Calibration of the Deterministic and Stochastic Volatility Libor Market Model

L. Molgedey
`lutz.molgedey@de.andersen.com`
ANDERSEN

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Abstract

This talk is organized in two parts. In the first part we present the calibration of the deterministic volatility Libor market model to at-the-money cap and swaption volatilities. In the second part we discuss a variant of the Libor market model with stochastic volatility. As for the deterministic volatility Libor market model we require the parameterization of the model to be as time homogenous as possible. Here, this is achieved by using time homogenous mean reversion levels and speeds for the stochastic volatilities of the respective forward rates. Correct (perfect) pricing of the (at-the-money) caplets corresponds then to non-stationary initial values of the forward rate volatilities. However, demanding a time homogenous model restricts possible caplet smile surfaces.

Contents

1. BGM/Jamshidian-Model (Deterministic Volatility Libor Market Model)

Lets consider a s-factor lognormal Libor market model for the forward Libor rate processes $L_{i=0,1,\ldots,N}$ with respect to a given tenor structure $0 = T_0 < T_1 < T_2 < \cdots < T_{N+1}$ in the terminal bond numeraire Q^{N+1}

$$\mathrm{d}\,L_i(t) = L_i(t)\left(\mu_i^{Q^{N+1}}(t)\mathrm{d}\,t + \mathrm{d}\,W_i(t)\right) \tag{1}$$

with

$$\mu_i^{Q^{N+1}}(t) = -\sum_{j=i+1}^{N} \frac{\delta_j L_j(t)\,\mathrm{cov}_{ij}(t)}{1+\delta_j L_j(t)} \quad , \tag{2}$$

$$\langle \mathrm{d}\,W_i(t)\mathrm{d}\,W_j(t)\rangle = \mathrm{cov}_{ij}(t)\mathrm{d}\,t \quad , \tag{3}$$

where $\delta_i = T_{i+1} - T_i$ are the day count fractions. A decomposition to a s-factor model ($s \le N$) yield to

$$\mathrm{d}\,L_i(t) = L_i(t)\left(-\sum_{j=i+1}^{N} \frac{\delta_j L_j(t)\,\mathrm{cov}_{ij}(t)}{1+\delta_j L_j(t)}\mathrm{d}\,t + \sum_{p=0}^{s}\gamma_{ip}(t)\mathrm{d}\,Z_p(t)\right) \quad , \tag{4}$$

where γ_{ip} are the instantaneous loadings for the factor $Z_p(t)$. In (4) $Z_p(t)$ is a standard one dimensional (uncorrelated) Wiener process and

$$\mathrm{cov}_{ij}(t) = \sum_{p=0}^{s}\gamma_{ip}(t)\gamma_{jp}(t) \tag{5}$$

is the instantaneous covariance of the Libor rate fluctuations.

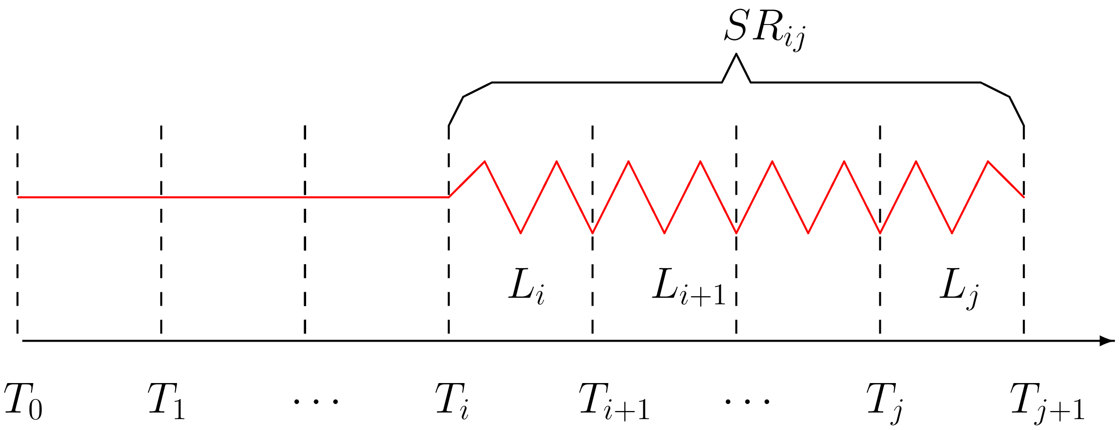


Figure 1: The reset and expiry times for the forward SWAP rate SR_{ij} and embedded forward Libor rates $L_{k=i,\ldots,j}$.

In terms of zero coupon bond prices $B(t,T)$ with maturity T the forward Libor rates are defined as

$$L_i(t) = \frac{1}{\delta_i}\left(\frac{B(t,T_i)}{B(t,T_{i+1})} - 1\right) \tag{6}$$

and the forward SWAP rates as

$$SR_{ij}(t) = \frac{B(t,T_i) - B(t,T_{j+1})}{\sum_{k=i}^{j}\delta_k B(t,T_{k+1})} = \frac{\sum_{k=i}^{j}\delta_k L_k(t) B(t,T_{k+1})}{\sum_{k=i}^{j}\delta_k B(t,T_{k+1})} \quad . \tag{7}$$

2. Forward Libor Rate Volatility and Calibration to Cap Volatilities

Since the Wiener processes are uncorrelated the instantaneous forward Libor rate volatility is given by

$$\sigma_i^{\mathrm{inst}}(t)^2 = \sum_{p=1}^{s} \gamma_{ip}(t)^2 \quad . \tag{8}$$

Spot caplet volatilities are calculated by integration of the squared instantaneous forward Libor rate volatilities (i.e. integration of the variance of the Libor fluctuations)

$$\left(\sigma_i^{\mathrm{Black}}\right)^2 T_i \quad = \quad \int_0^{T_i} \mathrm{d}\,t \left(\sigma_i^{\mathrm{inst}}(t)\right)^2 \quad . \tag{9}$$

However the solution of the inverse problem of obtaining the forward Libor rate volatilities from market quotes for flat cap volatilities is not unique. There are infinitely ways of closing the equations (see Rebonato). One way is to assume time homogeneity

$$\begin{aligned} \gamma_{ip}(t) &= \gamma_p(T_i - t) & (10)\\ \sigma_i^{\mathrm{inst}}(t) &= \sigma^{\mathrm{inst}}(T_i - t) & (11)\\ \mathrm{cov}_{ij}(t) &= \mathrm{cov}(T_i - t, T_j - t) \quad . & (12) \end{aligned}$$

The caplet volatilities are given then by

$$\begin{aligned} \left(\sigma_i^{\mathrm{Black}}\right)^2 T_i &= \int_0^{T_i} \mathrm{d}\,t \left(\sigma^{\mathrm{inst}}(T_i - t)\right)^2 & (13)\\ &= \int_0^{T_i} \mathrm{d}\,t \left(\sigma^{\mathrm{inst}}(t)\right)^2 \quad . & (14) \end{aligned}$$

Cap prices can then be obtained in the standard way by summing the caplet prices. These cap prices should be consistent with the cap prices obtained from market quotes for flat cap volatilities (the volatilities for all caplets

of the quoted cap are identical). However the direct bootstrapping of the market quotes is numerical instable and sensitive to noise in the data.

