Optimal Portfolios When Stock Prices Follow an Exponential Lévy Process

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Abstract. We investigate some portfolio problems that consist of maximizing expected terminal wealth under the constraint of an upper bound for the risk, where we measure risk by the variance, but also by the Capital-at-Risk (CaR). The solution of the mean-variance problem has the same structure for any price process which follows an exponential Lévy process. The mean-CaR involves a quantile of the corresponding wealth process of the portfolio. We derive a weak limit law for its approximation by a simpler Lévy process, often the sum of a drift term, a Brownian motion and a compound Poisson process. Certain relations between a Lévy process and its stochastic exponential are investigated.

Key words: Capital-at-risk, downside risk measure, exponential Lévy process, portfolio optimization, stochastic exponential, Value-at-Risk, weak limit law for Lévy processes.

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1 Introduction

It is well-known that the normal distribution does not describe the behaviour of asset returns in a very realistic way. One reason for this is that the distribution of real data is often leptokurtic, i.e. it exhibits more small values than a normal law and has often semi-heavy tails, in other words its curtosis is higher than the curtosis of the normal distribution. Eberlein and Keller [7] show for instance the fit of the generalized hyperbolic distribution to financial data in a very convincing way. Normal mixture models like the normal inverse Gaussian and the variance gamma model play an increasing role also in the financial industry. Consequently, to replace in the classical geometric Brownian motion the Wiener process by some general Lévy process is an important generalization of the traditional Black-Scholes model.

This paper can be viewed as a continuation of Emmer, Klüppelberg, and Korn [9], where a portfolio optimization problem was solved based on the Value-at-Risk as risk measure. Now we investigate portfolio optimization problems, when the price processes are governed by general exponential Lévy processes, comparing the optimal solutions with variance and Value-at-Risk as risk measures, respectively. We explain some basic theory of Lévy processes and refer to Protter [18] and, in particular, Sato [21] for relevant background.

The characteristic function of $L(t)$ is for each $t > 0$ given by

$$E \exp(isL(t)) = \exp(t\Psi(s)), \quad s \in \mathbb{R}^d,$$

where Ψ has Lévy-Khintchine representation

$$\Psi(s) = ia_L's - \frac{1}{2}s'\beta_L'\beta_L s + \int_{\mathbb{R}^d} (e^{is'x} - 1 - is'x1_{\{|x|\le 1\}})\nu_L(dx)\,,$$

with $a_L \in \mathbb{R}^d$, $\beta_L'\beta_L$ is a non-negative definite symmetric $d \times d$-matrix, and ν_L is a measure on $\mathbb{R}^d$ satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d}(|x|^2 \wedge 1)\nu(dx) < \infty$. The term corresponding to $xI_{\{|x|\le 1\}}$ represents a centering without which the integral may not converge. The characteristic triplet $(a_L, \beta_L'\beta_L, \nu_L)$ characterizes the Lévy process. We often write $(a, \beta'\beta, \nu)$ instead of $(a_L, \beta_L'\beta_L, \nu_L)$, if it is clear which Lévy process is concerned. Throughout this paper we denote by $\mathbb{R}^d$ the d-dimensional Euclidean space. Its elements are column vectors and for $x \in \mathbb{R}^d$ we denote by x' the transposed vector; analogously, for a matrix β we denote by β' its transposed matrix. We further denote by $|x| = (\sum_{i=1}^d x_i^2)^{1/2}$ the Euclidean norm of $x \in \mathbb{R}^d$.

According to Sato [21], Chapter 4, the following holds. For each ω in the probability space define $\Delta L(t,\omega) = L(t,\omega) - L(t-,\omega)$. For each Borel set $B \subset [0,\infty) \times \mathbb{R}^{d*}$ ($\mathbb{R}^{d*} = \mathbb{R}^d \setminus \{0\}$) set

$$M(B,\omega) = \#\{(t, \Delta L(t,\omega)) \in B\}\,.$$

Lévy's theory says that M is a Poisson random measure with intensity

$$m(dt, dx) = dt\nu(dx)\,,$$

where ν is the Lévy measure of the process L.

For $B = [a,b] \times A$ for $0 \le a < b < \infty$ and a Borel set A in $\mathbb{R}^{d*}$

$$M(B,\omega) = \#\{(t, \Delta L(t,\omega)) : a \le t \le b, \Delta L(t,\omega) \in A\}$$

counts jumps of size in A which happen in the time interval $[a,b]$. According to the above, this is a Poisson random variable with mean $(b-a)\nu(A)$.

With this notation, the Lévy-Khintchine representation corresponds to the representation

$$\begin{aligned} L(t) &= at + \beta W(t) + \sum_{0<s\le t} \Delta L(s) 1_{\{|\Delta L(s)|>1\}} \\ &\quad + \int_0^t \int_{|x|\le 1} x(M(dx,ds) - \nu(dx)ds)\,, \quad t \ge 0\,. \end{aligned} \tag{1.1}$$

This means that $L(t)$ has a Gaussian component $\beta W(t)$ and a pure jump part with Lévy measure ν, having the interpretation that a jump of size x occurs at rate $\nu(dx)$. To ensure the finiteness of the integral on the rhs, the small jumps are compensated by their expectation. This representation reduces in the finite variation case to

$$L(t) = \gamma t + \beta W(t) + \sum_{0<s\leq t} \Delta L(s), \quad t \geq 0,$$

where $\gamma = a - \int_{|x|\leq 1} x\nu(dx)$; i.e. $L(t)$ is the independent sum of a drift term, a Gaussian component and a pure jump part.

The paper is organized as follows. In Section 2 we introduce a multivariate Lévy Black-Scholes model and calculate the terminal wealth of a portfolio and its moments provided they exist. In Section 3 we use these results for a portfolio optimization that consists of maximizing the expected terminal wealth of a portfolio within a well-defined set of strategies under some constraint on the variance. In Section 4 we introduce the Capital-at-Risk (CaR), which is defined via a low quantile (Value-at-Risk) of the wealth process, and discuss for the one-dimensional case methods for its calculation and approximation. In Section 5 we optimize portfolios, where we replace the variance by the CaR. We work out real life examples as the normal inverse Gaussian and variance gamma model. Here we do not obtain closed form analytic solutions, but solve the optimization problem by an approximation and numerical algorithms. Section 6 is devoted to the proof of the weak limit theorem which we need for the approximation of the quantile of the wealth process. It involves some new results on the stochastic exponential of a Lévy process.

2 The market model

We consider a standard Black-Scholes type market consisting of a *riskless bond* and several *risky stocks*, which follow exponential Lévy processes. Their respective prices $(P_0(t))_{t\geq 0}$ and $(P_i(t))_{t\geq 0}$, $i = 1, \ldots, d$, evolve according to the equations

$$P_0(t) = e^{rt} \quad \text{and} \quad P_i(t) = p_i \exp(b_i t + \sum_{j=1}^{d} \sigma_{ij} L_j(t)), \quad t \geq 0. \tag{2.1}$$

Here $(L(t))_{t\geq 0} = (L_1(t), \ldots, L_d(t))_{t\geq 0}$ is a d-dimensional Lévy process (stationary independent increments with cadlag sample paths). We assume the $L_i, i = 1, \ldots, d$, to be independent. L has characteristic triplet $(a, \beta'\beta, \nu)$, where $a \in \mathbb{R}^d$, β is an arbitrary d-dimensional diagonal matrix. We introduce β as a diagonal matrix into the model to allow for some extra flexibility apart from $\sigma = (\sigma_{ij})_{1\leq i,j\leq d}$. This also includes the possibility of a pure jump process (for $\beta_i = 0$). Since the components of βW are independent Wiener processes with different variances possible, we allow for different scaling factors for the Wiener process and the non-Gaussian components. By the independence of the components we obtain for the Lévy measure ν of L and a d-dimensional rectangle $A = \times_{i=1}^d (a_i, b_i] \subset \mathbb{R}^d$ that $\nu(A) = \sum_{i=1}^d \nu_i(a_i, b_i]$, where ν_i is the Lévy measure of L_i for $i = 1, \ldots, d$; i.e. the Lévy measure is supported on the union of the coordinate axes (see Sato [21], E12.10, p. 67). Thus the probability that two components have a jump at the same time point is zero; i.e. jumps of different components occur a.s. at different times.

The quantity $r \in \mathbb{R}$ is the *riskless interest rate* and $\sigma = (\sigma_{ij})_{1 \le i,j \le d}$ is an invertible matrix, $b \in \mathbb{R}^d$ can be chosen such that each stock has the desired appreciation rate. Since the assets are on the same market, they show some dependence structure which we model by a linear combination of the same Lévy processes $L_1, \ldots, L_d$ for each asset price. This means the dependence structure of the market is the same as that of the Black-Scholes market in Emmer et al. [9].

We need the corresponding SDE in order to derive the wealth process. By Itô's formula, P_i is the solution to the SDE

$$
\begin{aligned}
dP_i(t) &= P_i(t-)(b_i dt + d\widehat{L}_i(t)) && (2.2)\\
&= P_i(t-)\Bigg(\big(b_i + \frac{1}{2}\sum_{j=1}^d (\sigma_{ij}\beta_{jj})^2\big)dt + \sum_{j=1}^d \sigma_{ij} dL_j(t) && (2.3)\\
&\quad + \exp(\sum_{j=1}^d \sigma_{ij}\Delta L_j(t)) - 1 - \sum_{j=1}^d \sigma_{ij}\Delta L_j(t) \Bigg)\,, t > 0\,, P_i(0) = p_i\,,
\end{aligned}
$$

i.e. $\widehat{L}_i$ is such that $\exp(\sum_{j=1}^d \sigma_{ij} L_j(t)) = \mathcal{E}(\widehat{L}_i)$, where $\mathcal{E}$ denotes the stochastic exponential of a process (see Protter [18] for background on stochastic analysis). From this representation we see that jumps of $\widehat{L}_i$ occur at the same time as those of $(\sigma L)_i = \sum_{j=1}^d \sigma_{ij} L_j$, but a jump of size $\sum_{j=1}^d \sigma_{ij}\Delta L_j$ is replaced by one of size $\exp(\sum_{j=1}^d \sigma_{ij}\Delta L_j) - 1$ leading to the term $\exp(\sum_{j=1}^d \sigma_{ij}\Delta L_j) - 1 - \sum_{j=1}^d \sigma_{ij}\Delta L_j$ in formula (2.2), whereas the Brownian component remains the same as in $(\sigma L)_i$.

Remark 2.1. It is not difficult to start with a general d-dimensional Lévy process with arbitrary characteristic triplet and calculate the moments of X^π; i.e. corresponding results of Proposition 2.6. Also the portfolio optimization problem (3.1) can be solved in this general case. This has been pointed out by a referee. We prefer, however, to work with a linear dependence structure, since it allows for nice formulae and can also be interpreted easily. The general case can be found in Emmer [8].

The following Lemma describes the relation between the characteristic triplets of a Lévy process and its stochastic exponential, which we need in the sequel.

