The Relation Between Implied And Realised Probability Density Functions

By Iliana Anagnou, Mascia Bedendo, Stewart Hodges and Robert Tompkins

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Realised Probability Density Functions

OUTLINE OF PRESENTATION

1. Motivation for the Research

2. Review of the Literature

3. Alternative Methods for Estimating Risk Neutral Densities

4. Testing Forecast Ability of Risk Neutral Densities

5. Empirical Results

6. Conclusions and Suggestions for Future Research

The Relation Between Implied And Realised Probability Density Functions

1. Motivation

A number of financial regulators have suggested that risk neutral densities (RND) associated with options markets could provide useful indicators of future market turbulence.
... but do such RNDs provide an unbiased forecast of realised probability density functions?

We test the ability of RNDs for options on the S&P 500 and the British Pound / US Dollar to predict future probability densities.
We consider five approaches to estimate the RNDs, which are consistent with approaches proposed and used by financial regulators.
We also provide a number of new testing procedures to assess the efficiency and unbiasness of the forecasts.
These tests provide more power than the usual Komolgorov/Smirnov tests.

2. Review Of The Literature

A. Information Content of Implied Volatilities & Risk Neutral Density Functions.

- Canina & Figlewski (1993), Jorion (1995), Christensen & Prabhala (1998)
- Bates (1991, 2000), Gemmill & Saflekos (1999)
- Dumas, Fleming & Whaley (1998)
- Bahra (1997), McManus (1999), Campa, et. al. (1999)
- Bliss & Panigirtzoglou (2001), Weinberg (2001)

B. Alternative Approaches to Risk Neutral Density Estimation

- Black & Scholes (1973) plus adjustments
  - Jarrow & Rudd (1982), Corrado & Su (1996, 1997)
- Implied Distribution Approach
  - Dupire (1992)., Derman & Kani (1994), Rubinstein (1994)
- Alternative Functional Forms
  - Bates (1991), Ritchey (1990), Bahra (1997), Madan, Carr & Chang (1998)
- Non-parametric Approaches
  - Breeden & Litzenberger (1978), Shimko (1991, 1993), Ait-Sahalia & Lo (1998), Jackwerth (1999)

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2. Review of The Literature

3. Testing Density Forecasts

- Probability Integral Transform (PIT) Approach
    - Fisher (1930), Rosenblatt (1952), Dawid (1984), Diebold, Gunther & Tay (1998)
- Kolmogorov- Smirnov
    - Cnkovic & Drachman (1997), Berkowitz (1999), De Raaij & Raunig (2000)

3. Empirical Results

A. Data Sources For Empirical Research

- Standard & Poor's 500 Options & Futures, March 86 – September 2000;
- US Dollar/British Pound Options & Futures, March 85 – September 2000;

B. Determination of Risk Neutral Densities

- Summary Statistic Tables for RND estimates
- Figures for RND estimates
- S&P 500
- British Pound / US Dollar
- Skewness and Excess Kurtosis Plots

C. Are RNDs good Predictors?

- Tables of Results

D. Interpreting the Results

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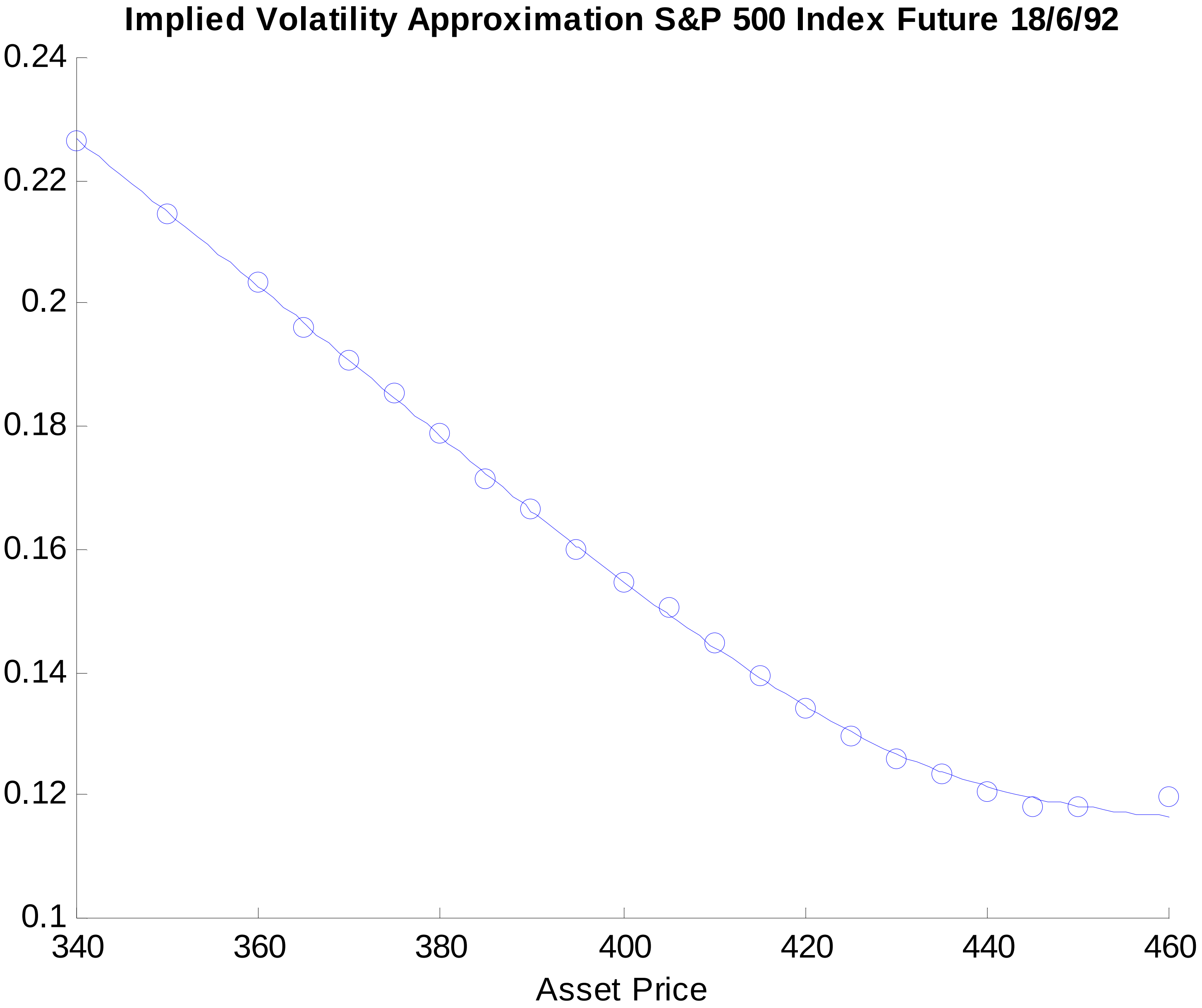

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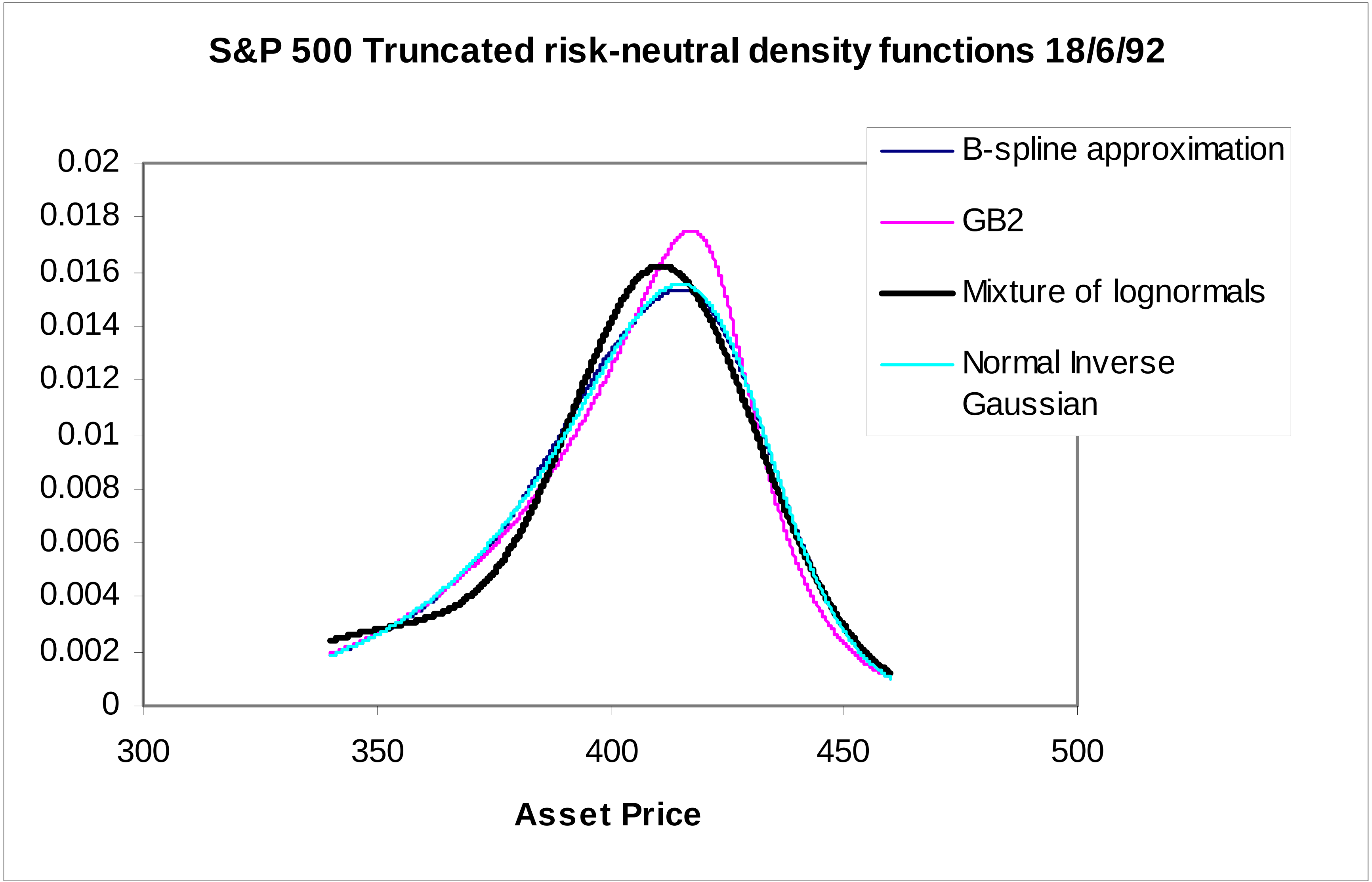