A good method is to start with a smooth function for the time homogeneous instantaneous forward Libor rate volatility (”Rebonato functional form”). One functional form for a humped curve is

$$\sigma^{\mathrm{inst}}(t) = (a + b\,t)\,\mathrm{e}^{-c\,t} + d \quad . \tag{15}$$

The closed form for the spot caplet volatilities is obtained by integration (using Mathematica)

$$\begin{aligned} (\sigma_i^{\mathrm{Black}})^2 T_i = \frac{1}{4c^3}\Bigg(& \left(b^2 + 2\,a\,b\,c + 2\,a^2\,c^2 + 8\,b\,c\,d + 8\,a\,c^2\,d + 4\,c^3\,d^2\,T_i\right) & (16)\\ & - \mathrm{e}^{-c\,T_i}\left(8\,b\,c\,d + 8\,a\,c^2\,d + 8\,b\,c^2\,d\,T_i\right) & (17)\\ & - \mathrm{e}^{-2\,c\,T_i}\left(b^2 + 2\,a\,b\,c + 2\,a^2\,c^2 + 2\,b^2\,c\,T_i + 4\,a\,b\,c^2\,T_i + 2\,b^2\,c^2\,T_i^2\right)\Bigg) \quad . & (18) \end{aligned}$$

Perfect calibration to at-the-money caplet volatilities is only possible by introducing a non time homogeneous scaling factor k_i:

$$\sigma_i^{\mathrm{inst}}(t) = k_i * g(T_i - t) \tag{19}$$

and a time homogeneous function g(t) as above. The scaling factor should be as close as possible to one.

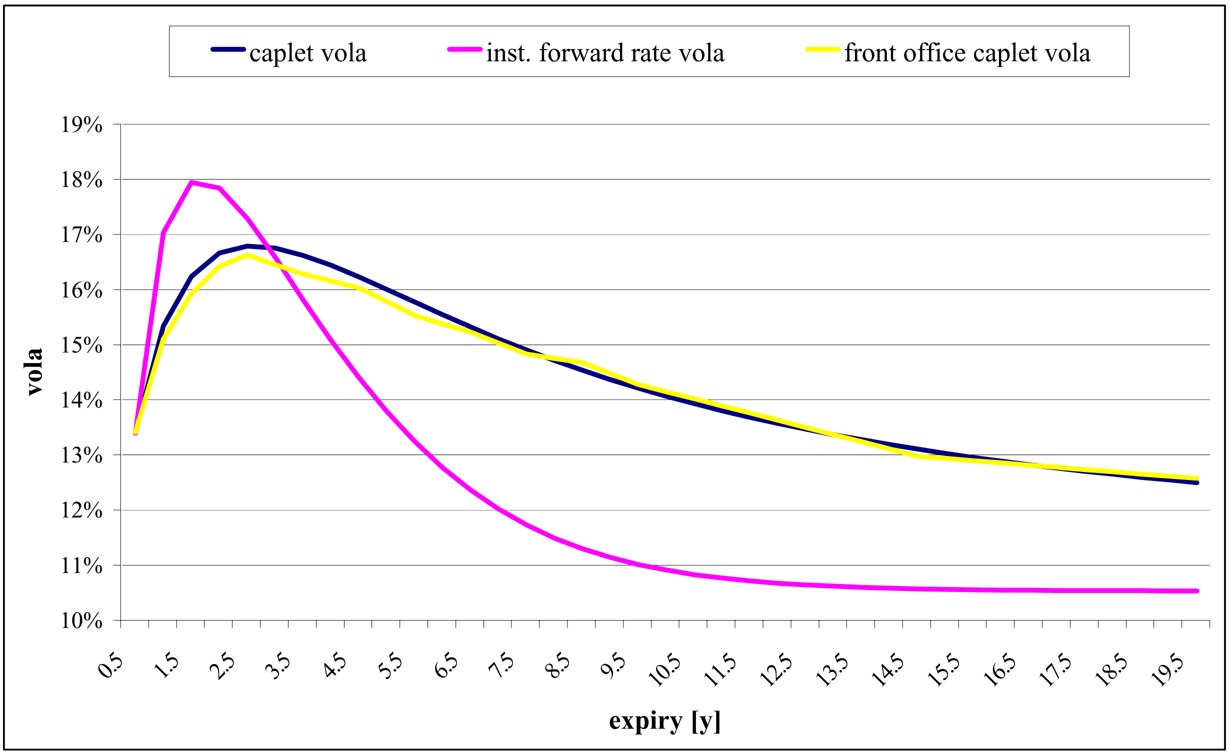


Figure 2: The fitted forward Libor rate volatility (without scaling factor) and resulting spot caplet volatility calibrated to market quotes. For comparison the caplet volatility of the front office (obtained by bootstapping) is shown.