Lemma 2.2. (Goll and Kallsen [10])
If L is a real-valued Lévy process with characteristic triplet (a, β^2, ν), then also $\widehat{L}$ defined by $e^L = \mathcal{E}(\widehat{L})$ is a Lévy process with characteristic triplet $(\widehat{a}, \widehat{\beta}^2, \widehat{\nu})$

given by

$$\begin{aligned}\widehat{a} - a &= \frac{1}{2}\beta^2 + \int \big((e^x - 1)1_{\{(|e^x-1|<1\}} - x1_{\{|x|<1\}}\big)\nu(dx)\\ \widehat{\beta}^2 &= \beta^2\\ \widehat{\nu}(\Lambda) &= \nu(\{x \in \mathbb{R} | e^x - 1 \in \Lambda\}) \text{ for any Borel set } \Lambda \subset \mathbb{R}^* .\end{aligned}$$

In the following Lemma the relation between the characteristic triplets of a d-dimensional Lévy process L and its linear transformation $\pi' L$ is given for $\pi \in \mathbb{R}^d$.

Lemma 2.3. (Sato [21], Prop. 11.10)
If L is a d-dimensional Lévy process with characteristic triplet $(a, \beta'\beta, \nu)$, then $\pi' L$ is for $\pi \in \mathbb{R}^d$ a one-dimensional Lévy process with characteristic triplet $(a_\pi, \beta^2_\pi, \nu_\pi)$ given by

$$\begin{aligned}a_\pi &= \pi' a + \int \pi' x (1_{\{|\pi' x|<1\}} - 1_{\{|x|<1\}})\nu(dx)\\ \beta^2_\pi &= |\pi'\beta|^2\\ \nu_\pi(\Lambda) &= \nu(\{x \in \mathbb{R}^d | \pi' x \in \Lambda\}) \text{ for any Borel set } \Lambda \subset \mathbb{R}^* .\end{aligned}$$

Let $\pi(t) = (\pi_1(t) \dots \pi_d(t))' \in \mathbb{R}^d$ be an admissible portfolio process, i.e. $\pi(t)$ is the fraction of the wealth $X^\pi(t)$, which is invested in asset i (see Korn [12], Section 2.1 for relevant definitions). The fraction of the investment in the bond is $\pi_0(t) = 1 - \pi(t)'\underline{1}$, where $\underline{1} = (1, \dots, 1)'$ denotes the vector (of appropriate dimension) having unit components. Throughout the paper, we restrict ourselves to constant portfolios; i.e. $\pi(t) = \pi$, $t \in [0, T]$, for some fixed *planning horizon* T. This means that the fractions in the different stocks and the bond remain constant on $[0, T]$. The advantages of this restriction are discussed in Emmer et al. [9] and Sections 3.3 and 3.4 of Korn [12]. In order to avoid negative wealth we require that $\pi \in [0, 1]^d$, hence shortselling is not allowed in this model. We also require $\pi'\underline{1} \le 1$; see Remark 2.4 below. This means that borrowing money is not permitted.

Denoting by $(X^\pi(t))_{t\ge 0}$ the *wealth process*, it follows the dynamic

$$dX^\pi(t) = X^\pi(t-)\Big(((1 - \pi'\underline{1})r + \pi' b)dt + \pi' d\widehat{L}(t)\Big) , \quad t > 0 , \quad X^\pi(0) = x ,$$

where $x \in \mathbb{R}$ denotes the *initial capital* of the investor. Using Itô's formula, this SDE has solution

$$\begin{aligned}X^\pi(t) &= x \exp(t(r + \pi'(b - r\underline{1})))\mathcal{E}(\pi'\widehat{L}(t)) &(2.4)\\ &= x \exp(a_X t + \pi'\sigma\beta W(t))\widetilde{X}^\pi(t) , \quad t \ge 0 , &(2.5)\end{aligned}$$

where a_X is as in Lemma 2.5 below and, setting $\ell(x) := \ln(1 + \pi'(e^{\sigma x} - \underline{1}))$,

$$\begin{aligned}\ln \widetilde{X}^\pi(t) &= \int_0^t \int_{\mathbb{R}^d} \ell(x)1_{\{|\ell(x)|>1\}} M_L(ds, dx)\\ &\quad + \int_0^t \int_{\mathbb{R}^d} \ell(x)1_{\{|\ell(x)|\le 1\}}(M_L(ds, dx) - ds\nu_L(dx)) , \quad t \ge 0 .\end{aligned}$$

Remark 2.4. Note that a jump $\Delta L(t)$ of L leads to a jump $\Delta \ln X^\pi(t)$ of $\ln X^\pi$ of size $\ln(1+\pi'(e^{\sigma\Delta L(t)}-\underline{1}))$ and thus $\Delta \ln X^\pi(t) > \ln(1-\pi'\underline{1})$, hence we require $\pi'\underline{1} \leq 1$.

The wealth process is again an exponential Lévy process. We calculate the characteristic triplet of its logarithm by means of Lemmas 2.2 and 2.3.

Lemma 2.5. *Consider model (2.1) with Lévy process L and characteristic triplet $(a, \beta'\beta, \nu)$. Define for the $d \times d$-matrix $\sigma\beta$ the vector $[\sigma\beta]^2$ with components*

$$[\sigma\beta]_i^2 = \sum_{j=1}^d (\sigma_{ij}\beta_{jj})^2, \quad i = 1, \ldots, d.$$

The process $\ln X^\pi$ is a Lévy process with characteristic triplet (a_X, β_X^2, ν_X) given by (again we set $\ell(x) := \ln(1+\pi'(e^{\sigma x}-\underline{1}))$)

$$\begin{aligned} a_X &= r + \pi'(b + [\sigma\beta]^2/2 - r\underline{1} + \sigma a) - |\pi'\sigma\beta|^2/2 \\ &\quad + \int_{\mathbb{R}^d} \big(\ell(x)1_{\{|\ell(x)| \leq 1\}} - \pi'\sigma x 1_{\{|x| \leq 1\}}\big)\nu(dx)\,, \\ \beta_X^2 &= |\pi'\sigma\beta|^2\,, \\ \nu_X(A) &= \nu(\{x \in \mathbb{R}^d | \ell(x) \in A\}) \text{ for any Borel set } A \subset \mathbb{R}^*\,. \end{aligned}$$

For the calculation of moments of the wealth process we need the existence of the moment generating function in some neighbourhood of 0. This corresponds to an analytic extension of the characteristic function. If this extension is possible (see Theorem 25.17 of Sato [21] for conditions), for all $k \in \mathbb{N}$, such that the k-th moment exists,

$$E[(X^\pi(t))^k] = \exp((k a_X + k^2\beta_X^2/2)t) E[(\widetilde{X}^\pi(t))^k], \quad t \geq 0\,,$$

and

$$E[(\widetilde{X}^\pi(t))^k] = \exp(\widetilde{\mu}_k t)\,, \quad t \geq 0\,, \tag{2.6}$$

where

$$\widetilde{\mu}_k = \int_{\mathbb{R}^d} \big((1+\pi'(e^{\sigma x}-\underline{1}))^k - 1 - k\ell(x)1_{\{|\ell(x)|\leq 1\}}\big)\,\nu(dx)$$

and ν is the Lévy measure of L. In particular,

$$E[\widetilde{X}^\pi(t)] = \exp\left(t\int_{\mathbb{R}^d} \big(\pi'(e^{\sigma x}-\underline{1}) - \ell(x)1_{\{|\ell(x)|\leq 1\}}\big)\nu(dx)\right), t \geq 0\,.$$

Proposition 2.6. *Assume in the situation of equation (2.1) that $L(1)$ has moment generating function $\widehat{f}(s) = E\exp(s'L(1))$ such that $\widehat{f}(e_i'\sigma) < \infty$ for $i = 1, \ldots, d$, where e_i is the i-th d-dimensional unit vector. Let $X^\pi(t)$ be as in equation (2.4). Then for $t \geq 0$*

$$E[X^\pi(t)] = x\exp(t(r + \pi'(b - r\underline{1} + \ln \widehat{f}(\sigma))))\,, \tag{2.7}$$

$$\mathrm{var}(X^\pi(t)) = x^2\exp(2t(r + \pi'(b - r\underline{1} + \ln \widehat{f}(\sigma))))(\exp(t\pi' A\pi) - 1), \tag{2.8}$$

where $\ln \widehat{f}(\sigma) = (\ln \widehat{f}(e_1'\sigma), \ldots, \ln \widehat{f}(e_d'\sigma))'$ *and* $A = (A_{ij})_{1 \le i,j \le d}$ *with*

$$A_{ij} = \ln \widehat{f}((e_i + e_j)'\sigma) - \ln \widehat{f}(e_i'\sigma) - \ln \widehat{f}(e_j'\sigma)\,, \quad 1 \le i, j, \le d\,.$$

Proof. Recall that $(a, \beta'\beta, \nu)$ is the characteristic triplet of L. By equation (2.6) and Lemma 2.5 we obtain for $t \ge 0$:

$$E\,[X^\pi(t)] = \tag{2.9}$$
$$x \exp\left(t\,(r + \pi'(b - r\underline{1} + \frac{1}{2}[\sigma\beta]^2 + \sigma a + \int_{\mathbb{R}^d} (e^{\sigma x} - \underline{1} - \sigma x 1_{\{|x|<1\}})\nu(dx)))\right),$$

$$\begin{aligned} \operatorname{var}(X^\pi(t)) &= x^2 \exp\Big(2t\,\big(r + \pi'(b - r\underline{1} + \frac{1}{2}[\sigma\beta]^2 + \sigma a \\ &\quad + \int_{\mathbb{R}^d} (e^{\sigma x} - \underline{1} - \sigma x 1_{\{|x|<1\}})\nu(dx))\big)\Big) \\ &\quad \times \left(\exp\left(t\left(|\pi'\sigma\beta|^2 + \int_{\mathbb{R}^d} (\pi'(e^{\sigma x} - \underline{1}))^2 \nu(dx)\right)\right) - 1\right). \end{aligned} \tag{2.10}$$

On the other hand we calculate

$$\begin{aligned} \widehat{f}(e_i'\sigma) &= E \exp(e_i'\sigma L(1)) \\ &= \exp\left(\left(\sigma a + [\sigma\beta]^2/2 + \int_{\mathbb{R}^d} (e^{\sigma x} - \underline{1} - \sigma x 1_{\{|x|<1\}})\nu(dx)\right)_i\right) \end{aligned}$$

and

$$\pi' A \pi = |\pi'\sigma\beta|^2 + \int_{\mathbb{R}^d} (\pi'(e^{\sigma x} - \underline{1}))^2 \nu(dx)).$$

Plugging this into (2.9) and (2.10) we obtain (2.7) and (2.8). □

Remark 2.7. Note that for $l = 1, \ldots, d\,(i = \sqrt{-1})$

$$\begin{aligned} \ln \widehat{f}(e_l'\sigma) &= \ln E \exp(\sum_{j=1}^d \sigma_{lj} L_j(1)) \;=\; \sum_{j=1}^d \ln \widehat{f}_j(\sigma_{lj}) \\ &= \ln E[\mathcal{E}(\widehat{L}_l)(1)] \;=\; \sum_{j=1}^d \Psi_j(-i\sigma_{lj}) \end{aligned}$$

by the independence of $L_1, \ldots, L_d$. This implies in particular

$$E\mathcal{E}(\pi'\widehat{L}(t)) = \prod_{l=1}^d (E[\mathcal{E}(\widehat{L}_l(t))])^{\pi_l}.$$