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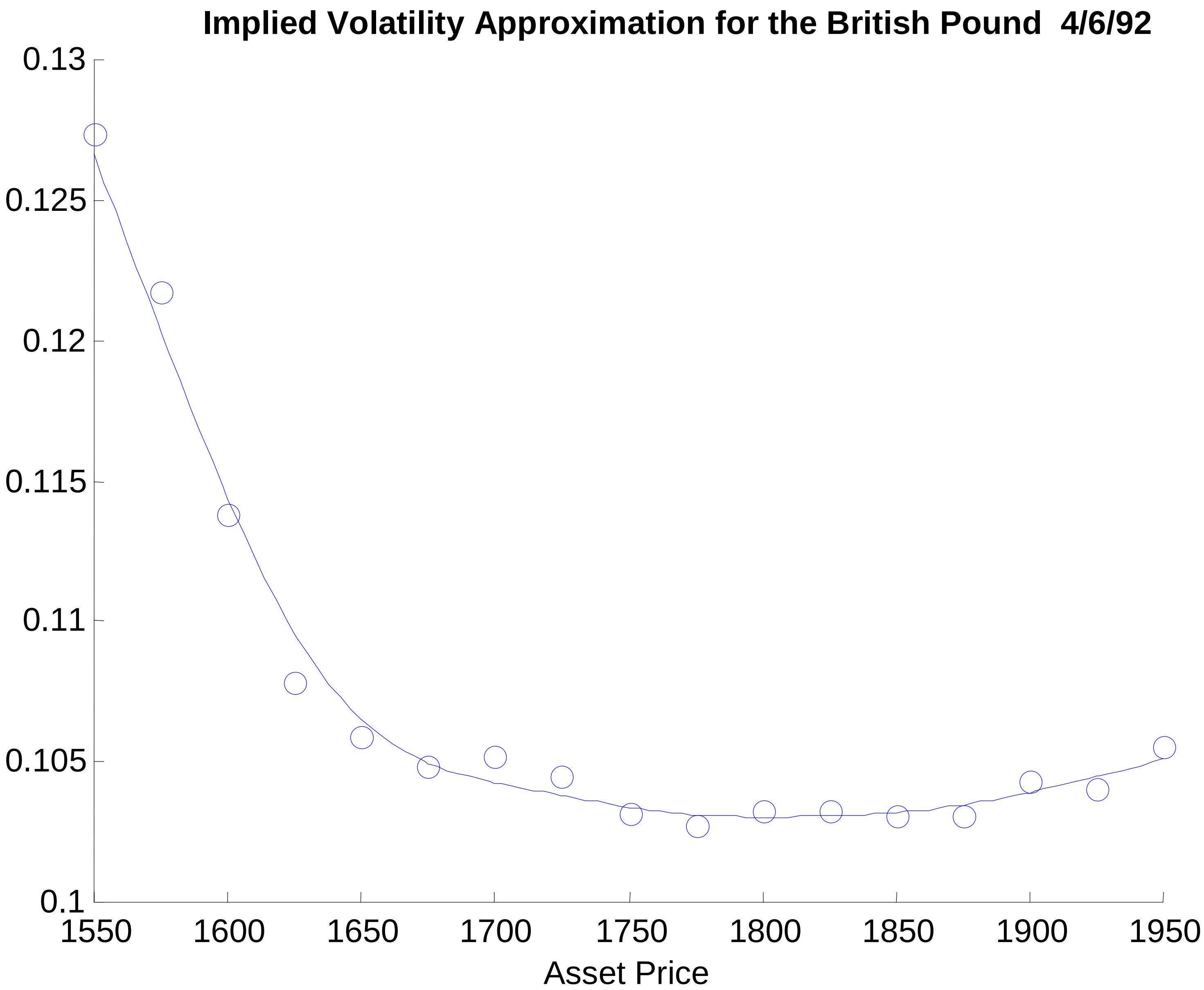

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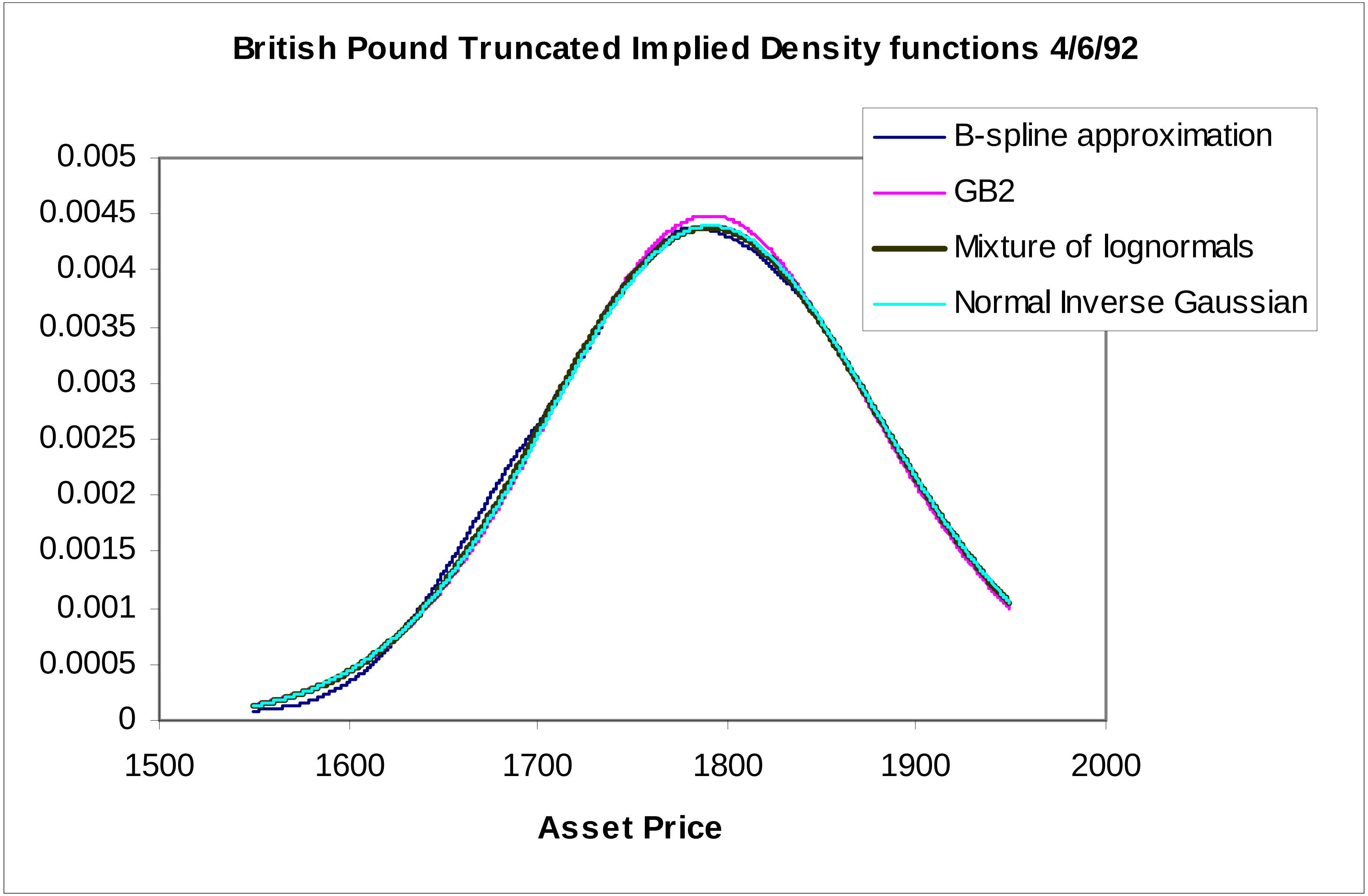

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4. Alternative Methods for Estimating Risk Neutral Densities

- Cubic B-Spline Functions
- Generalised Beta Approach
- Mixture of Lognormal Distributions
- Normal Inverse Gaussian

Estimating RNDs using Cubic B-Splines

The risk-neutral density is equal to the second derivative of the option price with respect to the strike price:

$$\varphi_T(K) = \frac{\partial^2 C(K,T)}{\partial K^2} \qquad \textbf{(1)}$$

Shimko (1993) used least squares to fit a quadratic function to the volatility smile.

We follow de Boor (1987) and estimate a cubic smoothing spline to minimise a smoothness criterion, such that

$$E(f) = \sum_{i=1}^{n} (C_i - C(f(x_i))^2 \leq tol \quad . \qquad \textbf{(20)}$$

Estimating RNDs using Generalised Beta

We employ the generalised beta of the second kind (*GB2*). *GB2* is defined (McDonald, 1984) as follows:

$$GB2(y;a,b,p,q) = \frac{|a| y^{ap-1}}{b^{ap}\mathrm{B}(p,q)\left[1+(y/b)^{a}\right]^{p+q}}, \qquad \textbf{(3)}$$

for $y > 0$, and $GB2(y;a,b,p,q) = 0$ otherwise.