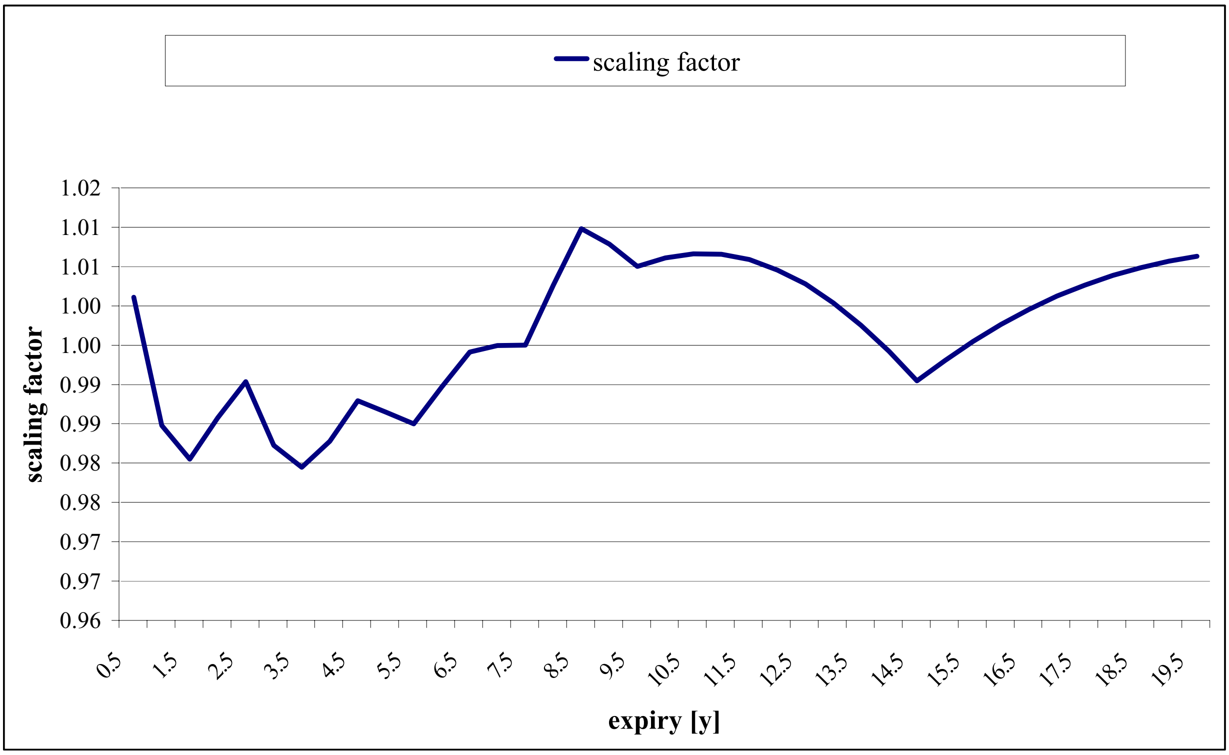


Figure 3: Scaling factor to ensure perfect calibration to caplet volotilies.

3. Forward SWAP Rate Volatility and Calibration to Swaption Volatilities

In a multi factor framework, according to equation (7) forward Libor and forward swap rates can not be simultaneous lognormal distributed. However, the forward Libor rates are highly correlated (this is the reason why one can use a model with only a few factors). Therefor such an assumption is a good approximation. Following this approximation one can use Ito's lemma to calculate swaption volatilities from caplet volatilities and Libor correlation structure. Since the SWAP rate are approximately lognormal distributed the resulting swaption volatilities are good approximations for the Black76 swaption volatilities.

Applying Ito's lemma to the forward SWAP rates gives

$$\left(\sigma^{\text{inst}}_{\text{SR}_{ij}}(t)\text{SR}_{ij}(t)\right)^2 = \sum_{k,l=i}^{j} L_k(t)\frac{\partial \text{SR}_{ij}}{\partial L_k}(t)\,\text{cov}_{kl}(t)\frac{\partial \text{SR}_{ij}}{\partial L_l}(t)L_l(t) \tag{20}$$

or approximately

$$\left(\sigma^{\text{inst}}_{\text{SR}_{ij}}(t)\text{SR}_{ij}(0)\right)^2 \approx \sum_{k,l=i}^{j} L_k(0)\frac{\partial \text{SR}_{ij}}{\partial L_k}(0)\,\text{cov}_{kl}(t)\frac{\partial \text{SR}_{ij}}{\partial L_l}(0)L_l(0) \quad . \tag{21}$$

This is again a good approximation if the yield curve fluctuations are mainly driven by parallel shifts (the first factor is most dominating). The spot european swaption volatilities are then obtained by integration (like for the caplet volatilities)

$$\begin{aligned}\left(\sigma^{\text{Black}}_{\text{SR}_{ij}}\right)^2 T_i &= \int_0^{T_i} \mathrm{d}\,t \left(\sigma^{\text{inst}}_{\text{SR}_{ij}}(t)\right)^2 && (22)\\ &\approx \int_0^{T_i} \mathrm{d}\,t \frac{1}{(\text{SR}_{ij}(0))^2}\sum_{k,l=i}^{j} L_k(0)\frac{\partial \text{SR}_{ij}}{\partial L_k}(0)\,\text{cov}_{kl}(t)\frac{\partial \text{SR}_{ij}}{\partial L_l}(0)L_l(0) && (23)\end{aligned}$$

$$= \int_0^{T_i} \mathrm{d}\,t \sum_{k,l=i}^{j} W^k_{ij}\,\text{cov}_{kl}(t)W^l_{ij} \tag{24}$$

with

$$W^k_{ij} = \frac{L_k(0)}{\text{SR}_{ij}(0)}\frac{\partial \text{SR}_{ij}}{\partial L_k}(0) \quad . \tag{25}$$

3.1. Rebonato approximation

Assuming that in equation (7) the forward Libor rates are independent of the Bond values one gets

$$\frac{\partial \text{SR}_{ij}}{\partial L_k}(t) \approx \frac{\delta_k B(t,T_{k+1})}{\sum_{l=i}^{j}\delta_l B(t,T_{l+1})} \tag{26}$$

This is a (very) good approximation for flat yield curves. Together with equation (7) it results to

$$\begin{aligned} W^k_{ij} &= \frac{1}{\text{SR}_{ij}(0)}\frac{\delta_k L_k(0)B(0,T_{k+1})}{\sum_{l=i}^{j}\delta_l B(0,T_{l+1})} && (27)\\ &= \frac{B(0,T_k)-B(0,T_{k+1})}{B(0,T_i)-B(0,T_{j+1})} \quad . && (28)\end{aligned}$$

3.2. Hull White approximation

The partial derivatives in equation (20) can be calculated exactly. The Bond values are related to the forward Libor rates according equation (6):

$$B(t,T_k) = B(t,T_{j+1})\prod_{l=k}^{j}(1+\delta_l L_l(t)) \quad . \tag{29}$$

Equation (7) can then be rewritten as

$$\begin{aligned} SR_{ij}(t) &= \frac{B(t,T_i) - B(t,T_{j+1})}{\sum_{k=i}^{j} \delta_k B(t,T_{k+1})} && (30) \\ &= \frac{\prod_{l=i}^{j}(1+\delta_l L_l(t)) - 1}{\sum_{k=i}^{j} \delta_k \prod_{l=k+1}^{j}(1+\delta_l L_l(t))} && (31) \end{aligned}$$

or

$$\ln\left(SR_{ij}(t)\right) = \ln\left(\prod_{l=i}^{j}(1+\delta_l L_l(t)) - 1\right) - \ln\left(\sum_{k=i}^{j} \delta_k \prod_{l=k+1}^{j}(1+\delta_l L_l(t))\right) \quad . \tag{32}$$