Remark 2.8. For $d = 1$ our portfolio consists of one bond and one stock only.
(a) Formula (2.8) reduces to

$$\begin{aligned} \operatorname{var}(X^\pi(t)) &= x^2 \exp(2t(r + \pi(b - r + \ln \widehat{f}(\sigma)))) \\ &\quad \times \left(\exp(\pi^2 t(\ln(\widehat{f}(2\sigma)) - 2\ln \widehat{f}(\sigma))) - 1\right). \end{aligned} \tag{2.11}$$

Moreover, we can set w.l.o.g. $\sigma = 1$. In this case the Lévy density f_X of the process $\ln X^\pi$ can be calculated from the Lévy density f_L of ν_L as

$$f_X(x) = f_L\left(\ln\left(\frac{e^x - 1}{\pi} + 1\right)\right)\frac{e^x}{e^x - (1-\pi)} 1_{\{x > \ln(1-\pi)\}}, \quad x \in \mathbb{R}.$$

(b) In the case of a jump part of finite variation we obtain for $t \geq 0$,

$$E\left[X^\pi(t)\right] = x\exp(t(r + \pi(b - r + \frac{1}{2}\beta^2 + \gamma + \widehat{\mu}))), \tag{2.12}$$

$$\begin{aligned} \operatorname{var}\left(X^\pi(t)\right) &= x^2 \exp\left(2t\big(r + \pi(b - r + \gamma + \widehat{\mu} + \frac{1}{2}\beta^2)\big)\right) \\ &\quad \times \left(\exp\left(\pi^2 t(\beta^2 + \widehat{\mu}_2 - 2\widehat{\mu})\right) - 1\right), \end{aligned} \tag{2.13}$$

for $\widehat{\mu} = \int(e^x - 1)\nu(dx)$, $\widehat{\mu}_2 = \int(e^{2x} - 1)\nu(dx)$, and $\gamma = a - \int_{|x|<1} x\nu(dx)$.

3 Optimal portfolios under variance constraints

In this section we consider the following optimization problem using the variance as risk measure

$$\max_{\{\pi \in [0,1]^d \,|\, \pi'\underline{1} \leq 1\}} E[X^\pi(T)] \quad \text{subject to} \quad \operatorname{var}(X^\pi(T)) \leq C\,, \tag{3.1}$$

where T is some given planning horizon and C is a given bound for the risk.

Theorem 3.1. *Let L be a Lévy process with representation (1.1). Then the optimal solution of problem (3.1) is given by*

$$\pi^* = \varepsilon^* \frac{((\sigma\beta)(\sigma\beta)')^{-1}(b - r\underline{1} + \ln\widehat{f}(\sigma))}{|(\sigma\beta)^{-1}(b - r\underline{1} + \ln\widehat{f}(\sigma))|} \tag{3.2}$$

(provided $\pi^* \in [0,1]^d$ and $\pi^{*\prime}\underline{1} \leq 1$), where $\ln\widehat{f}(\sigma)$ is defined in Proposition 2.6 and ε^* is the unique positive solution of

$$rT + |(\sigma\beta)^{-1}(b - r\underline{1} + \ln\widehat{f}(\sigma))|\varepsilon T + \frac{1}{2}\ln\left(\frac{x^2}{C}\left(\exp(Tf(\varepsilon)) - 1\right)\right) = 0\,, \tag{3.3}$$

and

$$f(\varepsilon) = \varepsilon^2\left(1 + \int_{\mathbb{R}^d}\left(\frac{((\sigma\beta)(\sigma\beta)')^{-1}(b - r\underline{1} + \ln\widehat{f}(\sigma))'}{|(\sigma\beta)^{-1}(b - r\underline{1} + \ln\widehat{f}(\sigma))|}(e^{\sigma x} - \underline{1})\right)^2 \nu(dx)\right).$$

Remark 3.2. If the solution to (3.3) does not satisfy $\pi^* \in [0,1]^d$ and $\pi^*\underline{1} \leq 1$, then the problem can be solved by the Lagrange method using some numerical optimization algorithm, for example the SQP method (sequential quadratic programming) (see e.g. Nocedal and Wright [16] or Boggs and Tolle [6]). If for $d = 1$ the solution of (3.3) leads to $\pi^* > 1$, the optimal portfolio is $\pi^* = 1$.

Proof of Theorem 3.1. Following the proof of Proposition 2.9 of Emmer et al. [9], where the same optimization problem has been solved for geometric Brownian motion, we obtain (3.2) as the portfolio with the largest terminal wealth over all portfolios satisfying $|\pi'\sigma\beta| = \varepsilon$. Plugging (3.2) into the explicit form (2.10) of the variance of the terminal wealth the constraint has the same form as in Proposition 2.9 of Emmer et al. [9]. Hence the result follows from a comparison of constants. The only difference to the optimization problem in [9] is the constraint $\pi^* \in [0,1]^d$ and $\pi^*\underline{1} \le 1$, which we took care of. $\square$

Remark 3.3. In the finite variation case and for $d = 1$ where we choose w.l.o.g. $\sigma = 1$, the rhs of (3.3) can be rewritten as

$$rT + \pi\left(b - r + \gamma + \widehat{\mu} + \frac{1}{2}\beta^2\right)T + \frac{1}{2}\ln\left(\frac{x^2}{C}\left(\exp\left(\pi^2\left(\beta^2 + \widehat{\mu}_2 - 2\widehat{\mu}\right)T\right) - 1\right)\right)$$

with $\widehat{\mu}$, $\widehat{\mu}_2$ and γ as in Remark 2.8(b).

In the following we consider some examples in order to understand the influence of the jumps on the choice of the optimal portfolio. For simplicity we take $d = 1$ in these examples and hence we choose w.l.o.g. $\sigma = 1$. For some figures and discussions of the examples we refer to Emmer [8].

Example 3.4. *(Exponential Brownian motion with jumps)*
Let $Y_1, Y_2, \dots$ be iid random variables with distribution p on $\mathbb{R}^*$ and $(N(t))_{t\ge0}$ a Poisson process with intensity $c > 0$, independent of the Y_i. Then $\overline{L}(t) := \sum_{i=1}^{N(t)} Y_i$, $t \ge 0$, defines a compound Poisson process with Lévy measure $\nu(dx) = cp(dx)$. The Lévy process $(L(t))_{t\ge0}$ is taken as the sum of a Brownian motion with drift $(\beta W(t) + \gamma t)_{t\ge0}$ and the compound Poisson process $(\overline{L}(t))_{t\ge0}$.

If $\widehat{g}(s) = Ee^{sY} < \infty$, then

$$\widehat{f}(s) = E\exp(s\overline{L}(1)) = \exp(c(\widehat{g}(s) - 1))\,.$$

If $\widehat{g}(1)$ or $\widehat{g}(2)$ exists, then by Remark 2.8(b),

$$\widehat{\mu} = c(\widehat{g}(1) - 1) \text{ and } \widehat{\mu}_2 = c(\widehat{g}(2) - 1)\,.$$

The drift $\gamma = -\frac{1}{2}\beta^2 - \widehat{\mu}$ is chosen such that the asset price has the same expectation as in the Black-Scholes model in Emmer et al. [9], Section 2. By (2.5), (2.12) and (2.13) we obtain for $t \ge 0$ (for ease of notation we set $r = 0$)

$$\begin{aligned} X^\pi(t) &= x\exp\left(t\big(\pi(b - \widehat{\mu}) - \frac{\pi^2\beta^2}{2}\big) + \pi\beta W(t)\right)\prod_{i=1}^{N(t)}(1 + \pi(e^{Y_i} - 1))\,, \\ E[X^\pi(t)] &= x\exp(t\pi b)\,, \\ \operatorname{var}(X^\pi(t)) &= x^2\exp(2t\pi b)\left(\exp(\pi^2 t(\beta^2 + c(\widehat{g}(2) - 2\widehat{g}(1) + 1))) - 1\right). \end{aligned}$$

The exponential compound Poisson process ($\beta = 0$) and the exponential Brownian motion ($c = 0$) are special cases of this example as well as the jump diffusion in Emmer et al. [9], where the Y_i were deterministic.

Example 3.5. *(Exponential normal inverse Gaussian (NIG) Lévy process)*
The NIG Lévy process has been introduced by Barndorff-Nielsen [3] and [4] and investigated further in Barndorff-Nielsen and Shephard [5]. It belongs to the class of generalized hyperbolic Lévy processes. The NIG Lévy model is a normal variance-mean mixture model such that

$$L(1) \quad = \quad \rho + \lambda\zeta^2 + \zeta W(1),$$

where $(W(t))_{t\geq 0}$ is a standard Brownian motion, $\zeta^2 \sim IG(\delta^2, \xi^2 - \lambda^2)$ and $\xi \geq |\lambda| \geq 0$, $\delta > 0$, $\rho \in \mathbb{R}$. This process is uniquely determined by the distribution of the increment $L(1)$ whose density is given by

$$nig(x; \xi, \lambda, \rho, \delta) := \frac{\xi}{\pi} \exp\left(\delta\sqrt{\xi^2 - \lambda^2} + \lambda(x - \rho)\right) \frac{K_1(\delta\xi g(x-\rho))}{g(x-\rho)}\,, \quad x \in \mathbb{R}\,,$$

where $g(x) = \sqrt{\delta^2 + x^2}$ and $K_1(x) = \frac{1}{2}\int_0^\infty \exp(-x(y + y^{-1})/2)dy$, $x > 0$, is the modified Bessel function of the third kind of order one. Note that for $s > 0$ the density of $L(t+s) - L(t)$, $t \geq 0$, is given by $nig(x, \xi, \lambda, s\rho, s\delta)$. The parameter ξ is a steepness parameter, i.e. for larger ξ we get less large and small jumps and more jumps of middle size, δ is a scale parameter, λ is a symmetry parameter and ρ a location parameter. For $\rho = \lambda = 0$ (symmetry around 0) the characteristic triplet $(0, 0, \nu)$ of a NIG Lévy process is given by

$$\nu(dx) = \frac{\delta\xi}{\pi}|x|^{-1}K_1(\xi|x|)dx\,, \quad x \in \mathbb{R}^*\,.$$

Since $\int_{|x|\leq 1}|x|\nu(dx) = \infty$, the sample paths of L are a.s. of infinite variation in any finite interval. The moment generating function of $L(1)$ is for the NIG distribution given by

$$\widehat{f}(s) = E\exp(sL(1)) = \exp(\delta(\xi - \sqrt{\xi^2 - s^2}))\,,$$

(see e.g. Raible [19], Example 1.6) We use (2.5), (2.7) and (2.11) to obtain for $t \geq 0$ (again we set $r = 0$),

$$\begin{aligned}
X^\pi(t) &= x\exp(t\,\pi\, b)\widetilde{X}^\pi(t)\,,\\
E\left[X^\pi(t)\right] &= x\exp(t(\pi(b + \delta(\xi - \sqrt{\xi^2 - 1}))))\,,\\
\mathrm{var}\left(X^\pi(t)\right) &= x^2\exp(2t\left(\pi(b + \delta(\xi - \sqrt{\xi^2 - 1})))\right))\\
&\quad \times\left(\exp\left(\delta\pi^2 t\left(2\sqrt{\xi^2 - 1} - \xi - \sqrt{\xi^2 - 4}\right)\right) - 1\right)\,.
\end{aligned}$$

