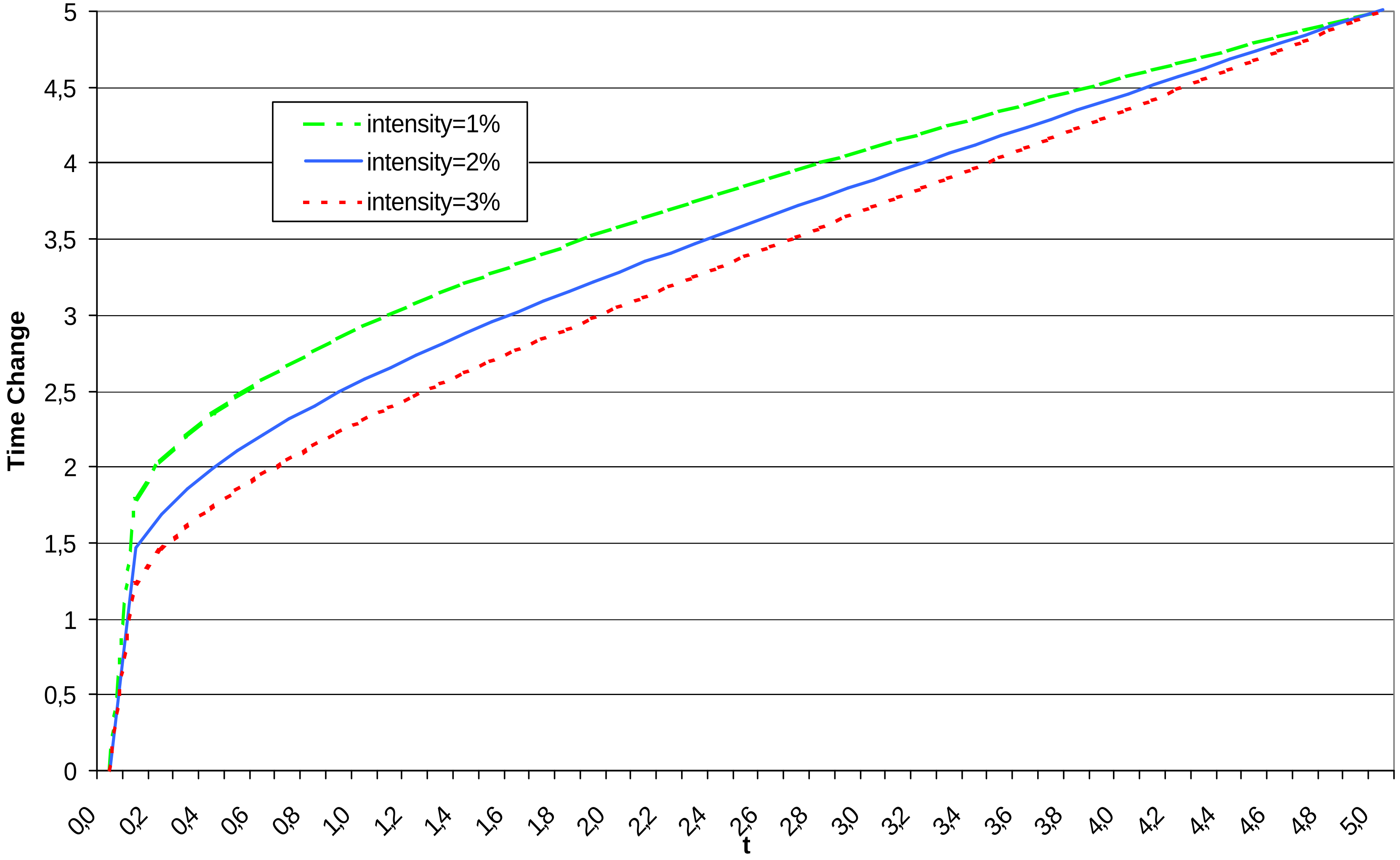
Space-time scaling property of Wiener process ⟶ some degree of freedom in determining the default threshold K_i. For fixed final time horizon $t_0 > 0$ require