The expression for its distribution function and h^th^ moment respectively is,

$$\frac{\left(\dfrac{(y/b)^{a}}{1+(y/b)^{a}}\right)^{p}}{pB(p,q)}\,{}_2F_1\left[\begin{matrix} p,1-q; \\ p+1; \end{matrix}\ \frac{(y/b)^{a}}{1+(y/b)^{a}}\right], \quad \mathrm{E}(y^{h}) = \frac{b^{h}\mathrm{B}(p+h/a,q-h/a)}{\mathrm{B}(p,q)} \qquad \textbf{(4)}$$

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where $\mathrm{B}(p,q)$ denotes the beta function, as given by $\mathrm{B}(p,q)=\int_0^1 t^{p-1}(1-t)^{q-1}d_t$ and ${}_2F_1$ represents the generalised hypergeometric series defined by:

$$ {}_pF_q\begin{bmatrix} a_1,\ldots,a_p;x \\ b_1,\ldots,b_q; \end{bmatrix} = \sum_{i=0}^{\infty}\frac{(a_1)_i\ldots(a_p)_i}{(b_1)_i\ldots(b_p)_i}\frac{x^i}{i!}\,. \qquad \textbf{(5)} $$

the mean as well as the shape of the GB2 is determined by the four parameters of the distribution. The parameter b affects directly the mean of the GB2,

$$ \mathrm{E}(y)=\frac{b\mathrm{B}(p+1/a,q-1/a)}{\mathrm{B}(p,q)} \qquad \textbf{(6)} $$

Under the assumption that the risk neutral distribution belongs to the distribution family of the generalised beta of the second kind (*GB2*), the values of the call and put options are obtained by numerical integration (using equations **(7.1) – (7.2)**).

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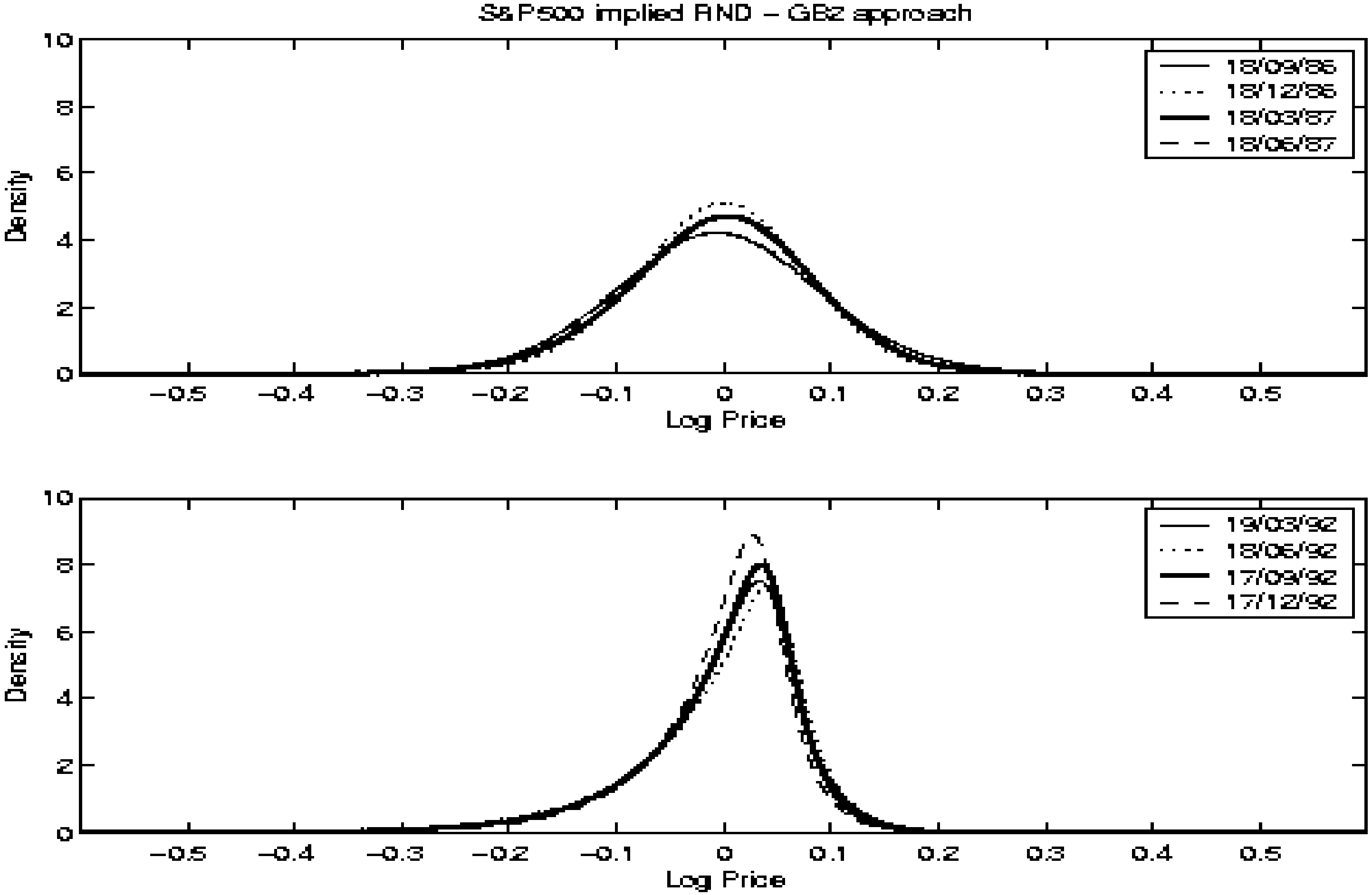

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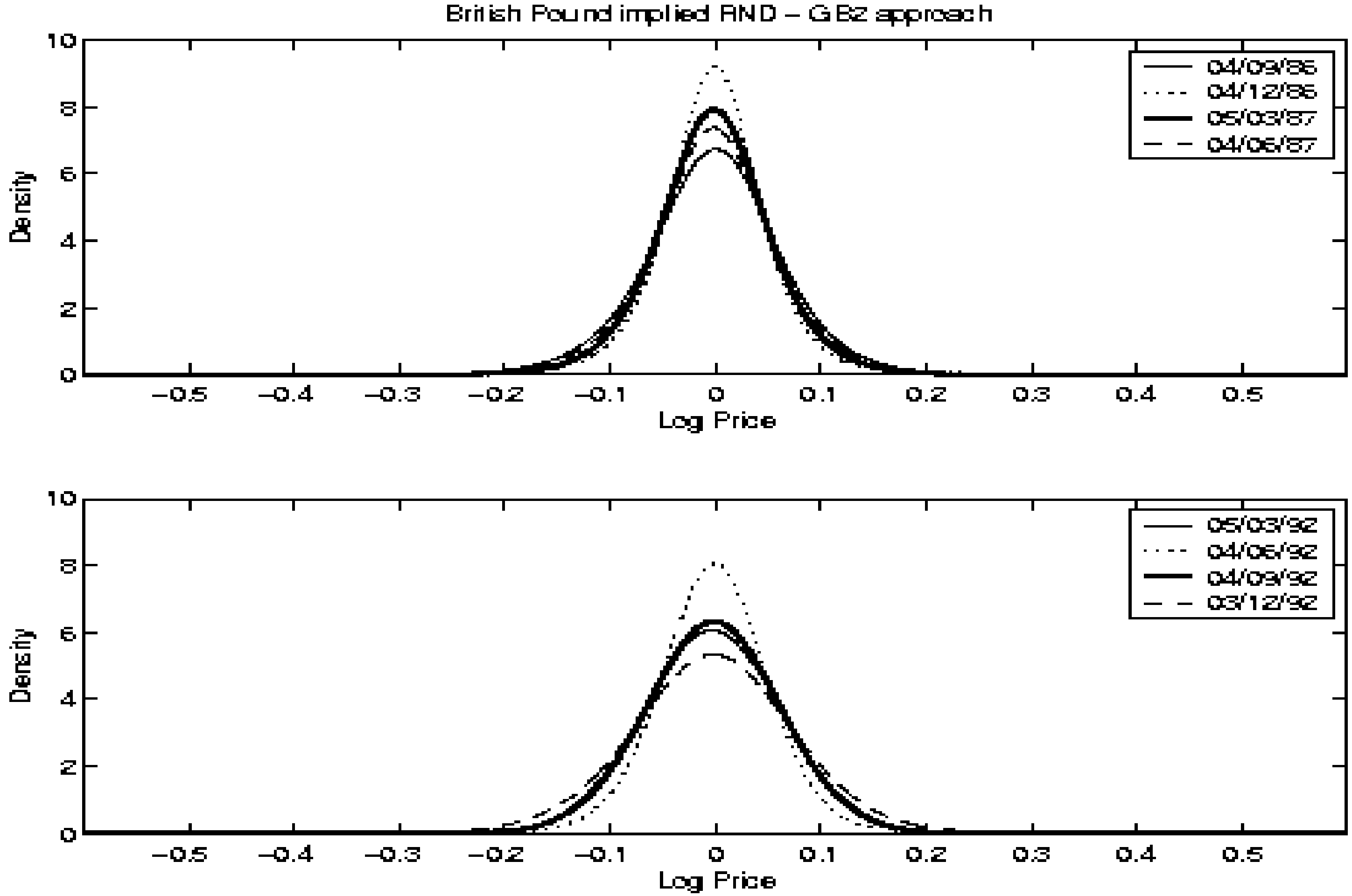

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Estimating RNDs using Two Lognormal Mixture

Here we assume that the density function $q(S_T)$ is given by a mixture of two lognormal density functions, that is,

$$q(S_T) = \sum_{i=1}^{2} \left[\theta_i L(\alpha_i, \beta_i; S_T)\right] \quad \textbf{(13)}$$

where $L(\alpha_i, \beta_i; S_T)$ is the i^{th} lognormal density with parameters α_i and β_i

Given the assumption made on $q(S_T)$, equations (1) and (2) can be expressed as follows:

$$c(X,\tau) = e^{-r\tau} \int_X^{\infty} \left[\theta L(\alpha_1, \beta_1; S_T) + (1-\theta) L(\alpha_2, \beta_2; S_T)\right](S_T - X)\, dS_T \quad \textbf{(15.1)}$$

$$p(X,\tau) = e^{-r\tau} \int_0^{X} \left[\theta L(\alpha_1, \beta_1; S_T) + (1-\theta) L(\alpha_2, \beta_2; S_T)\right](X - S_T)\, dS_T \quad \textbf{(15.2)}$$

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The estimates for the five parameters α_1, β_1, α_2, β_2, θ, are obtained by minimising the deviation of the theoretical prices for both calls and from the market prices, across the available range of strikes.