Example 3.6. *(Exponential variance gamma (VG) Lévy process)*
This normal-mean mixture model is of the same structure as the NIG model and has been suggested by Madan and Seneta [14]. Its non-symmetric version can be found in Madan, Carr and Chang [13]:

$$L(1) = \mu - \delta\zeta^2 + \zeta\varepsilon\,,$$

where $\mu, \delta \in \mathbb{R}$, ε is a standardnormal rv and $\zeta^2 \sim \Gamma(\xi, \theta)$ for parameters $\xi, \theta > 0$; i.e. ζ^2 has density

$$h(x; \xi, \theta) = \frac{x^{\xi-1}}{\Gamma(\xi)\theta^{\xi}} e^{-x/\theta}, \quad x > 0.$$

By conditioning on $\zeta^2(t)$ we obtain the characteristic function

$$\begin{aligned} E\exp(isL(t)) &= \exp(is\mu t)E[\exp(-(is\delta - s^2/2)\zeta^2(t)] \\ &= \frac{\exp(is\mu t)}{(1 - is\theta\delta + s^2\theta/2)^{\xi t}} \quad = \quad e^{t\Psi(s)}, \quad t \geq 0, \end{aligned}$$

where $\Psi(s) = i\mu s - \xi \ln(1 - is\theta\delta + s^2\theta/2)$. Thus $\mu = \gamma$, $\beta = 0$, hence L is a pure jump process with Lévy density

$$\nu(dx) = \frac{\xi}{|x|} \exp\left(-\sqrt{\frac{2}{\theta} + \delta^2}\,|x| - \delta x\right) dx, \quad x \in \mathbb{R}^*.$$

Since $\int_{|x|\leq 1} |x|\nu(dx) < \infty$, the sample paths of L are a.s. of finite variation in any finite interval; furthermore, those jumps are dense in $[0, \infty)$, since $\nu(\mathbb{R}) = \infty$; see Sato [21].

In order to calculate the wealth process and its mean and variance we use (2.4) and Remark 2.8(b). We observe that $E\exp(isL(1)) = e^{\Psi(s)}$ is analytic around 0, hence

$$\begin{aligned} \ln \widehat{f}(1) &= \Psi(-i) = \mu - \xi \ln(1 - \theta\delta - \theta/2) < \infty, \\ \ln \widehat{f}(2) &= \Psi(-2i) = 2\mu - \xi \ln(1 - 2\theta\delta - 2\theta) < \infty. \end{aligned}$$

Next we calculate

$$\begin{aligned} a &= \gamma + \xi \int\limits_{|x|\leq 1} \frac{x}{|x|} \exp\left(-\sqrt{\frac{2}{\theta} + \delta^2}\,|x| - \delta x\right) dx \\ &= \mu - \xi\theta\delta + \xi\theta\frac{c_2}{2}e^{c_1} - \xi\theta\frac{c_1}{2}e^{c_2}, \end{aligned}$$

where

$$c_1 = -\left(\sqrt{\frac{2}{\theta} + \delta^2} + \delta\right) \quad \text{and} \quad c_2 = -\left(\sqrt{\frac{2}{\theta} - \delta^2} + \delta\right).$$

We obtain for $t \geq 0$ (again we set $r = 0$)

$$\begin{aligned} X^\pi(t) &= x\exp(t\pi(b + \mu)) \prod_{s\leq t}(1 + \pi(e^{\Delta L(s)} - 1)) \\ E[X^\pi(t)] &= x(1 - \theta\delta - \theta/2)^{-\xi\pi t} \exp(t\pi(b + \mu))) \\ \operatorname{var}(X^\pi(t)) &= x^2(1 - \theta\delta - \theta/2)^{-2\xi\pi t} \exp(2t\pi(b + \mu)) \\ &\quad \times \left(\left(\frac{(1 - \theta\delta - \theta/2)^2}{1 - 2\theta\delta - 2\theta}\right)^{\xi\pi^2 t} - 1\right). \end{aligned}$$

Remark 3.7. Since Examples 3.5 and 3.6 have so many parameters, we can always attain the same expectation and variance for all three examples. But the shape of the distributions differs. Expectation and standard deviation are always increasing with the planning horizon T, which leads to a decreasing optimal portfolio. Note that the optimal portfolio is the same for all Lévy processes with the same mean and variance.

4 The Capital-at-Risk - calculation and approximation

In the portfolio problem of the last section, we replace now the variance by the Capital-at-Risk (CaR). Before we pose and solve the mean-CaR optimization problem, we define the CaR and indicate some properties. We further show how it can be determined (approximated) for a general Lévy process.

Definition 4.1. *Let x be the initial capital and T a given planning horizon. Let furthermore z_α be the α-quantile of the distribution of $\mathcal{E}(\pi\widehat{L}(T))$ for some portfolio $\pi \in [0,1]^d$, $\pi'\underline{1} \leq 1$, and $X^\pi(T)$ the corresponding terminal wealth. Then the* Value-at-Risk (VaR) *is given by*

$$\mathrm{VaR}(x,\pi,T) = \inf\{z \in \mathbb{R} : P(X^\pi(T) \leq z) \geq \alpha\} = xz_\alpha \exp((\pi'(b - r\underline{1}) + r)T)\,.$$

We define

$$\mathrm{CaR}(x,\pi,T) = xe^{rT} - \mathrm{VaR}(x,\pi,T) = xe^{rT}\left(1 - z_\alpha \exp(\pi'(b - r\underline{1})T)\right) \quad (4.1)$$

the Capital-at-Risk (CaR) *of the portfolio π (with initial capital x and time horizon T).*

The calculation of the CaR involves the quantile z_α of $\mathcal{E}(\pi\widehat{L}(T))$, which is quite a complicated object as we have seen in Lemma 2.5. To calculate its distribution explicitly is certainly not possible for Examples 3.5 and 3.6. One possibility would be to calculate the characteristic function of $\mathcal{E}(\pi\widehat{L}(T))$ using its characteristic triplet as given in Lemma 2.2. From this then one could approximate its density using the inverse Fast Fourier transform method, which is explained later in this section. However, the complicated expressions of its characteristic triplet in combination with the complicated integral in the Lévy-Khinchine formula seems to advise a different approach. As an alternative method we suggest an approximation method based on a weak limit theorem.

For simplicity we restrict ourselves to $d = 1$ and invoke a result by Asmussen and Rosinski [2]. The intuition behind is to approximate small jumps of absolute size smaller than ε by a simpler stochastic process, often by Brownian motion, such that the stochastic part of the Lévy process is approximated by an independent sum of a Brownian motion and a compound Poisson process. Before we study the applicability of such results to approximate quantiles of the wealth process, we explain the idea.

In a first step the small jumps with absolute size smaller than some $\varepsilon > 0$ are replaced by their expectation. This leads to the process

$$L_\varepsilon(t) = \mu(\varepsilon)t + \beta W(t) + N^\varepsilon(t)\,, \quad t \geq 0, \tag{4.2}$$

where

$$\mu(\varepsilon) = a - \int\limits_{\varepsilon \leq |x| \leq 1} x\nu(dx) \quad \text{and} \quad N^\varepsilon(t) \;=\; \sum_{s \leq t} \Delta L(s) 1_{\{|\Delta L(s)| \geq \varepsilon\}}\,.$$

Furthermore,

$$L(t) - L_\varepsilon(t) = \int_0^t \int\limits_{|x|<\varepsilon} x(M(dx,ds) - \nu(dx)ds)\,, \quad t \geq 0\,.$$

In a second step the contribution from the variation of small jumps is also incorporated. To this end we use the following representation

$$\begin{aligned} L(t) \;&=\; t\big(a - \int\limits_{\varepsilon<|x|\leq 1} x\nu(dx)\big) + \beta W(t) \\ &\quad + \sum_{0<s\leq t} \Delta L(s) 1_{\{|\Delta L(s)| \geq \varepsilon\}} + \int_0^t \int\limits_{|x|<\varepsilon} x(M(dx,ds) - \nu(dx)ds, \quad t \geq 0. \end{aligned}$$

In order to replace the small jumps for instance by some Gaussian term, we need that for $\varepsilon \to 0$ $(t \geq 0)$

$$\frac{1}{\sigma(\varepsilon)}(L(t) - (\mu(\varepsilon)t + \beta W(t) + N^\varepsilon(t))) = \sigma(\varepsilon)^{-1}(L(t) - L_\varepsilon(t)) \stackrel{D}{\rightarrow} W'(t)\,, \tag{4.3}$$

for some Brownian motion W', where

$$\sigma^2(\varepsilon) \;=\; \int\limits_{|x|<\varepsilon} x^2 \nu(dx), \quad \varepsilon > 0\,. \tag{4.4}$$

We denote by $\stackrel{D}{\rightarrow}$ weak convergence in $D[0,\infty)$ uniformly on compacta; see Pollard [17]. In the finite variation case (4.3) can be rewritten as $(t \geq 0)$

$$\frac{1}{\sigma(\varepsilon)} \left(\sum_{0<s\leq t} \Delta L(s) I(|\Delta L(s)| < \varepsilon) - E\left[\sum_{0<s\leq t} \Delta L(s) I(|\Delta L(s)| < \varepsilon) \right] \right) \stackrel{D}{\rightarrow} W'(t)\,.$$

This reminds us of the functional central limit theorem with Brownian motion as limit process. Here we can see that the standardized processes of the small jumps converge to Brownian motion as the jump size ε tends to 0. In fact, since Gaussian part and jump part are independent, the Brownian motion W' is independent of W, and this justifies the approximation in distribution

$$L(t) \approx \mu(\varepsilon)t + (\beta^2 + \sigma^2(\varepsilon))^{\frac{1}{2}} W(t) + N^\varepsilon(t)\,, \quad t \geq 0\,.$$

Proposition 4.2. *[Asmussen and Rosinski [2]]*
(a) A necessary and sufficient condition for (4.3) to hold is

$$\lim_{\varepsilon\to 0} \frac{\sigma(h\sigma(\varepsilon)\wedge\varepsilon)}{\sigma(\varepsilon)} = 1 \quad \forall h > 0\,. \tag{4.5}$$

(b) $\lim_{\varepsilon\to 0}\sigma(\varepsilon)/\varepsilon = \infty$ implies (4.5). If the Lévy measure has no atoms in some neighbourhood of 0, then condition (4.5) is equivalent to $\lim_{\varepsilon\to 0}\sigma(\varepsilon)/\varepsilon = \infty$.

We want to invoke such results to approximate quantiles of $\mathcal{E}(\pi\widehat{L}(T))$. We do this in two steps: firstly, we approximate $\mathcal{E}(\pi\widehat{L}(T))$, secondly, we use that convergence of distribution functions implies also convergence of their generalized inverses; see Proposition 0.1 of Resnick [20]. This gives us the approximation of the quantiles.