$$T^i_{t_0} = t_0, \tag{11}$$

which yields

$$K_i = \mathrm{N}^{(-1)}\left(\frac{F_i(t_0)}{2}\right)\sqrt{t_0}. \tag{12}$$

Calibrate time transformations for $F_i(t) = 1 - \exp(-\lambda_i\, t)$ and default intensities (hazard rates) $\lambda_1 = 1\%, \lambda_2 = 2\%, \lambda_3 = 3\%$. For $t_0 = 5$ obtain threshold barriers $K_1 = -4,406, K_2 = -3,731, K_3 = -3,306$.


Time Change
5
4,5
4
3,5
3
2,5
2
1,5
1
0,5
0
intensity=1%
intensity=2%
intensity=3%
0,0
0,2
0,4
0,6
0,8
1,0
1,2
1,4
1,6
1,8
2,0
2,2
2,4
2,6
2,8
3,0
3,2
3,4
3,6
3,8
4,0
4,2
4,4
4,6
4,8
5,0
t

Examples of the calibration to given dependency information (2) (event correlations ρ^E). Again assume

$$F_i(t) = 1 - \exp(-\lambda_i t),$$

$t_0 = 5$. Calibration results are the corresponding correlations of the Wiener processes (asset correlation).

$\rho^E/(\lambda_1,\lambda_2)$	(1%,1%)	(1%,2%)	(1%,3%)
0,00%	0,00%	0,00%	0,00%
5,00%	18,51%	16,27%	15,28%
10,00%	31,59%	28,82%	27,68%
15,00%	41,96%	39,23%	38,29%
20,00%	50,60%	48,16%	47,63%
25,00%	57,98%	55,99%	55,99%
30,00%	64,40%	62,92%	63,55%
35,00%	70,03%	69,11%	70,46%
40,00%	74,98%	74,66%	76,82%
45,00%	79,35%	79,64%	82,77%
50,00%	83,21%	84,12%	88,49%
55,00%	86,58%	88,15%	
60,00%	89,53%	91,79%	
65,00%	92,07%		

$\rho^E/(\lambda_1,\lambda_2)$	(2%,2%)	(2%,3%)	(3%,3%)
0,00%	0,00%	0,00%	0,00%
5,00%	13,98%	12,97%	11,94%
10,00%	25,52%	24,07%	22,48%
15,00%	35,43%	33,85%	31,94%
20,00%	44,13%	42,59%	40,52%
25,00%	51,87%	50,47%	48,32%
30,00%	58,78%	57,59%	55,44%
35,00%	64,99%	64,05%	61,92%
40,00%	70,56%	69,91%	67,82%
45,00%	75,55%	75,20%	73,16%
50,00%	80,01%	79,96%	77,97%
55,00%	83,96%	84,22%	82,27%
60,00%	87,43%	88,00%	86,07%
65,00%	90,45%	91,33%	89,40%

4.4 Implementation

Calibrate model analytically as shown and perform Monte Carlo simulation: Choose time discretization $0 = s_0 < s_1 < \cdots < s_m = t_0$ simulate the random variables $W^i_{T^i_{s_j}}, i = 1, \ldots, n, j = 1, \ldots, m$. The first time

$$\min\{s_j : W^i_{T^i_{s_j}} < K_i\}$$

overestimates the true default time τ_i! $\longrightarrow$
Apply Brownian bridge technique to capture the probability of defaults in between the grid points s_j.
For $\alpha = W^i_{T^i_{s_j}} > K_i$, $\beta = W^i_{T^i_{s_{j+1}}} > K_i$ the process $W^i_u, u \in [T^i_{s_j}, T^i_{s_{j+1}}]$ is a Brownian bridge with starting point α and end point β.