The total sum of squared errors for call and put options is minimised by means of a non-linear least squares optimisation routine. The minimisation problem then becomes:

$$\underset{\alpha_1,\alpha_2,\beta_1,\beta_2,\theta}{Min} \sum_i \left[c(X_i,\tau)-\hat{c}_i\right]^2 + \sum_i \left[p(X_i,\tau)-\hat{p}_i\right]^2 + \left[\theta e^{\alpha_1+0.5\beta_1^2} + (1-\theta)e^{\alpha_2+0.5\beta_2^2} - e^{r\tau}S\right]^2 \quad \textbf{(16)}$$

subject to β_1, β_2 >0 and $0\leq\theta\leq1$.

The computation is facilitated by closed-form solutions for the pricing of options **(17.1) – (18)** of Bahra (1997).

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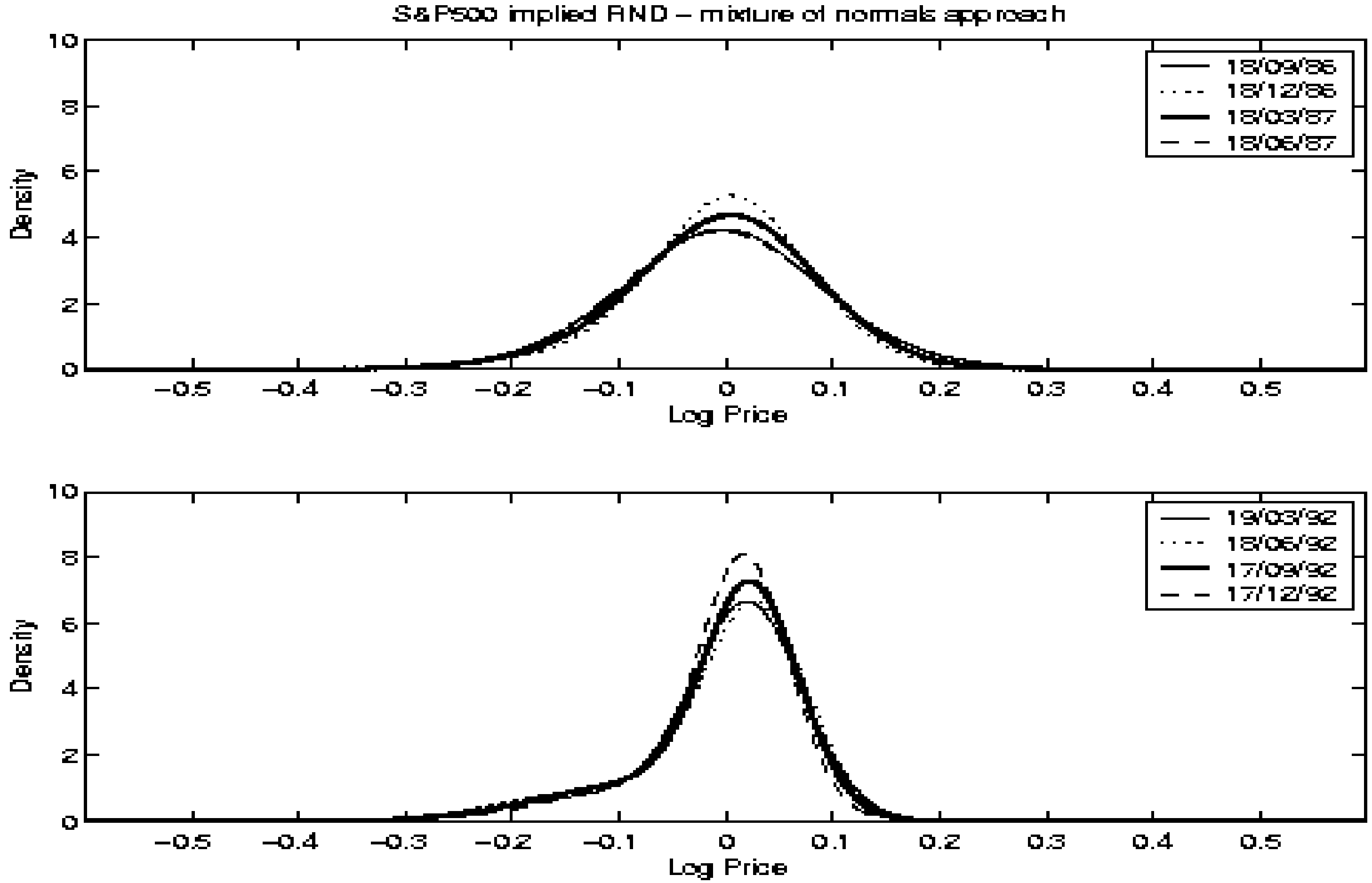

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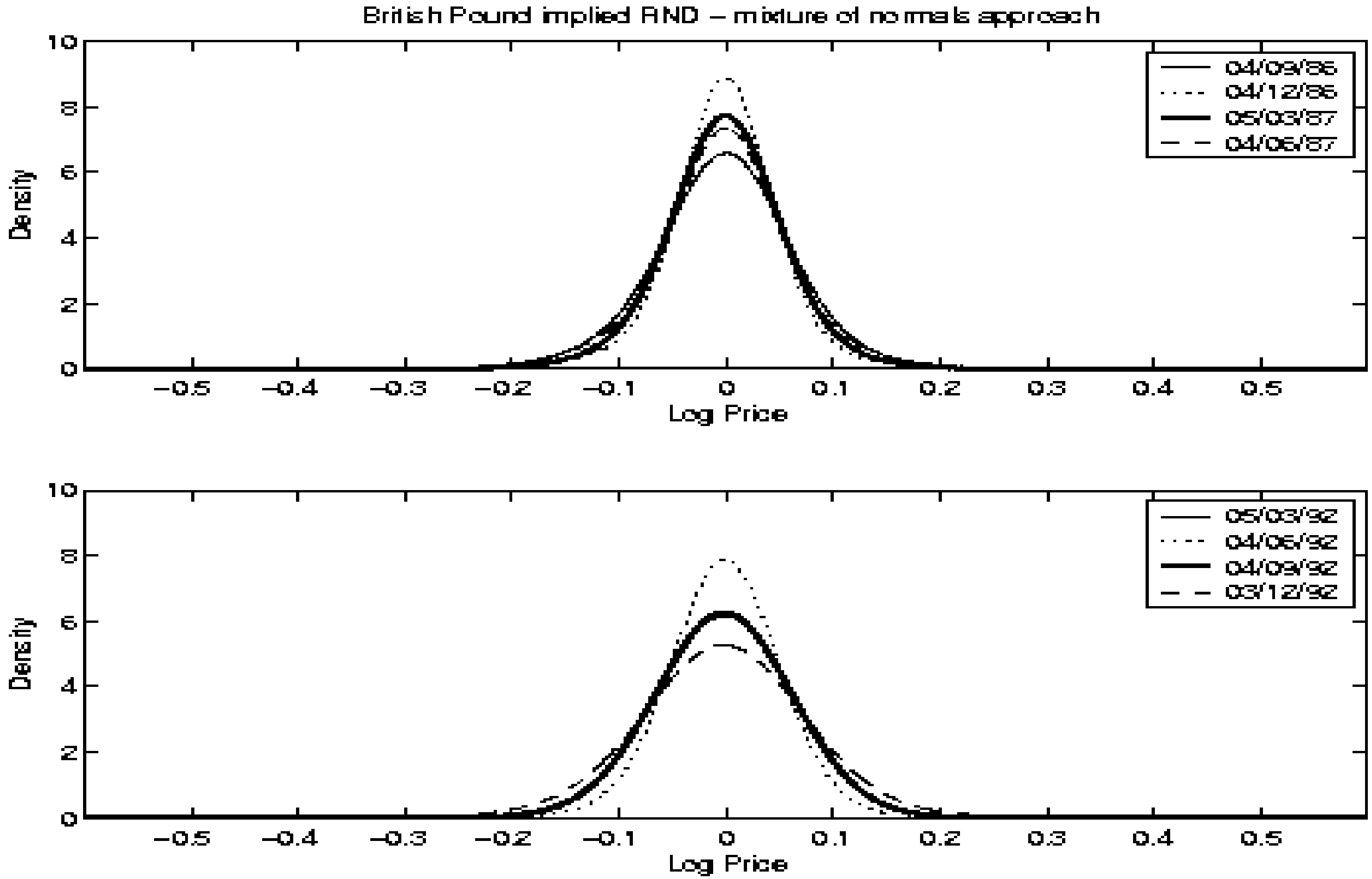

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Estimating RND using a Normal Inverse Gaussian Approach

Many recent studies [Barndorff-Nielsen 1997, etc.] have suggested that log asset returns can be well modelled by the Normal Inverse Gaussian (NIG) distribution, a special case of the generalised hyperbolic (GH) distribution. The NIG density function is given by:

$$NIG\left(x;\alpha,\beta,\mu,\delta\right)=\pi^{-1}\alpha e^{\delta\sqrt{\left(\alpha^2-\beta^2\right)}-\beta\mu}\,q\left(\frac{x-\mu}{\delta}\right)^{-1}K_1\left\{\delta\alpha q\left(\frac{x-\mu}{\delta}\right)\right\}e^{\beta x} \qquad \textbf{(9)}$$

where $q(x)=\sqrt{\left(1+x^2\right)}$, $\mu \in$ **R**, $\delta > 0$, $0 \le |\beta| \le \alpha$ and K_1 is the modified Bessel function of third order and index 1. The moment generating function of a NIG possesses a nice and neat expression:

$$M\left(u;\alpha,\beta,\mu,\delta\right)=\exp\left[\delta\left\{\sqrt{\alpha^2-\beta^2}-\sqrt{\left(\alpha^2-(\beta+u)^2\right)}\right\}+\mu u\right] \qquad \textbf{(10)}$$

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As an intuitive explanation of the parameters, α represents a steepness parameter, β is an asymmetry parameter, δ describes the scale and μ the location of the distribution. A further advantage of adopting a NIG specification consists in its property of being closed under convolution, not shared by alternative GH distributions. We estimate the parameters of the implied NIG RNDs by solving a non-linear least squares minimisation problem of the kind in (8), where:

$$C_i(K_i) = e^{-rT} \int_K^{+\infty} (S_0 e^{x_T} - K_i) NIG(x_T; \alpha, \beta, \mu, \delta) dx_T, \quad i = 1..n \qquad \textbf{(11.1)}$$

$$P_j(K_j) = e^{-rt} \int_0^K (K_j - S_0 e^{x_T}) NIG(x_T; \alpha, \beta, \mu, \delta) dx_T, \quad j = 1..m \qquad \textbf{(11.2)}$$

under the constraint that the mean of the implied distribution for the underlying equals the forward price, exactly as before.