Lemma 4.3. *Recall model (2.1) and (2.2) for $d=1$ and $\sigma=1$; i.e. $L=\ln\mathcal{E}(\widehat{L})$ and $\widehat{L}$ are Lévy processes with Lévy measures ν and $\widehat{\nu}$ respectively. Then for $0<\varepsilon<1$*

$$\begin{aligned}
\sigma^2(\varepsilon) &= \int_{(-\varepsilon,\varepsilon)} x^2\nu(dx) &= \int_{(e^{-\varepsilon}-1,e^{\varepsilon}-1)} (\ln(1+x))^2\widehat{\nu}(dx)\,,\\
\widehat{\sigma}^2(\varepsilon) &= \int_{(-\varepsilon,\varepsilon)} x^2\widehat{\nu}(dx) &= \int_{(\ln(1-\varepsilon),\ln(1+\varepsilon))} (e^x-1)^2\nu(dx)\,,
\end{aligned}$$

Proof. The transformation from L to $\widehat{L}$ only affects the jumps, which are related by $\Delta L(s) = \ln(1+\Delta\widehat{L}(s))$ for $s\geq 0$. We calculate

$$\begin{aligned}
\sigma^2(\varepsilon) &= E\left[\sum_{s\leq 1}(\Delta L(s))^2 1_{\{|\Delta L(s)|<\varepsilon\}}\right]\\
&= E\left[\sum_{s\leq 1}(\ln(1+\Delta\widehat{L}(s)))^2 1_{\{e^{-\varepsilon}-1<\Delta\widehat{L}(s)<e^{\varepsilon}-1\}}\right]\\
&= \int_{(e^{-\varepsilon}-1,e^{\varepsilon}-1)} (\ln(1+x))^2\widehat{\nu}(dx)\,.
\end{aligned}$$

The calculation of $\widehat{\sigma}^2$ is analogous. □

We formulate the following main result of this section. The proof is postponed to Section 6.

Theorem 4.4. *Let Z^ε, $\varepsilon>0$, be real-valued Lévy processes without Gaussian component and $Y^\varepsilon=\ln\mathcal{E}(Z^\varepsilon)$ their logarithmic stochastic exponentials with characteristic triplets $(a_Z,0,\nu_Z)$ and $(a_Y,0,\nu_Y)$ as defined in Lemma 2.2; for* *notational convenience we suppress ε. Let $g : \mathbb{R} \to \mathbb{R}^+$ with $g(\varepsilon) \to 0$ as $\varepsilon \to 0$. Let V be a Lévy process. Then equivalent are as $\varepsilon \to 0$,*

$$\frac{Z^\varepsilon(t)}{g(\varepsilon)} \quad \xrightarrow{D} \quad V(t)\,, \quad t \geq 0\,, \tag{4.6}$$

$$\frac{Y^\varepsilon(t)}{g(\varepsilon)} \quad \xrightarrow{D} \quad V(t)\,, \quad t \geq 0\,.$$

We apply this result to approximate $\ln \mathcal{E}(\pi \widehat{L})$ for $\pi \in (0, 1]$ as follows:

Corollary 4.5. *Let L be a Lévy process and L_ε the process given in (4.2). Let furthermore $\mathcal{E}^{\leftarrow}(e^L) = \widehat{L}$ be such that $\mathcal{E}\widehat{L} = e^L$ with characteristic triplet given in Lemma 2.2. Let L_ε and $\widehat{L}_\varepsilon$ enjoy the same relationship as L and $\widehat{L}$. Then*

$$\sigma(\varepsilon)^{-1}(L(t) - L_\varepsilon(t)) \xrightarrow{D} V(t)\,, \quad t \geq 0\,, \tag{4.7}$$

is equivalent to

$$(\pi\sigma(\varepsilon))^{-1}\big(\ln \mathcal{E}(\pi \widehat{L}(t)) - \ln \mathcal{E}(\pi \widehat{L}_\varepsilon(t))\big) \xrightarrow{D} V(t)\,, \quad t \geq 0.$$

For the proof of this corollary we need the following Lemma.

Lemma 4.6. *Let L be a Lévy process and L_ε as defined in (4.2). Then*

$$\ln \mathcal{E}(\pi \mathcal{E}^{\leftarrow}(\exp(L(t) - L_\varepsilon(t)))) = \ln \mathcal{E}(\pi \widehat{L}(t)) - \ln \mathcal{E}(\pi \widehat{L}_\varepsilon(t)), \quad t \geq 0.$$

Proof. First note that

$$L(t) - L_\varepsilon(t) = \int_0^t \int_{|x|<\varepsilon} x(M(dx, ds) - \nu(dx)ds)\,, \quad t \geq 0\,,$$

Now calculate by Itô's formula $\ln \mathcal{E}(\pi \mathcal{E}^{\leftarrow}(\exp(L(t) - L_\varepsilon(t))))$, $\ln \mathcal{E}(\pi \widehat{L}(t))$ and $\ln \mathcal{E}(\pi \widehat{L}_\varepsilon(t))$. In the latter case we obtain and

$$\begin{aligned} \ln \mathcal{E}(\pi \widehat{L}_\varepsilon(t)) \quad &= \quad \pi t \big(a - \int_{\varepsilon<|x|\leq 1} x\nu(dx) + \frac{1}{2}(1-\pi)\beta^2\big) + \pi\beta W(t) \\ &\quad + \int_0^t \int_{|x|>\varepsilon} \ln(1 + \pi(e^x - 1))M(dx, ds)\,, \quad t \geq 0\,. \end{aligned} \tag{4.8}$$

Calculating the difference of the last two expressions leads to the assertion. □

Proof of Corollary 4.5. Setting $g(\varepsilon) := \sigma(\varepsilon)$ and $Y^\varepsilon := L - L_\varepsilon$ in Theorem 4.4 we obtain that (4.7) holds if and only if

$$\sigma(\varepsilon)^{-1}\mathcal{E}^{\leftarrow}(\exp(L(t) - L_\varepsilon(t))) \xrightarrow{D} V(t)\,, \quad t \geq 0\,. \tag{4.9}$$

Applying Theorem 4.4 to $g(\varepsilon) := \pi\sigma(\varepsilon)$ and $Z_\varepsilon := \pi\mathcal{E}^{\leftarrow}(\exp(L - L_\varepsilon))$ leads to the equivalence of (4.9) and

$$(\pi\sigma(\varepsilon))^{-1} \ln \mathcal{E}(\pi\mathcal{E}^{\leftarrow}(\exp(L(t) - L_\varepsilon(t)))) \xrightarrow{D} V(t), \quad t \geq 0.$$

Lemma 4.6 leads to the assertion of the Corollary. □

From this corollary and (4.8) we conclude the following approximation for $\ln \mathcal{E}(\pi\widehat{L})$, which is needed for the calculation of the CaR in Definition 4.1.

Proposition 4.7. *With the same notation as above we have*

$$\begin{aligned} \ln \mathcal{E}(\pi\widehat{L}(t)) &\approx \ln \mathcal{E}(\pi\widehat{L}_\varepsilon(t)) + \pi\sigma(\varepsilon)V(t) \\ &= \gamma_\pi^\varepsilon t + \pi\beta W(t) + M_\pi^\varepsilon(t) + \pi\sigma(\varepsilon)V(t)\,, \quad t \geq 0. \end{aligned}$$

If V is a Brownian motion, then

$$\begin{aligned} \ln \mathcal{E}(\pi\widehat{L}(t))) &\approx \gamma_\pi^\varepsilon t + \pi(\beta^2 + \sigma^2(\varepsilon))^{1/2} W(t) + M_\pi^\varepsilon(t)\,, \quad t \geq 0,, \\ \gamma_\pi^\varepsilon &= \pi(\mu(\varepsilon) + \frac{1}{2}\beta^2(1-\pi))\,, \\ M_\pi^\varepsilon(t) &= \sum_{s \leq t} \ln(1 + \pi(e^{\Delta L(s) 1_{\{|\Delta L(s)| > \varepsilon\}}} - 1))\,; \end{aligned}$$

i.e. M_π^ε is a compound Poisson process with jump measure

$$\nu_{M_\pi^\varepsilon}(\Lambda) = \nu_L(\{x : \ln(1 + \pi(e^x - 1)) \in \Lambda\} \backslash (-\varepsilon, \varepsilon))$$

for any Borel set $\Lambda \subset \mathbb{R}$. Moreover, if the Lévy measure ν_L has a Lebesgue density f_L, a density of the Lévy measure ν_M of the process M_π^ε is given by

$$f_M(x) = f_L\left(\ln\left(\frac{e^x - 1}{\pi} + 1\right)\right) \frac{e^x}{e^x - (1-\pi)} 1_{\{x > \ln(1-\pi)\}} 1_{\{|\ln((e^x - 1)/\pi + 1)| > \varepsilon\}}$$

and thus M_π^ε has Poisson intensity $\int_{\mathbb{R}} f_M(x)dx$; the density of the jump sizes of M_π^ε is given by $f_M(x) / \int f_M(y)dy, x \in \mathbb{R}$.

By Proposition 0.1 of Resnick [20] we obtain the corresponding approximation for the α-quantile z_α of $\mathcal{E}(\pi\widehat{L}(T))$, where T is some fixed planning horizon.

Proposition 4.8. *With the quantities as defined in Proposition 4.7 we obtain*

$$z_\alpha \approx z_\alpha^\varepsilon(\pi) = \inf\{z \in \mathbb{R} : P(\gamma_\pi^\varepsilon T + M_\pi^\varepsilon(T) + \pi\beta W(T) + \pi\sigma_L(\varepsilon)V(T) \leq \ln z) \geq \alpha\}\,.$$

Moreover, if V is a Brownian motion, then

$$z_\alpha \approx z_\alpha^\varepsilon(\pi) = \inf\{z \in \mathbb{R} : P(\gamma_\pi^\varepsilon T + M_\pi^\varepsilon(T) + \pi(\beta^2 + \sigma_L^2(\varepsilon))^{1/2} W(T) \leq \ln z) \geq \alpha\}\,.$$

We obtain

$$\begin{aligned} \mathrm{VaR}(x, \pi, T) &\approx x z_\alpha^\varepsilon(\pi) \exp((\pi(b - r) + r)T)\,, \\ \mathrm{CaR}(x, \pi, T) &\approx x \exp(rT) \left(1 - z_\alpha^\varepsilon(\pi) \exp((\pi(b - r) + r)T)\right)\,. \end{aligned}$$

Provided we know the distribution of the Lévy process V, i.e. its characteristic triplet, we have reduced the problem of the calculation of a low quantile of $\ln \mathcal{E}(\pi \widehat{L}(T))$ to the calculation of a low quantile of the sum of the compound Poisson variable $M_\pi^\varepsilon(T)$, the normal variable $\pi\beta W(T)$, and the limit variable $\pi\sigma(\varepsilon)V(T)$. Here we see immediately two simplifications of the original problem by this approximation. Firstly, the process V is usually much simpler than the original Lévy process and, secondly, π is now only a linear factor, which simplifies numerical procedures considerably. Various examples have been investigated in detail in Emmer [8] using the Fast Fourier Transform method.

5 Optimal portfolios under CaR constraints

We consider now the following optimization problem using the Capital-at-Risk as risk measure.

$$\max_{\pi \in [0,1]} E[X^\pi(T)] \quad \text{subject to} \quad \mathrm{CaR}(x, \pi, T) \le C\,, \tag{5.1}$$

where T is some given planning horizon and C is a given bound for the risk.

Unfortunately, there is no analogue of Theorem 3.1. Due to the fact that, immediately by (2.7), the mean wealth $E[X^\pi(T)]$ is increasing in π, the optimal solution of (5.1) is the largest $\pi \in [0,1]$ that satisfies the CaR constraint. This portfolio can be found by simple numerical iteration. For relevant examples we calculate the approximation of Proposition 4.8, which leads jointly with an approximation of the Fast Fourier Transform and the numerical iteration to an (approximate) optimal solution of the mean-CaR portfolio problem. For figures and discussions we refer to Emmer [8].