Probability of crossing the boundary K_i during the time interval $[T^i_{s_j}, T^i_{s_{j+1}}]$

$$\begin{aligned}\mathbf{P}(\min_{u\in[s_j,s_{j+1}]} W^i_{T^i_u} < K_i) &= \mathbf{P}(\min_{u\in[T^i_{s_j},T^i_{s_{j+1}}]} W^i_u < K_i) \\ &= \exp\left(\frac{-2(\beta - K_i)(\alpha - K_i)}{\Delta}\right),\end{aligned}$$

with $\Delta = T^i_{s_{j+1}} - T^i_{s_j}$. For each time interval $[s_j, s_{j+1}]$ given that the threshold K_i has not been crossed before and that $W^i_{T^i_{s_{j+1}}} > K_i$ we draw an additional uniform random variable U^i_j and set

$$\tau_i = (s_j + s_{j+1})/2 \quad \text{if} \quad U^i_j < \exp\left(\frac{-2(\beta - K_i)(\alpha - K_i)}{\Delta}\right)$$

and proceed with our simulations if $U_j^i \geq \exp\left(\frac{-2(\beta-K_i)(\alpha-K_i)}{\Delta}\right)$.

5 Application example

Apply model to the pricing of a basket credit default swap.

Basket credit default swap: protection against the event of the kth default on a basket of n ($n \geq k$) underlying names. Premium (spread) s is paid as insurance fee until maturity or the event of the kth default in return for a compensation for the loss. Denote by s^{kth} the fair spread in a kth-to-default swap.

Example basket with $n = 5$ names, individual fair CDS spreads $s_1 = 0,80\%, s_2 = 0,90\%, s_3 = 1,00\%, s_4 = 1,10\%, s_5 = 1,20\%$ for all maturities. Assume recovery rate of $R = 15\%$. Derive distribution functions F_i of the default times τ_i, $i = 1, \ldots, 5$.

Price 5 years maturity kth-to-default basket default swap for various levels of correlation. Monte Carlo simulation on a monthly time grid with 10000 simulations. Compare results to alternative valuations in a normal copula model. The asset correlations in the copula model are calibrated to the same pairwise joint probabilities as the time change model.

fair basket default swap spreads in the time change model

ρ	10%	20%	30%	40%	50%	60%	70%
s^{first}	4,791%	4,563%	4,296%	3,953%	3,620%	3,252%	2,845%
s^{2nd}	0,625%	0,799%	0,941%	1,055%	1,131%	1,201%	1,259%
s^{3rd}	0,058%	0,130%	0,201%	0,296%	0,398%	0,523%	0,635%
s^{4th}	0,003%	0,015%	0,040%	0,076%	0,126%	0,202%	0,306%
s^{5th}	0,000%	0,003%	0,006%	0,013%	0,030%	0,057%	0,118%

fair basket default swap spreads in the normal copula model

ρ	10%	20%	30%	40%	50%	60%	70%
s^{first}	4,704%	4,442%	4,137%	3,806%	3,486%	3,147%	2,764%
s^{2nd}	0,670%	0,803%	0,941%	1,062%	1,151%	1,215%	1,257%
s^{3rd}	0,074%	0,137%	0,219%	0,320%	0,413%	0,523%	0,640%
s^{4th}	0,005%	0,016%	0,040%	0,084%	0,143%	0,222%	0,334%
s^{5th}	0,001%	0,003%	0,008%	0,016%	0,041%	0,075%	0,135%

For the first-to-default basket, time change model seems to produce slightly higher spreads. For the other cases the results are quite close.

This is supported by repeated simulations with different seeds for the random number generator. The seed variance of fair the first-to-default spread is less than 0.10% and much smaller for the other spreads. For the normal copula model we used a variance reduction technique based on stratified sampling whereas for the time change model we applied an in-sample orthogonalization of the increments of the Wiener processes.