The partial derivatives becomes

$$\frac{\partial \mathrm{SR}_{ij}}{\partial L_k}(t) = \mathrm{SR}_{ij}(t) \frac{\delta_k \alpha_k(t)}{1+\delta_k L_k(t)} \tag{33}$$

with

$$\begin{aligned} \alpha_k(t) &= \frac{\prod_{l=i}^{j}(1+\delta_l L_l(t))}{\prod_{l=i}^{j}(1+\delta_l L_l(t)) - 1} - \frac{\sum_{m=i}^{k-1} \delta_m \prod_{l=m+1}^{j}(1+\delta_l L_l(t))}{\sum_{m=i}^{j} \delta_m \prod_{l=m+1}^{j}(1+\delta_l L_l(t))} && (34) \\ &= \frac{B(t,T_i)}{B(t,T_i) - B(t,T_{j+1})} - \frac{\sum_{m=i}^{k-1} \delta_m B(t,T_{m+1})}{\sum_{m=i}^{j} \delta_m B(t,T_{m+1})} && (35) \\ &= \frac{B(t,T_i) \sum_{m=k}^{j} \delta_m B(t,T_{m+1}) + B(t,T_{j+1}) \sum_{m=i}^{k-1} \delta_m B(t,T_{m+1})}{(B(t,T_i) - B(t,T_{j+1})) \sum_{m=i}^{j} \delta_m B(t,T_{m+1})} && (36) \end{aligned}$$

and hence

$$W_{ij}^{k} = \frac{\delta_k L_k(0) \alpha_k(0)}{1+\delta_k L_k(0)} = \frac{(B(0,T_k) - B(0,T_{k+1}))\alpha_k(0)}{B(0,T_k)} \tag{37}$$

$$= \frac{B(0,T_i) \sum_{m=k}^{j} \delta_m B(0,T_{m+1}) + B(0,T_{j+1}) \sum_{m=i}^{k-1} \delta_m B(0,T_{m+1})}{B(0,T_k) \sum_{m=i}^{j} \delta_m B(0,T_{m+1})} \quad . \tag{38}$$

3.3. Calibration of the Implied Correlation

Here we assume to have a well calibrated instantaneous forward Libor rate volatility $\sigma_i^{\mathrm{inst}}(t)$. According to equation (5) and (8) the instantaneous correlation is given by

$$\rho_{ij}(t) = \frac{\mathrm{cov}_{ij}(t)}{\sigma_i^{\mathrm{inst}}(t)\sigma_j^{\mathrm{inst}}(t)} \quad . \tag{39}$$

Assuming time homogeneity (10) one obtains

$$\rho_{ij}(t) = \rho(T_i - t, T_j - t) = \frac{\mathrm{cov}(T_i - t, T_j - t)}{\sigma^{\mathrm{inst}}(T_i - t)\sigma^{\mathrm{inst}}(T_j - t)} \quad . \tag{40}$$

To calculate the spot swaption volatilities using the above approximation formulas one needs to calculate the integral over the instantaneous covariance

$$\begin{aligned} K_{kl}^{i} &= \int_0^{T_i} \mathrm{d}\,t\, \mathrm{cov}_{kl}(t) && (41) \\ &= \int_0^{T_i} \mathrm{d}\,t\, \mathrm{cov}(T_k - t, T_l - t) && (42) \\ &= \int_0^{T_i} \mathrm{d}\,t \sigma^{\mathrm{inst}}(T_k - t)\rho(T_k - t, T_l - t)\sigma^{\mathrm{inst}}(T_l - t) \quad . && (43) \end{aligned}$$

3.3.1. Functional Form of the Implied Correlation

There are several reasonable functional approaches for the implied correlation. The simples one is

$$\rho(\tau_1, \tau_2) = \beta_1 + (1 - \beta_1)\exp\left(-\beta_2 |\tau_1 - \tau_2|\right) \tag{44}$$

which together with (15) is still analytical tractable. However it is unrealistic to assume that long dated rates (e.g. 9 and 10 years) have the same correlation as short dated rates (e.g. 1 and 2 years). An more realistic extension is

$$\rho(\tau_1, \tau_2) = \mathrm{LongCorr} + (1 - \mathrm{LongCorr}) \exp\left(-\beta|\tau_1 - \tau_2|\right) \tag{45}$$

with

$$\begin{aligned} \mathrm{LongCorr} &\equiv \beta_1 + \beta_4(\tau_1 + \tau_2) + \beta_5|\tau_1 - \tau_2| && (46) \\ \beta &\equiv \beta_2 + \beta_3 \max(\tau_1, \tau_2) \quad . && (47) \end{aligned}$$

In this particular case the integral over the covariance has to be performed numerical. One problem with these kinds of functional forms imply a full rank correlation matrix. For monte carlo simulation one would like to have only a small number of factors. One solution is to use the first factors of PCA of the functional form. An other is to use directly a functional Ansatz for the factors.

3.3.2. Functional Ansatz for the Factor Loadings

Only a two factor model should be discussed here. The generalization to more factors is straightforward, but complex. In the two factor case we have

$$\sigma^{\text{inst}}(\tau)^2 = \gamma_1(\tau)^2 + \gamma_2(\tau)^2 \quad . \tag{48}$$

A unique way of parameterization is therefor

$$\begin{aligned} \gamma_1(\tau) &= \sigma^{\text{inst}}(\tau)\frac{1}{\sqrt{2}}\left(g(\tau) - \sqrt{1 - g(\tau)^2}\right) && (49) \\ \gamma_2(\tau) &= \sigma^{\text{inst}}(\tau)\frac{1}{\sqrt{2}}\left(g(\tau) + \sqrt{1 - g(\tau)^2}\right) \quad . && (50) \end{aligned}$$

The correlation function becomes

$$\rho(\tau_1, \tau_2) = g(\tau_1)g(\tau_2) + \sqrt{1 - g(\tau_1)^2}\sqrt{1 - g(\tau_2)^2} \quad . \tag{51}$$

A good Ansatz is

$$g(\tau) = g_\infty + (1 - g_\infty)\, \mathrm{e}^{-\beta\tau} \quad . \tag{52}$$

3.3.3. Free Form of the Implied Correlation with Smoothing Constrains

Normally, we have anyway to calculate numerically the remaining integral over the instantaneous covariance. Assuming time homogeneity one can replace the instantaneous volatilities and correlation by the stepwise constant parameters , i.e.

$$\begin{aligned} \gamma_p(T_i - T_m) &\longrightarrow b_{p,i-m} && (53) \\ \sigma^{\text{inst}}(T_i - T_m) &\longrightarrow s_{i-m} && (54) \\ \rho(T_k - T_m, T_l - T_m) &\longrightarrow \alpha_{k-m,l-m} \quad . && (55) \end{aligned}$$

