MathFinance Workshop 2002 - Frankfurt: Pricing and Hedging Generalized Passport Options with Collocation Finite Elements

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Abstract

Except for special cases, passport options and their extensions do not have closed-form solutions. Here we show how to derive approximate solutions using finite element methods. We also show that finite elements offer advantages in computing the hedge parameters.

The Product and its Extensions

- passport calls let the holder of the option hold all gains on a trading account, any losses are compensated by the writer of the option. The holder can take long and short positions of the same size.
- vacation calls are generalized passport options with different limits on long and short positions
- caps, floors and knock-out barriers can be applied to cheapen the option as usual

Assumptions

- the trading account is on an equity or a foreign currency, for which the usual BS assumptions hold
- definition of the trading gain: Summing up over all periods, the investor's total gain w is

$$w = \sum_{i=0}^{H-1} u(t_i)\,[S(t_{i-1}) - S(t_i)] \qquad (1)$$

Assuming continuous trading, i.e. $\lim_{i\to 0}(t_i - t_{i+1}) = 0$ the gain can be expressed as:

$$w(t) = \int_0^t u(s)dS(s) \qquad (2)$$

$$\Longleftrightarrow dw(t) = u(t)dS(t) \qquad (3)$$

$$\text{with } w(0) = 0 \qquad (4)$$

Assumptions (continued)

- different assumption on interest rates:
  - Ahn/Penaud/Wilmott: There is only one interest rate for borrowing and lending. The underlying does not pay any dividends.
  - Andersen/Andreasen/Brotherton-Ratcliffe: There is only one interest rate r for borrowing and lending. The underlying pays a continuous dividend yield γ.
  - Hyer/Lipton-Lifschitz/Pugachevsky: There are different rates for borrowing r and lending $\bar{r}$.

Pricing PDEs

- given optimal strategy u^*:

$$(r-\gamma)S\left(\frac{\partial V}{\partial S}+u^*\frac{\partial V}{\partial w}\right)-rV-\frac{\sigma^2S^2}{2}\left(\frac{\partial^2 V}{\partial S^2}+2u^*\frac{\partial^2 V}{\partial S\partial w}+(u^*)^2\frac{\partial^2 V}{\partial w^2}\right)=V_t \quad (5)$$

- unknown optimal strategy μ^*:

$$(r-\gamma)SV_S+rwV_w-rV-\frac{\sigma^2S^2}{2}\max_{a\le\mu^*\le b}\left[V_{SS}+2\mu^*V_{Sw}+(\mu^*)^2V_{ww}\right]=V_t \quad (6)$$

- unknown optimal strategy μ^* with different rates for borrowing and lending:

$$rSV_S-rV-\max_{a\le u^*\le b}\left([\bar{r}w+u^*(r-\bar{r})]+\frac{\sigma^2S^2}{2}\left[V_{SS}+2u^*V_{Sw}+(u^*)^2V_{ww}\right]\right)=V_t \quad (7)$$

Example 1

- passport call with $\gamma \neq r \quad \Longrightarrow \quad u^*$ is known:

$$(r-\gamma)S\left(\frac{\partial V}{\partial S}+u^*\frac{\partial V}{\partial w}\right)-rV-\frac{\sigma^2S^2}{2}\left(\frac{\partial^2 V}{\partial S^2}+2u^*\frac{\partial^2 V}{\partial S\partial w}+(u^*)^2\frac{\partial^2 V}{\partial w^2}\right)=V_t \quad (8)$$

with $x \equiv w/S$ reduction by one spatial variable:

$$\frac{\partial v}{\partial t}+(u^*-x)(r-\gamma)\frac{\partial v}{\partial x}+\frac{1}{2}(u^*-x)^2\sigma^2\frac{\partial^2 v}{\partial x^2}=\gamma v \quad (9)$$

eq. (9) has the following properties:

- nonlinear parabolic PDE
- one spatial variable plus time (to maturity)
- second order

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Example 2

- arithmetic Asian call with discrete monitoring

$$rSV_S - rV - \max_{a \le u^* \le b} \left([\bar{r}w + u^*(r - \bar{r})] + \frac{\sigma^2 S^2}{2} \left[V_{SS} + 2u^* V_{Sw} + (u^*)^2 V_{ww} \right] \right) = V_t \tag{10}$$

eq. (10) can be reduced to:

$$v_t + r(q - x)v_z + \frac{1}{2}(q - x)^2 \sigma^2 v_{xx} = 0 \tag{11}$$

with the final condition

$$v(T, x) = x^+ \tag{12}$$

eq. (11) has the following properties:

  - linear parabolic PDE
  - one spatial variable plus time (to maturity)
  - second order

Example 3

- passport call with a knock-out barrier on w:

$$(r-\gamma)SV_S + rwV_w - rV - \frac{\sigma^2 S^2}{2} \max_{-a\le\mu^*\le a} \left[V_{SS} + 2\mu^* V_{Sw} + (\mu^*)^2 V_{ww}\right] = V_t \quad (13)$$

no transformation to reduce number of variables is known. Eq. (13) has the following properties:

  - nonlinear parabolic PDE
  - two spatial variable plus time (to maturity)
  - second order
  - max-operator