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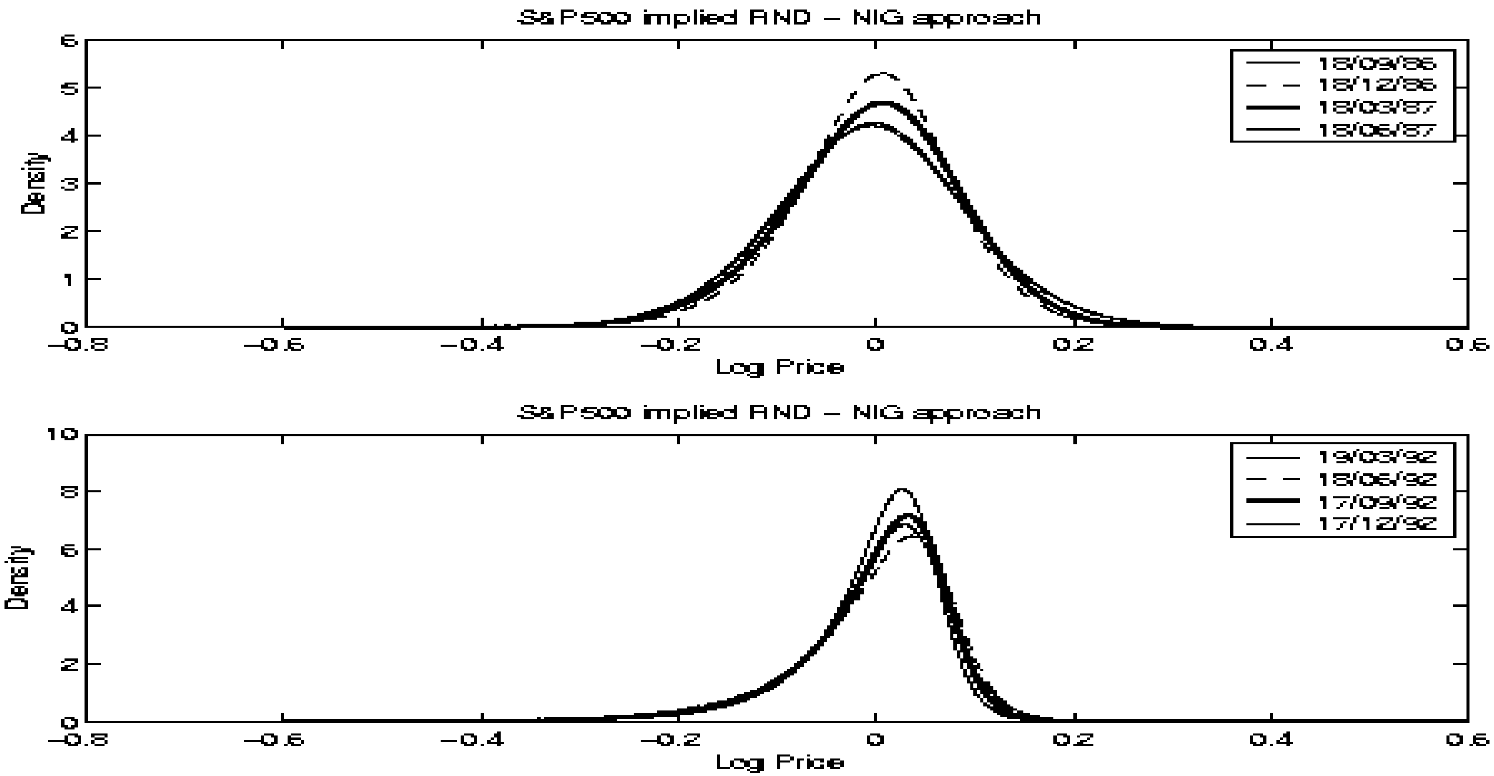

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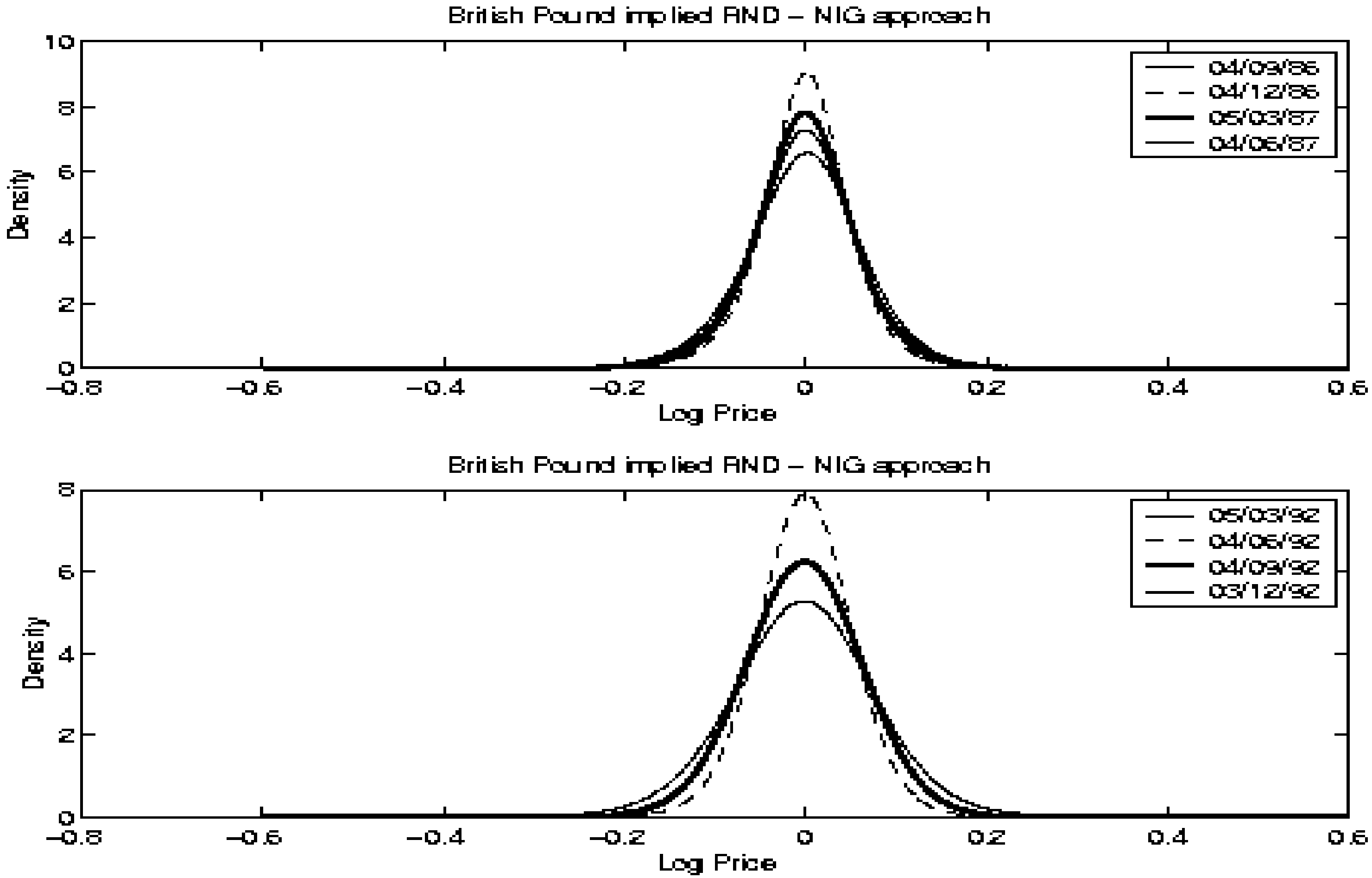

Table 1. Summary statistics for implied RNDs

	Mixture of lognormals				**Generalised Beta**				**NIG**			
	Mean	**St. Dev**	**Skew.**	**Kurt.**	**Mean**	**St. Dev**	**Skew.**	**Kurt.**	**Mean**	**St. Dev**	**Skew.**	**Kurt.**
British Pound												
Overall	-0.0017	0.0569	-0.0909	4.2137	-0.0017	0.0565	-0.0515	3.9291	-0.0017	0.0567	-0.0769	3.9793
	(0.0009)	(0.0141)	(0.1302)	(1.2258)	(0.0010)	(0.0139)	(0.1360)	(0.8149)	(0.0009)	(0.0140)	(0.1348)	(0.6584)
S&P 500												
	-0.0064	0.1108	-0.4418	4.1459	-0.0061	0.1102	-0.4886	4.3111	-0.0065	0.1119	-0.5627	5.2043
Pre-Crash Oct. 87	(0.0042)	(0.0324)	(0.4909)	(0.8723)	(0.0032)	(0.0302)	(0.4811)	(1.2335)	(0.0043)	(0.0334)	(0.6480)	(2.2628)
Post-Crash Oct.	-0.0050	0.0973	-1.2721	5.6351	-0.0043	0.0959	-1.3496	6.8340	-0.0054	0.1013	-1.7765	10.4181
87	(0.0040)	(0.0345)	(0.2805)	(1.1877)	(0.0028)	(0.0307)	(0.2986)	(0.9310)	(0.0050)	(0.0395)	(0.5693)	(4.4246)
Overall	-0.0053	0.0997	-1.1248	5.3709	-0.0046	0.0985	-1.1969	6.3864	-0.0056	0.1031	-1.5612	9.4931
	(0.0040)	(0.0343)	(0.4542)	(1.2688)	(0.0029)	(0.0308)	(0.4701)	(1.3799)	(0.0048)	(0.0385)	(0.7437)	(4.5737)

(Standard deviation in brackets)