Example 5.1. *(Exponential normal inverse Gaussian (NIG) Lévy process)* Recall the model as defined in Example 3.5, where we set again $\lambda = \rho = 0$. For the calculation of the CaR we use the approximation of Proposition 4.8. Setting $f_L(x) = f_{nig}(x) = \xi\delta K_1(\xi|x|)/(\pi|x|)$, $x \in \mathbb{R}$, the Lévy density of the NIG Lévy process, the intensity of the compound Poisson process M_π^ε and the density of its jump sizes can be calculated as explained in Proposition 4.7. Plugging f_{nig} into definition (4.4) we obtain

$$\sigma^2(\varepsilon) = \frac{\xi\delta}{\pi} \int\limits_{|x|<\varepsilon} |x| K_1(\xi|x|)dx\,, \quad \varepsilon > 0\,.$$

As shown in Asmussen and Rosinski [2] for the NIG Lévy process the normal approximation for small jumps is allowed since $\sigma(\varepsilon) \sim (2\delta/\pi)^{1/2}\varepsilon^{1/2}$ as $\varepsilon \to 0$. Since $\beta = 0$ the approximating Lévy process has a Gaussian component with variance $\sigma^2(\varepsilon)$. Moreover, $a = 0$, hence

$$\mu(\varepsilon) = -\frac{\xi\delta}{\pi} \int\limits_{\varepsilon \le |x| \le 1} \frac{x}{|x|} K_1(\xi|x|)dx\,, \quad \varepsilon > 0\,.$$

Such integrals can be evaluated by a polynomial approximation for the modified Bessel function of the third kind (see Abramowitz and Stegun [1], pp. 378-379).

Example 5.2. *(Exponential variance gamma (VG) Lévy process)*
(a) As mentioned in Asmussen and Rosinski [2], for the gamma process with $\nu(dx) = \xi x^{-1} e^{-x/\delta} dx$, $x > 0$, with $\delta, \xi > 0$, the normal approximation for small jumps fails. This is a consequence of Proposition 4.2, since

$$\lim_{\varepsilon \to 0} \frac{\sigma^2(\varepsilon)}{\varepsilon^2} = \lim_{\varepsilon \to 0} \frac{\xi}{\varepsilon^2} \int_0^{\varepsilon} x e^{-x/\delta} dx = \frac{\xi}{2}\,, \tag{5.2}$$

using for instance l'Hospital's rule. The limit relations of Theorem 4.4 hold, however, with Lévy process V having characteristic triplet $(a_V, 0, \nu_V)$, where

$$a_V = \xi(1 - \sqrt{2/\xi}) \wedge 0 \;\text{ and }\; \nu_V(dy) = \frac{\xi}{y} 1_{(0,\sqrt{2/\xi})}(y) dy.$$

Proposition 4.7 gives then the approximation for the small jumps.
We show that (4.7) holds. Set

$$D_\varepsilon(t) := \sigma(\varepsilon)^{-1}(L(t) - L_\varepsilon(t))\,, \quad t \geq 0\,,$$

By Pollard [17], Theorem V.19, (4.7) is equivalent to $D_\varepsilon(1) \xrightarrow{D} V(1)$, since D_ε are Lévy processes. By Kallenberg [11], Theorem 13.14 we need to show for the characteristic triplets $(a_D, 0, \nu_D)$ of the Lévy processes D_ε

$$\lim_{\varepsilon \to 0} \nu_D([x,z]) = \nu_V([x,z]) \text{ for any } 0 < x < z \tag{5.3}$$

$$\lim_{\varepsilon \to 0} \int\limits_{|y|<K} y^2 \nu_D(dy) = \int\limits_{|y|<K} y^2 \nu_V(dy) \text{ for each } K > 0 \tag{5.4}$$

$$\lim_{\varepsilon \to 0} a_D = a_V \tag{5.5}$$

First we prove (5.3). By the proof of Theorem 2.1 of Asmussen and Rosinski [2] for the process D_ε we have

$$a_D = -\frac{\xi}{\sigma(\varepsilon)} \int\limits_{\sigma(\varepsilon)\wedge\varepsilon < y < \varepsilon} e^{-y/\delta} dy$$

and Lévy measure $\nu_D(B) = \nu(\sigma(\varepsilon)B \cap (0,\varepsilon))$ for any Borel set $B \subset \mathbb{R}^*$. Hence V has Lévy measure $\nu_V(B) = \lim_{\varepsilon\to 0} \nu(\sigma(\varepsilon)B \cap (0,\varepsilon))$. For any interval $[x,z]$, $0 < x < z$, we calculate

$$\lim_{\varepsilon \to 0} \nu_D([x,z]) = \lim_{\varepsilon \to 0} \xi \int\limits_{\varepsilon\wedge\sigma(\varepsilon)x}^{\varepsilon\wedge\sigma(\varepsilon)z} y^{-1} e^{-y/\delta} dy = \xi \ln\left(\frac{z \wedge \sqrt{2/\xi}}{x \wedge \sqrt{2/\xi}}\right) = \nu_V([x,z])\,,$$

where we have used that $e^{-y/\delta} \to 1$ as $y \to 0$.

Next we prove (5.4). For each $K > 0$ we calculate $\int\limits_{|y|<K} y^2 \nu_V(dy) = \frac{\xi K^2}{2} \wedge 1$

giving with (5.2)

$$\lim_{\varepsilon\to 0} \int\limits_{|y|<K} y^2 \nu_D(dy) = \lim_{\varepsilon\to 0} \frac{\sigma^2(K\sigma(\varepsilon)\wedge\varepsilon)}{\sigma^2(\varepsilon)} = \frac{\xi K^2}{2} \wedge 1\,.$$

Similarly we calculate

$$a_V = \lim_{\varepsilon\to 0} a_D = \lim_{\varepsilon\to 0} -\frac{\xi}{\sigma(\varepsilon)} \int\limits_{\sigma(\varepsilon)\wedge\varepsilon<y<\varepsilon} e^{-y/\delta} dy = \xi(1-\sqrt{2/\xi}) 1_{\{1-\sqrt{2/\xi}<0\}}$$

which proves (5.5).

(b) For the exponential VG Lévy process the normal approximation for small jumps is not possible either, since by Example 3.6 and e.g. l'Hospital's rule

$$\begin{aligned}
\lim_{\varepsilon\to 0} \frac{\sigma^2(\varepsilon)}{\varepsilon^2} &= \lim_{\varepsilon\to 0} \frac{\xi}{\varepsilon^2} \int_{-\varepsilon}^{\varepsilon} \frac{x^2}{|x|} \exp(-\sqrt{\frac{2}{\theta}+\delta^2}|x| - \delta x) dx \\
&= \lim_{\varepsilon\to 0} \frac{\xi}{\varepsilon^2} \int_0^{\varepsilon} x(\exp(c_1 x) + \exp(c_2 x)) dx \quad = \quad \xi\,,
\end{aligned}$$

where $c_1 = -\left(\sqrt{\frac{2}{\theta}+\delta^2}+\delta\right) < 0$ and $c_2 = -\left(\sqrt{\frac{2}{\theta}+\delta^2}-\delta\right) < 0$.
As in part (a) we show (5.3)-(5.5) and obtain a limit process V with characteristic triplet $(0, 0, \nu_V)$, where

$$\nu_V(dy) = \frac{\xi}{y} 1_{(-1/\sqrt{\xi},\, 1/\sqrt{\xi})}(y) dy\,.$$

In the same way as for the normal approximation one can calculate quantiles for approximations of small jumps by the Lévy process V using the FFT method; see Mauthner [15].

6 Proof of Theorem 4.4

We first derive some auxiliary results. As usual we write

$$a\Lambda := \{ax \,|\, x\in\Lambda\},\ e^{\Lambda} := \{e^x \,|\, x\in\Lambda\},\ \text{and}\ \Lambda - 1 := \{x-1 \,|\, x\in\Lambda\}.$$

Lemma 6.1. *Let Z_ε and Y^ε be Lévy processes with characteristic triplets as in Theorem 4.4. Set*

$$E_\varepsilon := \frac{Z_\varepsilon}{g(\varepsilon)} \quad \text{and} \quad D_\varepsilon := \frac{Y^\varepsilon}{g(\varepsilon)}$$

Then E_ε is a Lévy process with characteristic triplet $(a_E, 0, \nu_E)$ and D_ε is a Lévy process with characteristic triplet $(a_D, 0, \nu_D)$, which both depend on ε. They *satisfy the following relations:*

$$\begin{aligned}
a_E &= \frac{1}{g(\varepsilon)}\big(a_Z - \int\limits_{g(\varepsilon)<|x|\le 1} x\nu_Z(dx)\big), \\
\nu_E(\Lambda) &= \nu_Z(g(\varepsilon)\Lambda) = \nu_Y(\{x|(e^x-1)/g(\varepsilon)\in\Lambda\}) \ \textit{for any Borel set } \Lambda\subset\mathbb{R}^*, \\
a_D &= \frac{1}{g(\varepsilon)}\big(a_Y - \int\limits_{g(\varepsilon)<|x|\le 1} x\nu_Y(dx)\big), \\
\nu_D(\Lambda) &= \nu_Y(g(\varepsilon)\Lambda) = \nu_Z(e^{g(\varepsilon)\Lambda}-1) \ \textit{for any Borel set } \Lambda\subset\mathbb{R}^*, \\
a_D - a_E &= \frac{1}{g(\varepsilon)}\int(\ln(x+1)1_{\{|\ln(x+1)|\le g(\varepsilon)\}} - x1_{\{|x|\le g(\varepsilon)\}})\nu_Z(dx). \\
&= \frac{1}{g(\varepsilon)}\int(x1_{\{|x|\le g(\varepsilon)\}} - (e^x-1)1_{\{|e^x-1|\le g(\varepsilon)\}})\nu_Y(dx).
\end{aligned}$$

Proof. Since E_ε and D_ε have no Gaussian component, $\beta_E=\beta_D=0$.
Using Lemmata 2.2 and 2.3 and setting $\pi = 1/g(\varepsilon)$ we obtain for any Borel set $\Lambda\subset\mathbb{R}^*$,

$$\nu_E(\Lambda) \quad = \quad \nu_Z(g(\varepsilon)\Lambda) = \nu_Y(\{x|(e^x-1)/g(\varepsilon)\in\Lambda\})$$

and analogously,

$$\nu_D(\Lambda) = \nu_Y(g(\varepsilon)\Lambda) = \nu_Z(\{x|\ln(x+1)/g(\varepsilon)\in\Lambda\}).$$

Moreover,

$$\begin{aligned}
a_E &= \frac{1}{g(\varepsilon)}a_Z + \frac{1}{g(\varepsilon)}\int x(1_{\{|x|\le g(\varepsilon)\}} - 1_{\{|x|\le 1\}})\nu_Z(dx) \\
&= \frac{1}{g(\varepsilon)}\big(a_Z - \int\limits_{g(\varepsilon)<|x|\le 1} x\nu_Z(dx)\big).
\end{aligned}$$