The integrals are then replaced by sums

$$\left(\sigma_i^{\text{Black}}\right)^2 T_i = \int_0^{T_i} \mathrm{d}\,t \left(\sigma^{\text{inst}}(T_i - t)\right)^2 = \sum_{m=1}^{i} \delta_{m-1} s_{i-m}^2 = K_{ii}^i \tag{56}$$

and

$$K_{kl}^i = \int_0^{T_i} \mathrm{d}\,t \sigma^{\text{inst}}(T_k - t)\rho(T_k - t, T_l - t)\sigma^{\text{inst}}(T_l - t) = \sum_{m=1}^{i} \delta_{m-1} s_{k-m}\alpha_{k-m,l-m}s_{l-m} \quad . \tag{57}$$

Strictly speaking, the time homogeneity assumption for the discrete version is only possible for δ_k to be constant for all k. The discrete version is therefor a index model. To account the full tenor structure one needs a complicated interpolation technique. In general the following equation is valid

$$s_{i-m}^2 = \frac{1}{\delta_{m-1}} \int_{T_i - T_m}^{T_i - T_{m-1}} \mathrm{d}\,t \left(\sigma^{\text{inst}}(t)\right)^2 \quad . \tag{58}$$

This implies only for a homogenous tenor structure

$$s_j^2 = \frac{1}{\delta} \int_{T_j}^{T_{j+1}} \mathrm{d}\,t \left(\sigma^{\text{inst}}(t)\right)^2 \quad . \tag{59}$$

In the discrete version the correlation matrix can be written as.

$$\alpha_{ij} = \sum_{p=1}^{s} b_{pi} b_{pj} \quad . \tag{60}$$

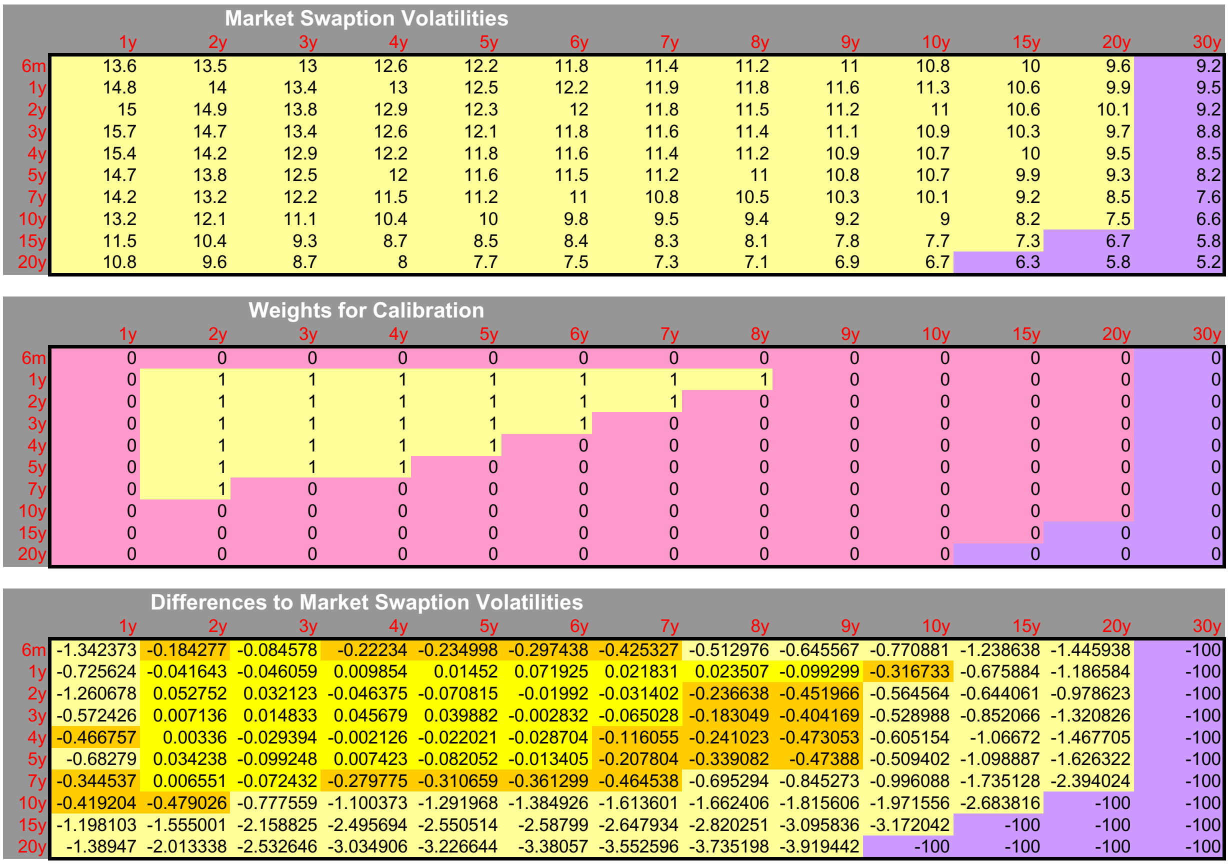
To calibrate the correlation one has to optimize the b-loadings. However, due to the properties of a correlation matrix, this is a constrained optimization. Using the following parameterization this can be reformulated as a unconstrained optimization (see Rebonato):

$$b_{pk} = \cos(\theta_{pk}) \prod_{q=1}^{p-1} \sin(\theta_{qk}) \quad ; \quad p = 1, \ldots, s-1 \tag{61}$$

$$b_{sk} = \prod_{q=1}^{s-1} \sin(\theta_{qk}) \quad . \tag{62}$$

With a full set of angles θ_{pk} the calibration to swaption volatilities is overdetermined. With additional smoothing cost functions one can avoid over fitting. One cost function could be

$$V^{\text{smoothing}} = \sum_{k=1,l=2}^{i} \left(\alpha_{k,l} - \alpha_{k,l-1}\right)^2 \quad . \tag{63}$$