Table 2 Summary statistics for RNDs parameters

		Mixture of lognormals			Generalised Beta			NIG	
		Mean	**St. Dev**		**Mean**	**St. Dev**		**Mean**	**St. Dev**
British Pound									
Overall	μ_1	7.3631	0.1043	a	32.6826	18.0630	α	46.0802	40.0090
	σ_1	0.0496	0.0314	b	1588.2151	156.0428	β	-4.7392	15.2142
	μ_2	7.3591	0.0951	P	1.6016	1.5059	μ	0.1599	0.2473
	σ_2	0.0798	0.0272	q	1.6478	1.4258	δ	0.0142	0.0679
	θ	0.7104	0.1828						
S&P 500									
Pre-Crash Oct. 87									
	μ_1	5.5941	0.1098	a	18.0844	13.4017	α	20.8044	9.6114
	σ_1	0.0818	0.0183	b	282.2138	24.9645	β	-3.3220	2.6088
	μ_2	5.5264	0.1317	P	1.8253	1.3471	μ	0.1992	0.0831
	σ_2	0.1434	0.0506	q	2.9418	2.2263	δ	0.0326	0.0331
	θ	0.6847	0.0789						
Post-Crash Oct. 87									
	μ_1	6.4558	0.5372	a	101.2404	20.7728	α	21.6629	7.6972
	σ_1	0.0567	0.0164	b	756.3794	423.6070	β	-14.1291	6.7670
	μ_2	6.3105	0.5118	P	0.1291	0.0477	μ	0.0854	0.0345
	σ_2	0.1242	0.0385	q	0.3734	0.2107	δ	0.0688	0.0372
	θ	0.7860	0.0587						
Overall									
	μ_1	6.3029	0.5905	a	86.5218	37.4625	α	21.5105	7.9885
	σ_1	0.0612	0.0192	b	671.9585	425.1849	β	-12.2117	7.4817
	μ_2	6.1714	0.5556	P	0.4288	0.8489	μ	0.1056	0.0635
	σ_2	0.1276	0.0411	q	0.7895	1.2368	δ	0.0624	0.0388

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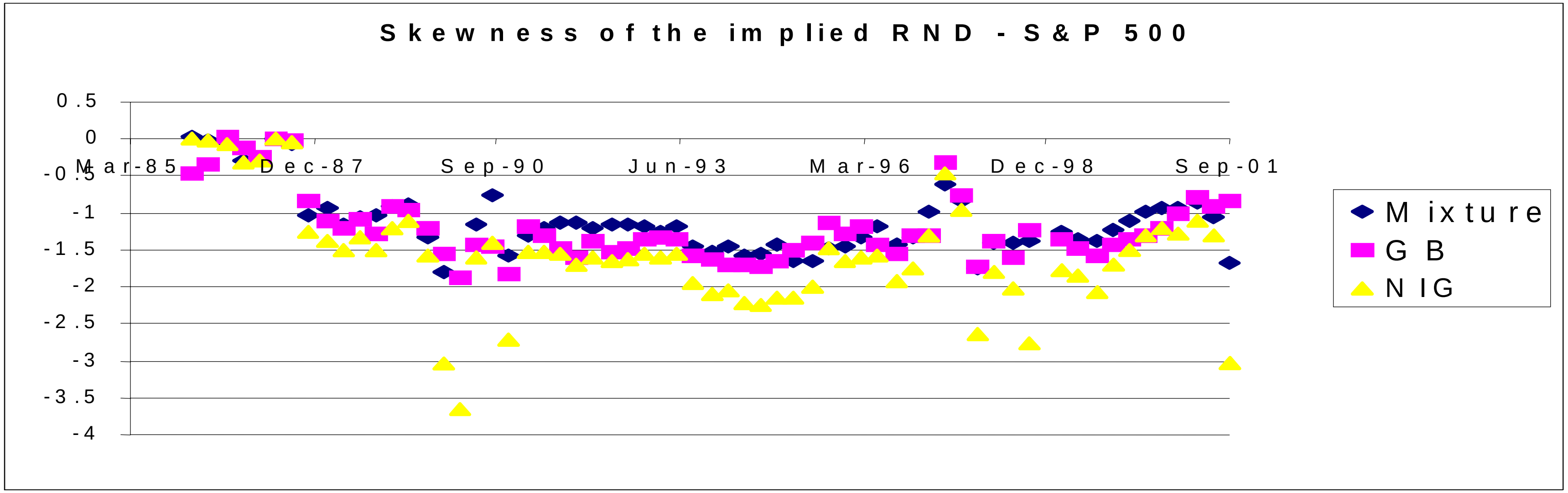

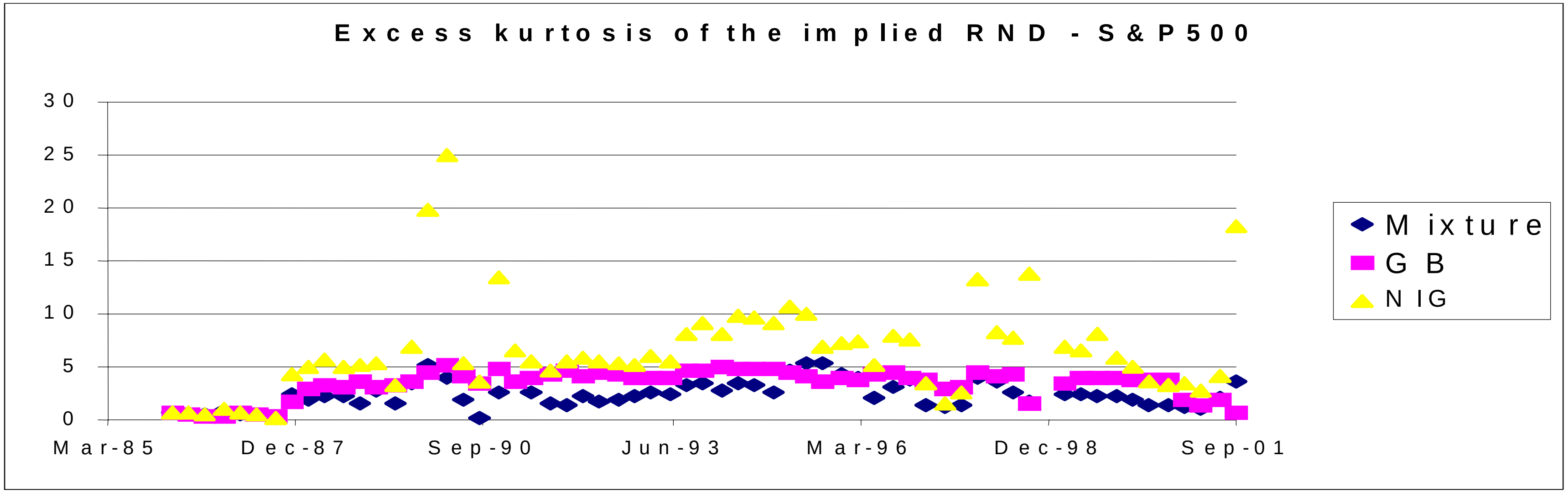

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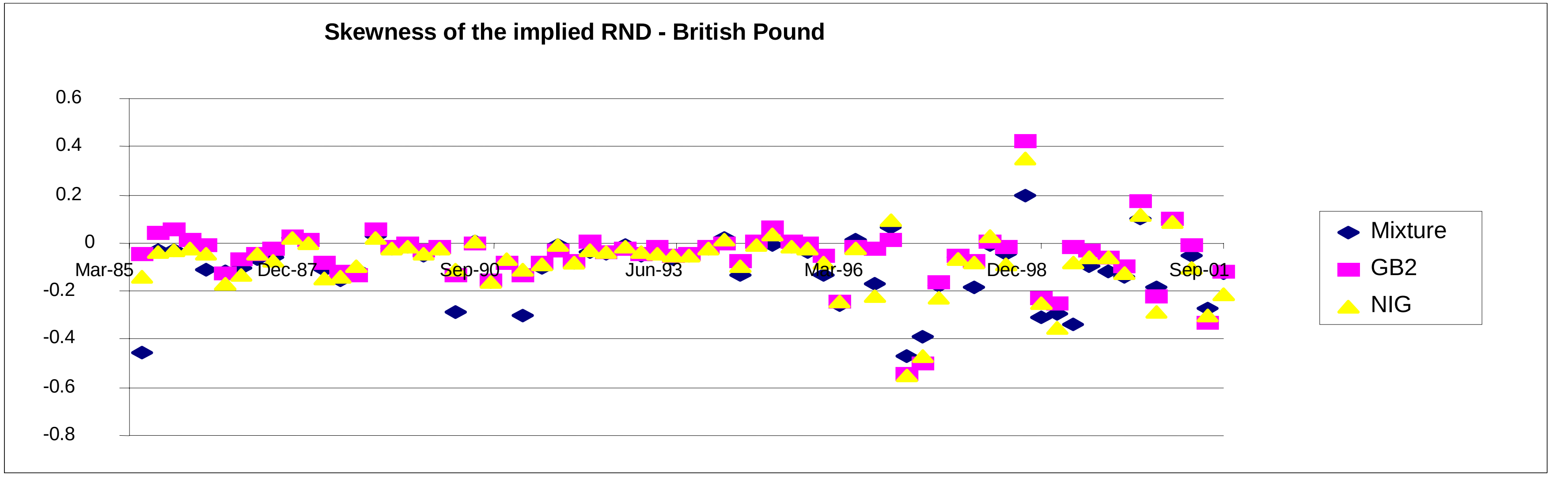