In a similar way we prove

$$a_D = \frac{1}{g(\varepsilon)}\big(a_Y - \int\limits_{g(\varepsilon)<|x|\le 1} x\nu_Y(dx)\big).$$

Using Lemma 2.2 we obtain

$$\begin{aligned}
a_D - a_E &= \frac{1}{g(\varepsilon)}\big(a_Y - a_Z + \int\limits_{g(\varepsilon)<|x|\le 1} x(\nu_Z-\nu_Y)(dx)\big) \\
&= \frac{1}{g(\varepsilon)}\int\left(\ln(x+1)1_{\{|\ln(x+1)|<g(\varepsilon)\}} - x1_{\{|x|<g(\varepsilon)\}}\right)\nu_Z(dx).
\end{aligned}$$

□

Lemma 6.2. *Let $K : \mathbb{R}^+ \to \mathbb{R}^+$ and $g : \mathbb{R}^+ \to \mathbb{R}^+$ be such that $g(\varepsilon) \to 0$ as $\varepsilon \to 0$. Then*

$$\lim_{\varepsilon\to 0} \frac{1}{g^2(\varepsilon)} \int\limits_{(-hg(\varepsilon),hg(\varepsilon))} x^2 \nu_Z(dx) = K(h) \quad \forall h > 0 \tag{6.1}$$

if and only if

$$\lim_{\varepsilon\to 0} \frac{1}{g^2(\varepsilon)} \int\limits_{A_{\varepsilon,h}} (\ln(x+1))^2 \nu_Z(dx) = K(h) \quad \forall h > 0\,,$$

where $A_{\varepsilon,h} := (\exp(-hg(\varepsilon)) - 1, \exp(hg(\varepsilon)) - 1)$ *for each* $\varepsilon, h > 0$.

Proof. Set $\nu = \nu_Z$. Let $h > 0$. Since $g(\varepsilon) \to 0$ as $\varepsilon \to 0$, there exists some $\tilde{\varepsilon} > 0$ such that $hg(\varepsilon) < 1$ for all $0 < \varepsilon < \tilde{\varepsilon}$. By a Taylor expansion we have for some $\theta \in (0,1)$

$$e^{hg(\varepsilon)} - 1 = hg(\varepsilon)e^{\theta hg(\varepsilon)}$$

and hence

$$e^{-1}hg(\varepsilon) < hg(\varepsilon) < e^{hg(\varepsilon)} - 1 < ehg(\varepsilon)$$

and, analogously,

$$-ehg(\varepsilon) < -hg(\varepsilon) < e^{-hg(\varepsilon)} - 1 < -e^{-1}hg(\varepsilon)\,.$$

This leads to

$$(-K_1 g(\varepsilon), K_1 g(\varepsilon)) \subseteq A_{\varepsilon,h} \subseteq (-K_2 g(\varepsilon), K_2 g(\varepsilon)) \tag{6.2}$$

for $K_1 = e^{-1}h$ and $K_2 = eh$.

Assume that (6.1) holds. Then by a Taylor expansion around 0 we have for some $\theta = \theta(x) \in (0,1)$

$$\ln(x+1) = x - \frac{x^2}{2(\theta x + 1)^2}$$

giving

$$\begin{aligned}
&\frac{1}{g^2(\varepsilon)} \int\limits_{A_{\varepsilon,h}} (\ln(x+1))^2 \nu(dx) \\
&= \frac{1}{g^2(\varepsilon)} \int\limits_{A_{\varepsilon,h}} x^2 \nu(dx) - \frac{1}{g^2(\varepsilon)} \int\limits_{A_{\varepsilon,h}} \frac{x^3}{(\theta x+1)^2} \nu(dx) + \frac{1}{g^2(\varepsilon)} \int\limits_{A_{\varepsilon,h}} \frac{x^4}{4(\theta x+1)^4} \nu(dx) \\
&= I_1(\varepsilon) - I_2(\varepsilon) + I_3(\varepsilon)\,. \qquad (6.3)
\end{aligned}$$

First note that with (6.1) and (6.2),

$$
\begin{aligned}
&|I_2(\varepsilon) - I_3(\varepsilon)| \\
&\leq \frac{1}{g^2(\varepsilon)} \left(\left| \int\limits_{A_{\varepsilon,h}} \frac{x^3}{(\theta x+1)^2} \nu(dx) \right| + \int\limits_{A_{\varepsilon,h}} \frac{x^4}{4(\theta x+1)^4} \nu(dx) \right) \qquad (6.4) \\
&\leq \frac{1}{g^2(\varepsilon)} \left(\int\limits_{(-K_2 g(\varepsilon), K_2 g(\varepsilon))} \left| \frac{x^3}{(\theta x+1)^2} \right| \nu(dx) + \int\limits_{(-K_2 g(\varepsilon), K_2 g(\varepsilon))} \frac{x^4}{4(\theta x+1)^4} \nu(dx) \right) \\
&\leq \left(\sup_{x \in (-K_2 g(\varepsilon), K_2 g(\varepsilon))} \left(\frac{|x|}{(\theta x+1)^2} + \frac{x^2}{4(\theta x+1)^4} \right) \right) \frac{1}{g^2(\varepsilon)} \int\limits_{(-K_2 g(\varepsilon), K_2 g(\varepsilon))} x^2 \nu(dx) \\
&\leq \left(\frac{K_2 g(\varepsilon)}{(1-K_2 g(\varepsilon))^2} + \frac{(K_2 g(\varepsilon))^2}{4(1-K_2 g(\varepsilon))^4} \right) \frac{1}{g^2(\varepsilon)} \int\limits_{(-K_2 g(\varepsilon), K_2 g(\varepsilon))} x^2 \nu(dx) \\
&\to \; 0\,, \quad \varepsilon \to 0\,. \qquad (6.5)
\end{aligned}
$$

Hence, setting $C_{\varepsilon,h}(s,t) = (-hg(\varepsilon)\exp(shg(\varepsilon)), hg(\varepsilon)\exp(thg(\varepsilon)))$,

$$
\lim_{\varepsilon \to 0} \frac{1}{g^2(\varepsilon)} \int\limits_{A_{\varepsilon,h}} (\ln(x+1))^2 \nu(dx) = \lim_{\varepsilon \to 0} I_1(\varepsilon) = \lim_{\varepsilon \to 0} \frac{1}{g^2(\varepsilon)} \int_{C_{\varepsilon,h}(-\theta_1,\theta_2)} x^2 \nu(dx)
$$

for some θ_1, $\theta_2 \in (0,1)$ using a Taylor expansion. Thus,

$$
\lim_{\varepsilon \to 0} \frac{1}{g^2(\varepsilon)} \int\limits_{C_{\varepsilon,h}(-1,-1)} x^2 \nu(dx) \;\leq\; \lim_{\varepsilon \to 0} I_1(\varepsilon) \;\leq\; \lim_{\varepsilon \to 0} \frac{1}{g^2(\varepsilon)} \int_{C_{\varepsilon,h}(1,1)} x^2 \nu(dx).
$$

Since $g(\varepsilon) \to 0$ as $\varepsilon \to 0$, we obtain for arbitrary $\varepsilon_0 > 0$ and all $\varepsilon < \varepsilon_0$ an upper bound for the right-hand side

$$
\lim_{\varepsilon \to 0} \frac{1}{g^2(\varepsilon)} \int\limits_{|x| < hg(\varepsilon)\exp(hg(\varepsilon_0))} x^2 \nu(dx) = K(h \exp(hg(\varepsilon_0)))\,.
$$

Since ε_0 can be chosen arbitrarily small, we obtain under condition (6.1)

$$
\lim_{\varepsilon \to 0} \frac{1}{g^2(\varepsilon)} \int\limits_{|x| < hg(\varepsilon)\exp(hg(\varepsilon))} x^2 \nu(dx) = K(h). \qquad (6.6)
$$

Similarly, we get a lower bound and hence

$$
\lim_{\varepsilon \to 0} \frac{1}{g^2(\varepsilon)} \int\limits_{|x| < hg(\varepsilon)\exp(-hg(\varepsilon))} x^2 \nu(dx) = K(h)
$$

and thus also

$$
\lim_{\varepsilon \to 0} I_1(\varepsilon) = K(h).
$$

For the converse first note that by (6.4)

$$
\begin{aligned}
&|I_2(\varepsilon) - I_3(\varepsilon)| \\
&\quad \le \left(\sup_{x\in A_{\varepsilon,h}} \left(\frac{|x|}{(\theta x+1)^2} + \frac{x^2}{4(\theta x+1)^4} \right) \right) \frac{1}{g^2(\varepsilon)} \int\limits_{A_{\varepsilon,h}} x^2 \nu(dx) \\
&\quad \le \left(\frac{\exp(g(\varepsilon)h) - 1}{\exp(-2g(\varepsilon)h)} + \frac{(\exp(g(\varepsilon)h)-1)^2}{4\exp(-4g(\varepsilon)h)} \right) I_1(\varepsilon) \qquad (6.7)
\end{aligned}
$$

and hence $|I_2(\varepsilon) - I_3(\varepsilon)| \le T(\varepsilon) I_1(\varepsilon)$ for some positive $T(\varepsilon) \to 0$ as $\varepsilon \to 0$. So by (6.3)

$$I_1(\varepsilon) \le \frac{1}{g^2(\varepsilon)} \int\limits_{A_{\varepsilon,h}} (\ln(x+1))^2 \nu(dx) + T(\varepsilon) I_1(\varepsilon)$$

and hence

$$I_1(\varepsilon)(1 - T(\varepsilon)) \le \frac{1}{g^2(\varepsilon)} \int\limits_{A_{\varepsilon,h}} (\ln(x+1))^2 \nu(dx).$$

Taking limsup results in $\limsup_{\varepsilon\to 0} I_1(\varepsilon) \le K(h)$. Then by (6.7) $|I_2(\varepsilon) - I_3(\varepsilon)| \to 0$ and by (6.3) we obtain $\lim_{\varepsilon\to 0} I_1(\varepsilon) = K(h)$ for each $h > 0$. Using the same argument as for (6.6),

$$
\begin{aligned}
K(h) &= \lim_{\varepsilon\to 0} I_1(\varepsilon) \;\le\; \lim \frac{1}{g^2(\varepsilon)} \int\limits_{|x|<hg(\varepsilon)\exp(hg(\varepsilon))} x^2 \nu(dx) \\
&= \lim \frac{1}{g^2(\varepsilon)} \int\limits_{|x|<hg(\varepsilon)} x^2 \nu(dx) \\
&= \lim \frac{1}{g^2(\varepsilon)} \int\limits_{|x|<hg(\varepsilon)\exp(-hg(\varepsilon))} x^2 \nu(dx) \\
&\le \lim_{\varepsilon\to 0} I_1(\varepsilon) \;=\; K(h)
\end{aligned}
$$

we obtain (6.1). □

The following Lemma can be considered as an inverse version to Lemma 6.2. Its proof is indeed quite similar and we refer the interested reader to Emmer [8].