Market Swaption Volatilities

	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y	15y	20y	30y
6m	13.6	13.5	13	12.6	12.2	11.8	11.4	11.2	11	10.8	10	9.6	9.2
1y	14.8	14	13.4	13	12.5	12.2	11.9	11.8	11.6	11.3	10.6	9.9	9.5
2y	15	14.9	13.8	12.9	12.3	12	11.8	11.5	11.2	11	10.6	10.1	9.2
3y	15.7	14.7	13.4	12.6	12.1	11.8	11.6	11.4	11.1	10.9	10.3	9.7	8.8
4y	15.4	14.2	12.9	12.2	11.8	11.6	11.4	11.2	10.9	10.7	10	9.5	8.5
5y	14.7	13.8	12.5	12	11.6	11.5	11.2	11	10.8	10.7	9.9	9.3	8.2
7y	14.2	13.2	12.2	11.5	11.2	11	10.8	10.5	10.3	10.1	9.2	8.5	7.6
10y	13.2	12.1	11.1	10.4	10	9.8	9.5	9.4	9.2	9	8.2	7.5	6.6
15y	11.5	10.4	9.3	8.7	8.5	8.4	8.3	8.1	7.8	7.7	7.3	6.7	5.8
20y	10.8	9.6	8.7	8	7.7	7.5	7.3	7.1	6.9	6.7	6.3	5.8	5.2

Weights for Calibration

	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y	15y	20y	30y
6m	0	0	0	0	0	0	0	0	0	0	0	0	0
1y	0	1	1	1	1	1	1	1	0	0	0	0	0
2y	0	1	1	1	1	1	1	0	0	0	0	0	0
3y	0	1	1	1	1	1	0	0	0	0	0	0	0
4y	0	1	1	1	1	0	0	0	0	0	0	0	0
5y	0	1	1	1	0	0	0	0	0	0	0	0	0
7y	0	1	0	0	0	0	0	0	0	0	0	0	0
10y	0	0	0	0	0	0	0	0	0	0	0	0	0
15y	0	0	0	0	0	0	0	0	0	0	0	0	0
20y	0	0	0	0	0	0	0	0	0	0	0	0	0

Differences to Market Swaption Volatilities

	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y	15y	20y	30y
6m	-1.342373	-0.184277	-0.084578	-0.22234	-0.234998	-0.297438	-0.425327	-0.512976	-0.645567	-0.770881	-1.238638	-1.445938	-100
1y	-0.725624	-0.041643	-0.046059	0.009854	0.01452	0.071925	0.021831	0.023507	-0.099299	-0.316733	-0.675884	-1.186584	-100
2y	-1.260678	0.052752	0.032123	-0.046375	-0.070815	-0.01992	-0.031402	-0.236638	-0.451966	-0.564564	-0.644061	-0.978623	-100
3y	-0.572426	0.007136	0.014833	0.045679	0.039882	-0.002832	-0.065028	-0.183049	-0.404169	-0.528988	-0.852066	-1.320826	-100
4y	-0.466757	0.00336	-0.029394	-0.002126	-0.022021	-0.028704	-0.116055	-0.241023	-0.473053	-0.605154	-1.06672	-1.467705	-100
5y	-0.68279	0.034238	-0.099248	0.007423	-0.082052	-0.013405	-0.207804	-0.339082	-0.47388	-0.509402	-1.098887	-1.626322	-100
7y	-0.344537	0.006551	-0.072432	-0.279775	-0.310659	-0.361299	-0.464538	-0.695294	-0.845273	-0.996088	-1.735128	-2.394024	-100
10y	-0.419204	-0.479026	-0.777559	-1.100373	-1.291968	-1.384926	-1.613601	-1.662406	-1.815606	-1.971556	-2.683816	-100	-100
15y	-1.198103	-1.555001	-2.158825	-2.495694	-2.550514	-2.58799	-2.647934	-2.820251	-3.095836	-3.172042	-100	-100	-100
20y	-1.38947	-2.013338	-2.532646	-3.034906	-3.226644	-3.38057	-3.552596	-3.735198	-3.919442	-100	-100	-100	-100

Figure 4: A crucial point in the calibration of the correlation matrix is the selection of the reference swaptions.

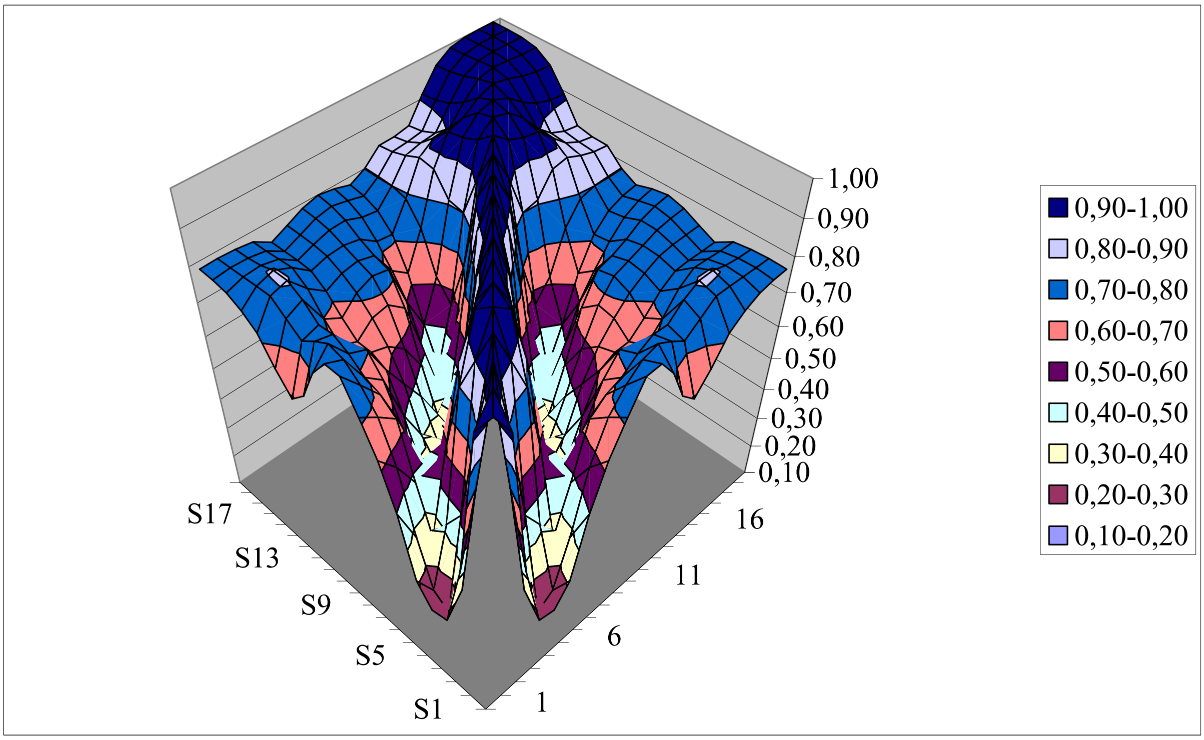


Figure 5: Implied correlation function obtained from a 5 factor model calibrated to cap and swaption volatilities.