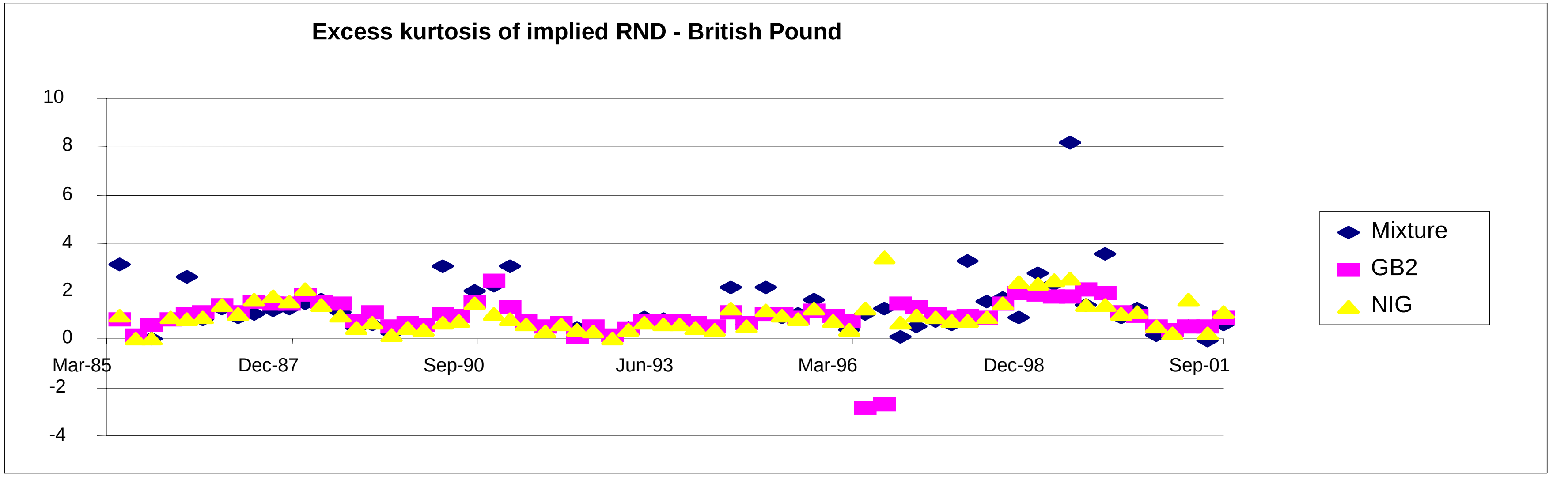

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5. Testing Forecast Ability of Risk Neutral Densities

The Probability Integral Transform (PIT) approach

Given a sequence of one-step ahead density forecasts $p_t(y_t|\Omega_t)$,

the probability integral transforms, z_t, of the realisation of the process y_t should be uniform on (0, 1) and i.i.d. , where

$$z_t = \int_{-\infty}^{y_t} p_t(u)du \qquad \textbf{(19)}$$

[see Diebold, Gunther and Tay (1998)].

For testing the body of the distribution, we apply the same test "normalised" between the truncation points.

Testing the Tails

Amongst the various "scoring rules", we chose to work with the Brier's (1950) quadratic probability score, also simply called the Brier score:

$$B_n = \frac{1}{n}\sum_t 2\left(P_t - R_t\right)^2 \qquad \textbf{(24)}$$

Clearly, B_n assumes values between 0 and 2 and more accurate probability forecasts are reflected in smaller values for the score. To assess whether B_n departs significantly from its expected value $\sum_t P_t(1-P_t)$, Seillier-Moiseiwitsch and Dawid (1993) suggest the test statistic:

$$Y_n^B = \sum_t (1-2P_t)(R_t - P_t) \Big/ \left[\sum_t (1-2P_t)^2 P_t(1-P_t)\right]^{1/2} \qquad \textbf{(25)}$$

which is asymptotically distributed as a standard normal.

Table 3a. Tail Tests

Test statistics			Right tail			Left tail			Both tails		
			Frequency	Probability forecast	Y_n^B	Frequency	Probability forecast	Y_n^B	Frequency	Probability forecast	Y_n^B
5% conf. level					1.6449			1.6449			1.6449
1% conf. level					2.3263			2.3263			2.3263
British Pound											
	B-spline		0.1194	0.0566	**2.1281*	0.0299	0.0431	*-0.3084*	0.1493	0.0997	*1.1111*
	Mixture lognormals	truncated	0.1194	0.0549	**2.2102*	0.0299	0.0430	*-0.3302*	0.1493	0.0980	*1.1484*
	Generalised Beta	truncated	0.1194	0.0541	**2.2410*	0.0299	0.0424	*-0.3099*	0.1493	0.0965	*1.1997*
	NIG	truncated	0.1194	0.0548	**2.2222*	0.0299	0.0433	*-0.3437*	0.1493	0.0982	*1.1419*
S&P500											
(unadjusted)	B-spline		0.0645	0.0777	*-0.0583*	0.0161	0.0343	*-0.6143*	0.0806	0.1121	*-0.4217*
	Mixture lognormals	truncated	0.0645	0.0780	*-0.2070*	0.0161	0.0364	*-0.6840*	0.0806	0.1144	*-0.5862*
	Generalised Beta	truncated	0.0645	0.0742	*-0.0660*	0.0161	0.0330	*-0.5761*	0.0806	0.1072	*-0.3949*
	NIG	truncated	0.0645	0.0757	*-0.1234*	0.0161	0.0337	*-0.6075*	0.0806	0.1094	*-0.4800*

* rejected at 5% confidence level.

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Table 3b. Tail Tests

Test statistics			**Right tail** Frequency	Probability forecast	Y_n^B	**Left tail** Frequency	Probability forecast	Y_n^B	**Both tails** Frequency	Probability forecast	Y_n^B
5% conf. level					1.6449			1.6449			1.6449
1% conf. level					2.3263			2.3263			2.3263
S&P500											
mean	B-spline		0.0168	0.0777	**-1.7875*	0.0161	0.0343	*-0.6143*	0.0287	0.1121	**-1.9654*
adjusted	Mixture lognormals	truncated	0.0168	0.0780	**-1.7916*	0.0161	0.0364	*-0.6840*	0.0287	0.1144	**-1.9775*
	Generalised Beta	truncated	0.0168	0.0742	**-1.7317*	0.0161	0.0330	*-0.5761*	0.0287	0.1072	**-1.9181*
	NIG	truncated	0.0168	0.0757	**1.7438*	0.0161	0.0337	*-0.6075*	0.0287	0.1094	**1.9215*
S&P500											
fully											
adjusted	Mixture lognormals	$\gamma = 1$	0.0645	0.0883	*-0.4540*	0.0161	0.0281	*-0.4091*	0.0806	0.1164	*-0.6212*
		$\gamma = 2$	0.0645	0.0995	*-0.6825*	0.0161	0.0218	*-0.1474*	0.0806	0.1213	*-0.6965*
	Generalised Beta	$\gamma = 1$	0.0645	0.0884	*-0.3176*	0.0161	0.0252	*-0.2915*	0.0806	0.1096	*-0.4314*
		$\gamma = 2$	0.0645	0.0954	*-0.5525*	0.0161	0.0196	*-0.0248*	0.0806	0.1149	*-0.5181*
	NIG	$\gamma = 1$	0.0645	0.0858	*-0.3681*	0.0161	0.0254	*-0.3045*	0.0806	0.1112	*-0.4961*
		$\gamma = 2$	0.0645	0.0966	*-0.5938*	0.0161	0.0195	*-0.0268*	0.0806	0.1161	*-0.5676*

* rejected at 5% confidence level.

Testing the Body of the RND functions

To test the **uniformity** of z_t series, we compute statistics, based on the vertical difference between the empirical distribution function and the theoretical distribution function of the null.

Let $z_{(1)} < z_{(2)} < \ldots < z_{(n)}$ be the values of the z_t series (both truncated and non truncated) arranged in ascending order.

Some measures are computed as the supremum of that vertical difference: the most famous amongst these measures is the Kolmogorov-Smirnov statistic, introduced by Kolmogorov (1933):

$$D = \max\left(D^{+}, D^{-}\right) \qquad \textbf{(26)}$$

where $D^{+} = \max_i \left\{ i/n - z_{(i)} \right\}$; $\quad D^{-} = \max_i \left\{ z_{(i)} - \left(i-1\right)/n \right\}$

A similar measure is the Kuiper statistic (Kuiper 1962), defined as:

$$V = D^{+} + D^{-} \qquad \textbf{(27)}$$

Additional Tests

Additional tests are the Watson U^2 statistic, introduced by Watson (1961) and the Anderson-Darling (1952) A^2 test, which are respectively defined as follows:

$$U^2 = \sum_i \{z_{(i)} - (2i-1)/(2n)\}^2 + 1/(12n) - n(\bar{z} - 0.5)^2 \qquad \textbf{(28)}$$

where $\bar{z} = \sum_i z_i / n$

$$A^2 = -n - (1/n)\sum_i (2i-1)[\log z_{(i)} + \log\{1 - z_{(n+1-i)}\}] \qquad \textbf{(29)}$$

Berkowitz (1999) shows that most of the testing procedures described above are not powerful enough for small samples. Berkowitz suggests to test the null hypothesis against a first-order autoregressive alternative given by:

$$x_t - \mu = \rho(x_{t-1} - \mu) + \varepsilon_t \qquad \textbf{(31)}$$

such that the null of i.i.d. (0,1) translates into $\mu = 0$, $\rho = 0$, and $\mathrm{var}(\varepsilon_t) = \sigma^2 = 1$.

Table 4a. Distributional Tests **H_0: z ~ i.i.d. U(0,1)**

Test statistics			**D**	**A^2**	U^2	**V**	**N_2**
5% conf. level			1.3580	2.4920	0.1870	1.7470	5.9915
1% conf. level			1.6280	3.8570	0.2680	2.0010	9.2103
British Pound							
	B-spline		1.1704	1.3305	*0.2101	*1.8189	5.2446
	Mixture lognormals	truncated	1.0711	0.9918	0.1537	1.5762	3.4711
		full	1.2019	1.5066	0.0899	1.3311	2.8267
	Generalised Beta	truncated	0.9929	0.9344	0.1378	1.4917	3.1033
		full	1.1875	1.5160	0.0798	1.3158	2.7967
	NIG	truncated	1.0421	0.9780	0.1489	1.5492	3.3900
		full	1.1865	1.5177	0.0849	1.3159	2.8274
S&P500 *(unadjusted)*	B-spline		*1.4018	*2.9740	*0.1997	*1.8423	*6.9183
	Mixture lognormals	truncated	*1.4222	*3.4280	0.1684	1.7401	*6.6247
		full	*1.3364	*2.5544	*0.1923	1.6642	5.7182
	Generalised Beta	truncated	1.2515	*3.1139	0.1284	1.4637	*6.1668
		full	1.2617	2.3233	0.1414	1.4659	5.2380
	NIG	truncated	1.3413	*3.1801	0.1474	1.6025	*6.2891
		full	1.3317	2.3951	0.1705	1.6101	5.4749

* rejected at 5% confidence level.
** rejected at 1% confidence level.