Lemma 6.3. *Let* $K : \mathbb{R}^+ \to \mathbb{R}^+$ *and* $g : \mathbb{R}^+ \to \mathbb{R}^+$ *be such that* $g(\varepsilon) \to 0$ *as* $\varepsilon \to 0$. *Then*

$$\lim_{\varepsilon\to 0} \frac{1}{g^2(\varepsilon)} \int\limits_{(-hg(\varepsilon),hg(\varepsilon))} x^2 \nu_Y(dx) = K(h) \quad \forall h > 0$$

if and only if

$$\lim_{\varepsilon\to 0} \frac{1}{g^2(\varepsilon)} \int\limits_{B_{\varepsilon,h}} (e^x - 1)^2 \nu_Y(dx) = K(h) \quad \forall h > 0\,,$$

where $B_{\varepsilon,h} := (\ln(1-hg(\varepsilon)), \ln(1+hg(\varepsilon)))$ *for each* $\varepsilon, h > 0$.

Now we can prove Theorem 4.4.

Proof of Theorem 4.4. Assume that (4.6) holds, i.e. $E_\varepsilon(t) = Z^\varepsilon(t)/g(\varepsilon) \xrightarrow{D} V(t)$, $t \geq 0$, as $\varepsilon \to 0$. Since E_ε are Lévy processes weak convergence of the processes is equivalent to $E_\varepsilon(1) \xrightarrow{D} V(1)$ (see e.g. Pollard [17], Theorem V.19). Let now $(a_E, 0, \nu_E)$ be the characteristic triplets of the Lévy processes E_ε as derived in Lemma 6.1 (recall that they depend on ε). Since $\beta_E = 0$, according to Kallenberg [11], Theorem 13.14, $E_\varepsilon(1) \xrightarrow{D} V(1)$ if and only if

$$\begin{aligned}
\lim_{\varepsilon\to 0} \int\limits_{|x|<h} x^2 \nu_E(dx) &= \beta_V^2 + \int\limits_{|x|<h} x^2 \nu_V(dx) \quad \forall h > 0\,, &(6.8)\\
\lim_{\varepsilon\to 0} \nu_E(\{|x| \geq c\}) &= \nu_V(\{|x| \geq c\}) \quad \forall c > 0\,, &(6.9)\\
\lim_{\varepsilon\to 0} a_E &= a_V\,. &(6.10)
\end{aligned}$$

So we assume that (6.8)-(6.10) hold.
Moreover, setting $D_\varepsilon = Y^\varepsilon/g(\varepsilon)$ with characteristic triplets $(a_D, 0, \nu_D)$, we have to show

$$\begin{aligned}
\lim_{\varepsilon\to 0} \int\limits_{|x|<h} x^2 \nu_D(dx) &= \beta_V^2 + \int\limits_{|x|<h} x^2 \nu_V(dx) \quad \forall h > 0\,, &(6.11)\\
\lim_{\varepsilon\to 0} \nu_D(\{|x| \geq c\}) &= \nu_V(\{|x| \geq c\}) \quad \forall c > 0\,, &(6.12)\\
\lim_{\varepsilon\to 0} a_D &= a_V\,. &(6.13)
\end{aligned}$$

To prove (6.11) we consider

$$\begin{aligned}
\int\limits_{|x|<h} x^2 \nu_D(dx) &= E\left[\sum_{s\leq 1} (\Delta D_\varepsilon(s))^2 1_{\{|\Delta D_\varepsilon(s)|<h\}}\right]\\
&= \frac{1}{g^2(\varepsilon)} E\left[\sum_{s\leq 1} (\ln(1+\Delta Z_\varepsilon(s)))^2 1_{\{\Delta Z_\varepsilon(s)\in A_{\varepsilon,h}\}}\right]\\
&= \frac{1}{g^2(\varepsilon)} \int\limits_{A_{\varepsilon,h}} (\ln(x+1))^2 \nu_Z(dx)\,, &(6.14)
\end{aligned}$$

where $A_{\varepsilon,h} = (e^{-\sigma(\varepsilon)h} - 1, e^{\sigma(\varepsilon)h} - 1)$. By (6.8) and Lemma 6.2, setting $K(h) = \beta_V^2 + \int_{|x|<h} x^2 \nu_V(dx)$ the right-hand side of (6.14) converges to $\beta_V^2 + \int_{|x|<h} x^2 \nu_V(dx)$ for each $h > 0$.

Now we prove (6.12). By Lemma 6.1 we have

$$\nu_D(\{|x| \geq c\}) \;=\; \nu_Z(e^{g(\varepsilon)\{|x|\geq c\}}-1)$$
$$=\;\; \nu_Z(e^{g(\varepsilon)\{|x|\geq c\}}-1\cap\{|x|\geq cg(\varepsilon)\})+\nu_Z(e^{g(\varepsilon)\{|x|\geq c\}}-1\cap\{|x|< cg(\varepsilon)\})$$

The first term converges to $\nu_V(\{|x|\geq c\})$, since by (6.9)

$$\nu_Z(\{|x|\geq cg(\varepsilon)\}) = \nu_E(\{|x|\geq c\}) \to \nu_V(\{|x|\geq c\})\,, \quad \varepsilon\to 0\,.$$

Since for any Borel set $\Lambda\subset\mathbb{R}^*$

$$\nu_Z(\Lambda)\inf_{x\in\Lambda}(\ln(x+1))^2 \leq \int\limits_{\Lambda}(\ln(x+1))^2\nu_Z(dx)$$

holds, we get

$$\begin{aligned}
&\nu_Z(e^{g(\varepsilon)\{|x|\geq c\}}-1\cap\{|x|< cg(\varepsilon)\})\\
=\;\;& \nu_Z(\{|x|< cg(\varepsilon)\}\backslash(e^{g(\varepsilon)\{|x|<c\}}-1))\\
\leq\;\;& \frac{1}{(cg(\varepsilon))^2}\int\limits_{\{|x|<cg(\varepsilon)\}\backslash(e^{g(\varepsilon)\{|x|<c\}}-1)}(\ln(1+x))^2\nu_Z(dx)\\
=\;\;& \frac{1}{(cg(\varepsilon))^2}\int\limits_{\{|x|<cg(\varepsilon)\}}(\ln(1+x))^2\nu_Z(dx)\\
&-\frac{1}{(cg(\varepsilon))^2}\int\limits_{\{|x|<cg(\varepsilon)\}\cap(e^{g(\varepsilon)\{|x|<c\}}-1)}(\ln(1+x))^2\nu_Z(dx)\\
\to\;\;& 0, \quad \varepsilon\to 0,
\end{aligned}$$

since both terms in the second last line tend to $K(c)/c^2$, where $K(h) = \beta_V^2+\int_{|x|<h}x^2\nu_V(dx)$. This can be seen as follows. For the first term we use Taylor's theorem in the same way as in the proof of Lemma 6.2 replacing $A_{\varepsilon,h}$ by $(-cg(\varepsilon),cg(\varepsilon))$. The second term tends to $K(c)/c^2$ using the same Taylor expansion and since by a Taylor expansion for e^x-1 around 0

$$\{|x|< cg(\varepsilon)\}\cap(e^{g(\varepsilon)\{|x|<c\}}-1) \;=\; (-cg(\varepsilon)e^{-\theta_1 cg(\varepsilon)}, cg(\varepsilon))$$

for some θ_1, $\theta_2\in(0,1)$.

Now we prove (6.13). By (6.10) we know that $a_E\to a_V$, hence we only need to show

$$|a_D-a_E|\to 0.$$

By Lemma 6.1 and the Taylor expansion we use in (6.3) setting $D_\varepsilon = \{|\ln(x+1)|< g(\varepsilon)\} = \{e^{-g(\varepsilon)}-1< x< e^{g(\varepsilon)}-1\} = \{e^{g(\varepsilon)\{|x|<1\}}-1\}$, we obtain for some $\theta \in (0,1)$

$$\begin{aligned}
|a_D - a_E| &= \frac{1}{g(\varepsilon)} \left| \int (\ln(x+1) 1_{D_\varepsilon})\, \nu_Z(dx) \right| \\
&= \frac{1}{g(\varepsilon)} \left| \int \left(\left(x - \frac{x^2}{2(\theta x+1)^2} \right) 1_{D_\varepsilon} - x 1_{\{|x|<g(\varepsilon)\}} \right) \nu_Z(dx) \right| \\
&= \frac{1}{g(\varepsilon)} \left| \int (x 1_{D_\varepsilon} - x 1_{\{|x|<g(\varepsilon)\}}) \nu_Z(dx) - \int\limits_{D_\varepsilon} \frac{x^2}{2(\theta x+1)^2} \nu_Z(dx) \right| .
\end{aligned}$$

From

$$\frac{1}{2(\theta x+1)^2} < \frac{1}{2e^{-2g(\varepsilon)}} \quad \text{for} \quad x \in (e^{-g(\varepsilon)} - 1, e^{g(\varepsilon)} - 1)$$

we conclude

$$\lim_{\varepsilon \to 0} \frac{1}{g(\varepsilon)} \int\limits_{D_\varepsilon} \frac{x^2}{2(\theta x+1)^2} \nu_Z(dx) = 0 \, .$$

We obtain

$$\begin{aligned}
&\limsup_{\varepsilon \to 0} |a_D - a_E|^2 \\
&= \limsup_{\varepsilon \to 0} \frac{1}{g^2(\varepsilon)} \left| \int x (1_{D_\varepsilon} - 1_{\{|x|<g(\varepsilon)\}}) \nu_Z(dx) \right|^2 \\
&= \limsup_{\varepsilon \to 0} \frac{1}{g^2(\varepsilon)} \left| \int\limits_{D_\varepsilon \setminus \{|x|<g(\varepsilon)\}} x \nu_Z(dx) - \int\limits_{\{|x|<g(\varepsilon)\} \setminus D_\varepsilon} x \nu_Z(dx) \right|^2 \\
&\le \limsup_{\varepsilon \to 0} \frac{1}{g^2(\varepsilon)} \int\limits_{D_\varepsilon \setminus \{|x|<g(\varepsilon)\}} x^2 \nu_Z(dx) + \limsup_{\varepsilon \to 0} \frac{1}{g^2(\varepsilon)} \int\limits_{\{|x|<g(\varepsilon)\} \setminus D_\varepsilon} x^2 \nu_Z(dx) \, .
\end{aligned}$$

Both terms converge to 0 as follows.

$$\lim_{\varepsilon \to 0} \frac{1}{g^2(\varepsilon)} \int\limits_{D_\varepsilon} x^2 \nu_Z(dx) = K(1)$$

by the proof of Lemma 6.2,

$$\lim_{\varepsilon \to 0} \frac{1}{g^2(\varepsilon)} \int\limits_{\{|x|<g(\varepsilon)\}} x^2 \nu_Z(dx) = K(1)$$

by (6.8) for $h = 1$, and

$$\lim_{\varepsilon \to 0} \frac{1}{g^2(\varepsilon)} \int\limits_{\{|x|<g(\varepsilon)\} \cap D_\varepsilon} x^2 \nu_Z(dx) = K(1),$$

where $K(h) = \beta_V^2 + \int\limits_{|x|<h} x^2 \nu_V(dx)$, since by a Taylor expansion of $e^x - 1$ around 0

$$\{|x| < g(\varepsilon)\} \cap D_\varepsilon \quad = \quad (-g(\varepsilon)e^{\theta_1 g(\varepsilon)}, g(\varepsilon))$$

for some θ_1, $\theta_2 \in (0,1)$ and using the same argumentation as in the proof of Lemma 6.2.

The other direction can be proved analogously. □

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