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4. Stochastic Volatility Extension of the Libor Market Model

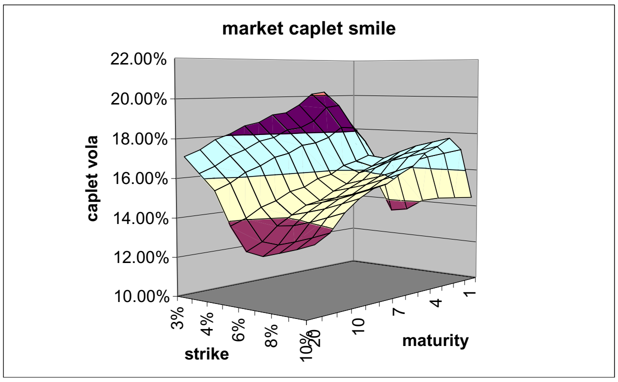

Figure 6: Market caplet volatility smile.

4.1. Smile/Skew modeling by other authors:

- Andersen and Andreasen: Constant Elasticity of Variance (CEV) Model
- Brigo and Mercurio: Mixture of Lognormal Model
- ???: Shifted Lognormal (SL) Model
- Rebonato: Stochastic parameters for the functional form of instantaneous volatility (a, b, c, d)

4.2. Stoachstic Volatility Extension

We will discuss a Libor rate model with Heston style stochastic volatility and CEV/SL:

$$
\begin{aligned}
\mathrm{d}\,L_i(t) &= \phi(L_i(t))\left(\mu_i^{Q^{N+1}}(t)\mathrm{d}\,t + g_i(t)\sqrt{V_i(t)}\mathrm{d}\,W_i(t)\right) && (64)\\
\mathrm{d}\,V_i(t) &= \lambda_i(t)\left(\bar{V}_i(t) - V_i(t)\right)\mathrm{d}\,t + \eta_i(t)\sqrt{V_i(t)}\mathrm{d}\,Z_i(t) \quad , && (65)
\end{aligned}
$$

where,

$\phi(L)$: function to model CEV or SL
$g_i(t)$: deterministic part of volatility
$\lambda_i(t)$: mean reversion speed
$\bar{V}_i(t)$: long term mean of V
$\eta_i(t)$: volatility of variance

The CEV model is obtained for $\phi(L) = L^\gamma$ $(0 < \gamma < 2)$, the SL model for $\phi(L) = L + \alpha$. In case of deterministic volatility a closed form solution for both models is known.

In the following we will model the skew with the $\phi(L)$ function and the smile with the Heston Stochastic Volatility equation.

The **assumption** that Z is uncorrelated to W allows a technical simplification. However, it is empirically justified that caplet volatilities are uncorrelated to Libor rates.

In case of zero correlation one can adapt the Hull & White (1987) result and simulate the stochastic volatility part only. The price C_i of the i-th caplet is then given by the weighted average of the solutions for deterministic volatility $DS(x)$ with the probability density of the realized variance $\Phi(x)$

$$
C_i = \int DS(x_i)\Phi(x_i)\mathrm{d}\,x_i \qquad (66)
$$

where the probability density $\Phi(s)$ can be calculated in a monte carlo simulation of

$$
x_i = \frac{1}{T_i}\int_0^{T_i} g_i(t)^2 V_i(t)\mathrm{d}\,t \qquad (67)
$$

In the following we will use the Shifted Lognormal Model only (the solution of the deterministic volatility problem is simply the Black76 solution with shifted forward rate and Strike).

4.3. Euler Simulation Schema for Stochastic Volatility Libor Market Model

Transforming stochastic differential equations to a form with additive Wiener process using Ito's lemma

$$
\begin{aligned}
\mathrm{d}\,\ln(L_i(t)+\alpha) &= \left(\mu_i^{Q^{N+1}}(t) - \frac{1}{2}g_i(t)^2 V_i(t)\right)\mathrm{d}\,t + g_i(t)\sqrt{V_i(t)}\mathrm{d}\,W_i(t) && (68)\\
\mathrm{d}\,2\sqrt{V_i(t)} &= \frac{1}{\sqrt{V_i(t)}}\left(\lambda_i(t)\left(\bar{V}_i(t) - V_i(t)\right) - \frac{1}{4}\eta^2\right)\mathrm{d}\,t + \eta_i(t)\mathrm{d}\,Z_i(t) \quad . && (69)
\end{aligned}
$$

one can obtain the following Euler difference schema

$$
\begin{aligned}
L_i(t_{k+1}) &= (L_i(t_k)+\alpha)\exp\left\{\left(\mu_i^{Q^{N+1}}(t_k) - \frac{1}{2}g_i(t_k)^2 V_i(t_k)\right)\Delta + g_i(t_k)\sqrt{V_i(t_k)}\sqrt{\Delta}W_i(k)\right\} - \alpha && (70)\\
V_i(t_{k+1}) &= \left(\sqrt{V_i(t_k)} + \frac{1}{2\sqrt{V_i(t_k)}}\left(\lambda_i(t_k)\left(\bar{V}_i(t_k) - V_i(t_k)\right) - \frac{1}{4}\eta_i(t_k)^2\right)\Delta + \eta_i(t)\sqrt{\Delta}Z_i(k)\right)^2 && (71)
\end{aligned}
$$

with $\Delta = t_{k+1} - t_k$ and $W_i(k), Z_i(k)$ drawn from Gaussian distribution $N(1,0)$. Note, due to the Hull & White result we only need to simulate the stochastic volatility part (equation 71).

4.4. Time Homogeneous Parameterization

As for the deterministic volatility Libor market model we require that all model parameters are time homogenous, i.e.

$$
g_i(t) = g(T_i - t) \qquad (72)
$$

$$
\begin{aligned}
\lambda_i(t) &= \lambda(T_i - t) \qquad (73)\\
\bar{V}_i(t) &= \bar{V}(T_i - t) \qquad (74)\\
\eta_i(t) &= \eta(T_i - t) \qquad (75)\\
\langle \mathrm{d}\, Z_i(t) \mathrm{d}\, Z_j(t) \rangle &= \rho(T_i - t, T_j - t) \mathrm{d}\, t \quad . \qquad (76)
\end{aligned}
$$

If "perfect" calibration to at-the-money or fixed strike caplet volatilities is required we can utilize the initial conditions of the hidden stochastic volatility process, i.e.

$$V_i(0) \neq \bar{V}_i(0) = \bar{V}(T_i) \quad . \tag{77}$$

4.5. Simples Implementation

In the following we will discuss the smallest model setup: the one stochastic volatility factor model. Additionally we reduce the number of model parameter to the minimum number and use the already above utilized functional form for the humped forward rate volatility curve.

$$
\begin{aligned}
g(t) &= (a + b\, t)\, \mathrm{e}^{-c\, t} + d \qquad (78)\\
\lambda(t) &= \lambda \qquad (79)\\
\bar{V}(t) &= 1 \qquad (80)\\
\eta(t) &= \eta \qquad (81)\\
\rho(t, t') &= 1 \qquad (82)
\end{aligned}
$$

Although we have a one factor model for the stochastic volatility part we still have one stochastic volatility equation for each maturity (only the wiener processes are identically).