Table 4b. Distributional Tests **H_0: z ~ i.i.d. U(0,1)**

Test statistics			**D**	**A^2**	U^2	**V**	**N_2**
5% conf. level			1.3580	2.4920	0.1870	1.7470	5.9915
1% conf. level			1.6280	3.8570	0.2680	2.0010	9.2103
S&P500							
mean adjusted	Mixture lognormals	truncated	0.7816	0.9685	0.1205	1.5038	3.8592
		full	1.0112	1.0023	*0.2023	1.6845	*6.7412
	G B	truncated	0.8347	1.2673	0.1697	1.5496	5.1874
		full	0.0995	1.8277	*0.2462	*1.8317	*7.2213
	NIG	truncated	0.8439	1.1201	0.1528	1.5872	4.8237
		full	1.0195	1.7389	*0.2279	*1.7473	*7.3278
S&P500							
fully adjusted							
γ = 1	Mixture lognormals	truncated	1.2257	2.2117	0.1224	1.5940	4.3194
		full	1.1319	1.4176	0.1444	1.5019	3.8273
	Generalised Beta	truncated	1.0661	1.9670	0.0888	1.3205	3.9606
		full	0.9959	1.2182	0.1003	1.2442	3.2466
	NIG	truncated	1.1028	2.0225	0.1048	1.4079	4.0920
		full	1.0089	1.2966	0.1259	1.3257	3.5912
γ = 2	Mixture lognormals	truncated	1.0328	1.3215	0.0875	1.4547	2.6036
		full	0.8366	0.7662	0.1079	1.2507	2.7586
	Generalised Beta	truncated	0.9099	1.1388	0.0606	1.2067	2.3155
		full	0.7939	0.5993	0.0733	1.1102	2.0760
	NIG	truncated	0.9011	1.1786	0.0725	1.2531	2.4574
		full	0.7867	0.6865	0.0945	1.1553	2.5204

* rejected at 5% confidence level.

Testing the Body of the RND functions

In order to test the hypothesis of **independence** across the observations, the likelihood ratio test can be expressed as:

$$LR1 = -2\left(L\left(\hat{\mu},\hat{\sigma}^2,0\right) - L\left(\hat{\mu},\hat{\sigma}^2,\hat{\rho}\right)\right) \sim \chi^2(1) \qquad \textbf{(33)}$$

Similarly, the likelihood ratio (LR) test for the joint hypothesis of independent observations with zero mean and unit variance, can be formulated as:

$$LR2 = -2\left(L(0,1,0) - L\left(\hat{\mu},\hat{\sigma}^2,\hat{\rho}\right)\right) \sim \chi^2(3) \qquad \textbf{(34)}$$

Although the LR tests seem adequate for our problem since they should possess good statistical power against general alternatives also for small sample sizes, they implicitly maintain the assumption of normality of the x_t series, instead of verifying it explicitly.

Assessing Normality

The normality of the transformed probability series is then assessed via the Jarque-Bera and the Doornik-Hansen tests. The Jarque-Bera statistic (J-B), probably the most common test for normality, is formulated as:

$$JB = n\left[\beta_1 / 6 + (\beta_2 - 3)^2 / 24\right] \quad \sim \quad \chi^2(2) \qquad \textbf{(35)}$$

where $\sqrt{\beta_1}$ is the sample skewness and β_2 the sample kurtosis.

Arguing that the statistics $\sqrt{\beta_1}$ and β_2 are not independent, except for very large sample sizes, Doornik and Hansen (1994) propose an alternative test for normality based on transformed measures of skewness (z_1) and kurtosis (z_2), expressed as:

$$DH = z_1^2 + z_2^2 \quad \sim \quad \chi^2(2) \qquad \textbf{(36)}$$

Table 5a. Distributional Tests **H_0: x ~ i.i.d. N(0,1)**

Test statistics			**μ**		**ρ**		**σ^2**	**J-B**	**D-H**	**LR1**	**LR2**
5% conf. level				(>0.05)		(>0.05)		(>0.05)	5.9915	3.8415	7.8147
1% conf. level				(>0.01)		(>0.01)		(>0.01)	9.2103	6.6349	11.3449
British Pound											
	B-spline		0.0678	(0.4667)	-0.1946	(0.1517)	0.6776	0.3098	2.9851	2.0746	6.4250
	Mixture lognormals	truncated	0.0642	(0.5213)	-0.1732	(0.2027)	0.7572	0.4140	2.5221	1.6315	4.2031
		full	0.1450	(0.2328)	-0.0316	(0.7959)	0.9859	*0.0364	*7.1201	0.0528	2.0036
	Generalised Beta	truncated	0.0702	(0.4885)	-0.1717	(0.2067)	0.7723	0.4591	2.3032	1.6030	3.9385
		full	0.1520	(0.2151)	-0.0280	(0.8186)	0.9929	0.0785	*5.9985	0.0391	2.1515
	NIG	truncated	0.0659	(0.5119)	-0.1732	(0.2028)	0.7635	0.4309	2.5294	1.6305	4.0930
		full	0.1487	(0.2235)	-0.0324	(0.7911)	0.9958	*0.0336	*7.2972	0.0555	2.0916
S&P500 *(unadjusted)*	B-spline		0.2361	(0.0684)	0.1094	(0.4198)	*0.6567	0.6796	1.2221	0.7357	*9.1551
	Mixture lognormals	truncated	*0.2881	(0.0386)	0.1014	(0.4549)	0.7470	0.6741	0.7546	0.6375	*8.4560
		full	0.2153	(0.0733)	0.0189	(0.8858)	0.7738	0.1659	3.5481	0.0218	5.4918
	Generalised Beta	truncated	0.2855	(0.0517)	0.1355	(0.3177)	0.7771	0.8716	0.3378	1.0994	*8.1476
		full	0.2234	(0.0820)	0.0567	(0.6655)	0.8186	0.3089	2.9475	0.1862	4.9505
	NIG	truncated	*0.2808	(0.0483)	0.1130	(0.4049)	0.7681	0.7830	0.6162	0.7801	*7.9044
		full	0.2134	(0.0833)	0.0310	(0.8134)	0.7969	0.1960	3.4374	0.0573	5.0050

* rejected at 5% confidence level.
** rejected at 1% confidence level.

Table 5b. Distributional Tests **H_0: x ~ i.i.d. N(0,1)**

Test statistics				μ	ρ	σ^2	**J-B**	**D-H**	**LR1**	**LR2**
5% conf. level				(>0.05)	(>0.05)		(>0.05)	5.9915	3.8415	7.8147
1% conf. level				(>0.01)	(>0.01)		(>0.01)	9.2103	6.6349	11.3449
S&P500										
mean adjusted		Mixture lognormals	truncated	-0.0167 (0.8821)	0.1741 (0.2164)	*0.6054	0.8065	0.0716	1.7302	*7.9393
			full	-0.0924 (0.3862)	0.0063 (0.9631)	*0.6437	0.0528	*5.9983	0.0173	5.3069
		Generalised Beta	truncated	-0.0241 (0.8183)	0.1894 (0.1743)	*0.5792	0.9215	0.6871	1.7879	*8.1876
			full	-0.0894 (0.3783)	0.0316 (0.8097)	*0.6397	*0.0392	*6.9745	0.0790	5.9825
		NIG	truncated	-0.0215 (0.8465)	0.1792 (0.1979)	*0.5983	0.8906	0.7503	1.7544	6.8452
			full	-0.0931 (0.3922)	0.0171 (0.8978)	*0.6428	*0.0412	*6.8825	0.0267	5.8162
S&P500										
fully adjusted										
	$\gamma = 1$	Mixture lognormals	truncated	0.2173 (0.1175)	0.1127 (0.4055)	0.7717	0.6829	0.7333	*3.8415	*7.8148
			full	0.1274 (0.2894)	0.0339 (0.7962)	0.7940	0.1682	3.5017	0.0675	3.2305
		Generalised Beta	truncated	0.2151 (0.1420)	0.1461 (0.2806)	0.8015	0.8760	0.3458	1.2534	5.7529
			full	0.1362 (0.2899)	0.0705 (0.5906)	0.8400	0.3096	2.9095	0.2877	2.7514
		NIG	truncated	0.2103 (0.1383)	0.1245 (0.3581)	0.7914	0.7937	0.6074	0.9215	5.5187
			full	0.1256 (0.3088)	0.0454 (0.7294)	0.8177	0.1986	3.3902	0.1204	2.8174
	$\gamma = 2$	Mixture lognormals	truncated	0.1485 (0.2876)	0.1238 (0.3605)	0.7968	0.6900	0.7199	0.8953	4.0339
			full	0.0426 (0.7273)	0.0480 (0.7142)	0.8146	0.1699	3.4655	0.1337	2.0007
		Generalised Beta	truncated	0.1467 (0.3195)	0.1562 (0.2482)	0.8264	0.8792	0.3561	1.4087	4.0092
			full	0.0519 (0.6905)	0.0832 (0.5252)	0.8616	0.3096	2.8824	0.4016	1.5770
		NIG	truncated	0.1417 (0.3210)	0.1357 (0.3160)	0.8152	0.8024	0.6039	1.0699	3.7808
			full	0.0412 (0.7422)	0.0587 (0.6540)	0.8387	0.2005	3.3537	0.2001	1.6650

* rejected at 5% confidence level.

6. Conclusions

1. Using a variety of methods to determine RNDs (including a NIG model), we find that RND forecasts are biased estimators of realised density functions for options on the S&P 500 and British Pound.

2. The major source of error for the S&P 500 was initially found to be the mean value. The forecast efficiency is still rejected under a simple multiplicative shift of the mean, but not when we transform using a power utility function.

3. In cases where we are unable to reject forecast efficiency it may nevertheless be dangerous to accept the forecasts as "good". Other evidence that implied volatilities look backwards more than forwards, and provide exaggerated forecasts of the future (c.f. Gemmill and Saflekos (2000)).

4. These considerations lead us to concur with Shiratsuka (2001) that the utility of RNDs for financial regulators is extremely limited. We contend that the use of RNDs as indicators of future market turbulence could actually prove to be counterproductive.