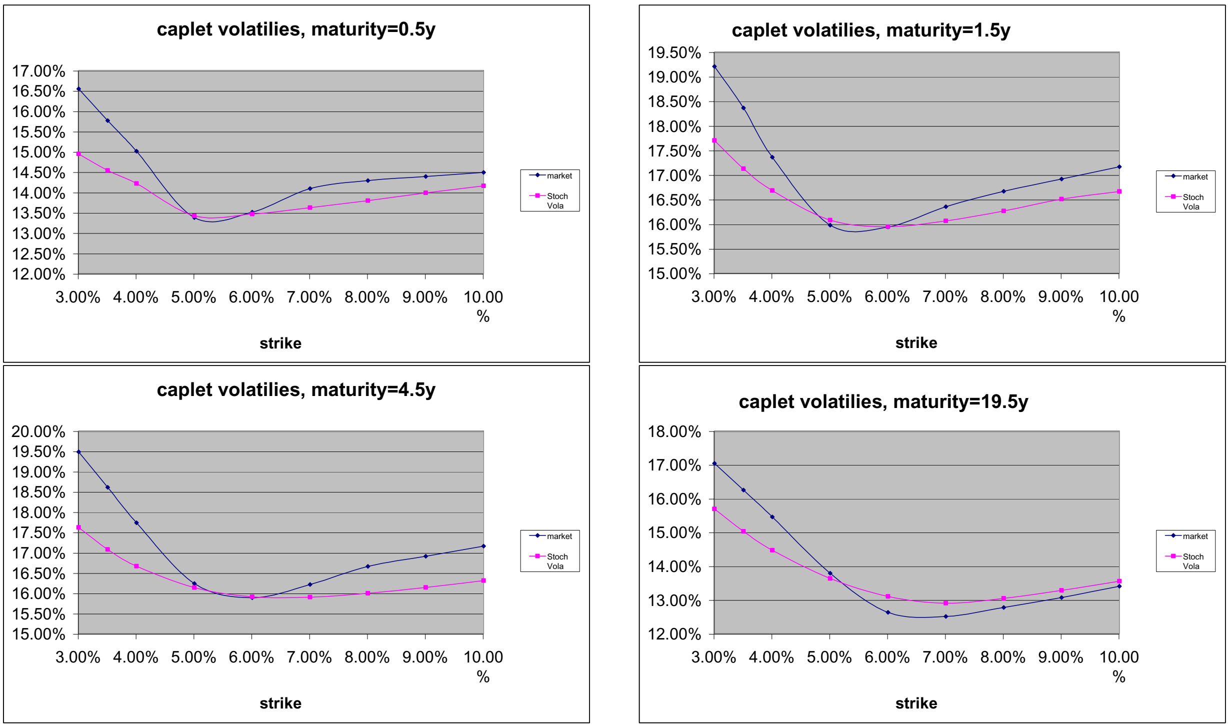

Figure 7: Calibration of the caption smile with one set of parameters. The initial values for the stochastic volatility function where chosen to fit the 5% strike values almost perfectly.

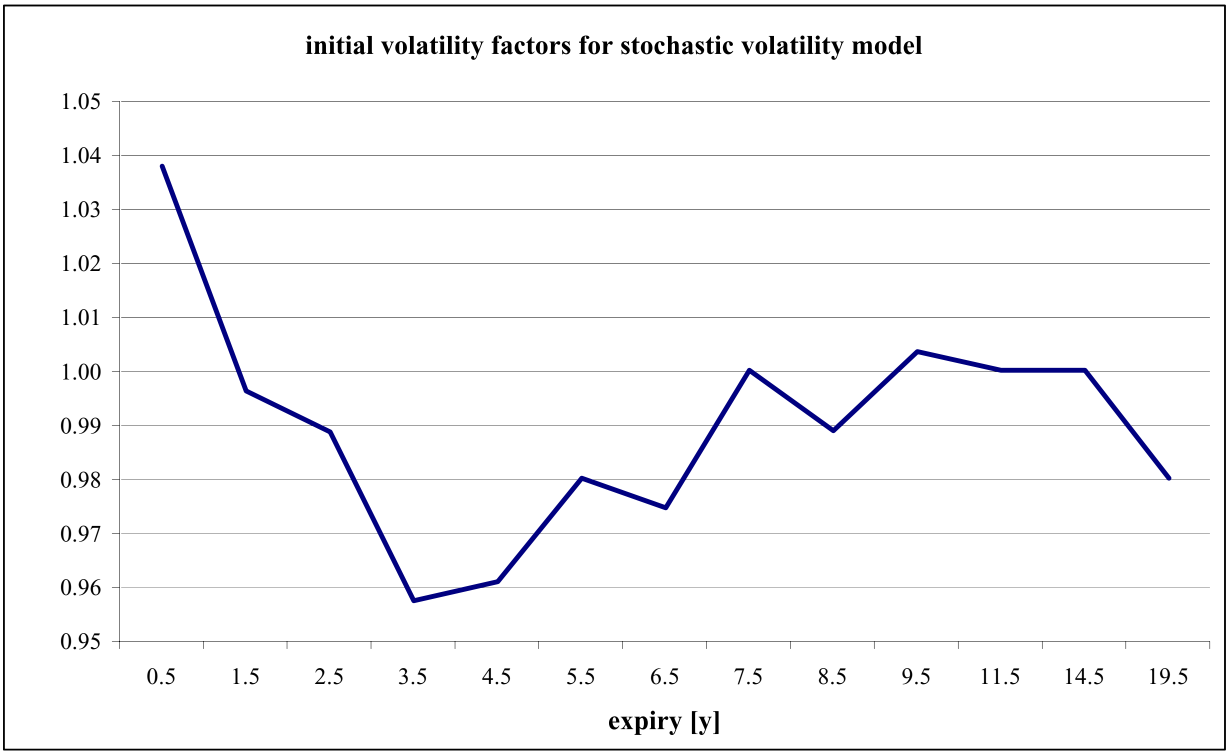

Figure 8: These initial values for the volatility process should be close to one. Deviations from one are needed for perfect fitting of the at-the-money or fixed strike caplet volatilities.