1D Collocation FE

First, we consider the stationary problem $L(u(x)) = f$, L being a non-linear differential operator. Time is integrated in a later step. We look for an approximate solution $\tilde{u}$ for the following problem:

$$L(u) = f, \;\; u(x_{min}) = u_{min}, \;\; u(x_{max}) = u_{max} \tag{14}$$

An approximate solution $\tilde{u}$ is of the following form:

$$\tilde{u} = \sum_{k=0}^{N+1} a_k \phi_k(x) \tag{15}$$

The task is to find values for a_i which make $\tilde{u}$ the "best" approximation. While the Galerkin finite element methods normally used in derivative pricing determine the a_i by solving

$$\int_{x_{min}}^{x_{max}} (L(\tilde{u}) - f)\, \phi_k \, dx = 0, \; k = 1, \ldots, N \tag{16}$$

collocation finite element methods take the following approach:

$$(L(\tilde{u}) - f)\,(\xi_k) = 0, \; k = 1, \ldots, N \tag{17}$$

1D Collocation FE (continued)

The collocation points for each element $[x_i; x_{i+1}]$ in the method used here are

$$\xi_{2i+1,2i+2} = x_i + \left(\frac{1}{2} \pm \frac{1}{2\sqrt{3}}\right)(x_{i+1} - x_i) \tag{18}$$

1D Collocation FE (continued)

The approximate function is defined by

$$
\begin{aligned}
\tilde{u} &= \sum_{k=0}^{N+1} a_k \phi_k \\
&= a_0 H_0 + a_1 S_0 + a_2 H_1 + a_3 S_1 + \ldots + a_N H_{\frac{N}{2}} + a_{N+1} S_{\frac{N}{2}} \\
\text{with} \quad H_k(x) &= 3\left(\frac{x - x_{k-1}}{x_k - x_{k-1}}\right)^2 - 2\left(\frac{x - x_{k-1}}{x_k - x_{k-1}}\right)^3 \quad \text{for } x_{k-1} \le x \le x_k \\
&= 3\left(\frac{x_{k+1} - x}{x_{k+1} - x_k}\right)^2 - 2\left(\frac{x_{k+1} - x}{x_{k+1} - x_k}\right)^3 \quad \text{for } x_k \le x \le x_{k+1} \\
&= 0 \text{ elsewhere} \\
\text{and} \quad S_k(x) &= -\frac{(x - x_{k-1})^2}{(x_k - x_{k-1})} + \frac{(x - x_{k-1})^3}{(x_k - x_{k-1})^2} \quad \text{for } x_{k-1} \le x \le x_k \\
&= \frac{(x_{k+1} - x)^2}{(x_{k+1} - x_k)} - \frac{(x_{k+1} - x)^3}{(x_{k+1} - x_k)^2} \quad \text{for } x_k \le x \le x_{k+1} \\
&= 0 \text{ elsewhere}
\end{aligned}
$$

1D Collocation FE (continued)

The integration of the boundary conditions is achieved by

$$a_0 = u\left(x_{min}\right) \tag{19}$$
$$a_N = u\left(x_{max}\right) \tag{20}$$

Finding the N parameters $a_1, \ldots, a_{N-1}, a_{N+1}$ results in a system of non-linear equations:

$$\begin{aligned} f_1(a_1, \ldots, a_{N-1}, a_{N+1}) &= 0 \\ &\vdots \\ f_N(a_1, \ldots, a_{N-1}, a_{N+1}) &= 0 \end{aligned} \tag{21}$$

This system of non-linear equations is solved via Newton's method.

1D Collocation FE (continued)

Spatial variables are discretized with finite elements while time is treated with finite differences. This can be visualised as the non-linear elliptic operator $L(u)$ evolving through time. Each equation corresponds to a collocation point. The dynamic counterpart to eq. (21) is given by

$$\dot{\boldsymbol{a}} = \begin{pmatrix} \dot{a}_1 \\ \vdots \\ \dot{a}_{N-1} \\ \dot{a}_{N+1} \end{pmatrix} = \begin{pmatrix} g_1\left(a_1, \ldots, a_{N-1}, a_{N+1}\right) \\ \vdots \\ g_N\left(a_1, \ldots, a_{N-1}, a_{N+1}\right) \end{pmatrix} \tag{22}$$

$$\Longrightarrow \quad \text{stiff system of non-linear } \textit{ordinary} \text{ differential equations} \tag{23}$$

Solving eq. (22) is possible with various time-stepping procedures:

- second order Adams-Moulton
- first order backward difference

2D Collocation FE

We again consider the stationary problem $L\left(u(x,y)\right) = f$, L being a non-linear differential operator. We search for an approximate solution $\tilde{u}(x,y)$ for the following problem:

$$\begin{aligned} L(u) &= f \;, && (24) \\ u(x_{min}) &= u_{min_x},\;\; u(x_{max}) = u_{max_x} \;, && (25) \\ u(y_{min}) &= u_{min_y},\;\; u(y_{max}) = u_{max_y} \;, && (26) \end{aligned}$$

That is, we consider a rectangular domain with Dirichlet conditions. This method can easily be generalized for non-rectangular domains and Neumann and mixed boundary condition. An approximate solution $\tilde{u}$ can take the following form:

$$\tilde{u} = \sum_{k=1}^{N} a_k \phi_k(x,y) \;, \tag{27}$$

Again, the task is to find values for a_i which make $\tilde{u}$ the “best” approximation.

2D Collocation FE (continued)

While the Galerkin finite element methods sometimes used in two-asset derivative pricing determine the a_i's by solving

$$\int_{x_{min}}^{x_{max}} \int_{y_{min}}^{y_{max}} \left(L(\tilde{u}) - f\right) \phi_k \, dy \, dx = 0, \; k = 1, \ldots, N \; , \tag{28}$$

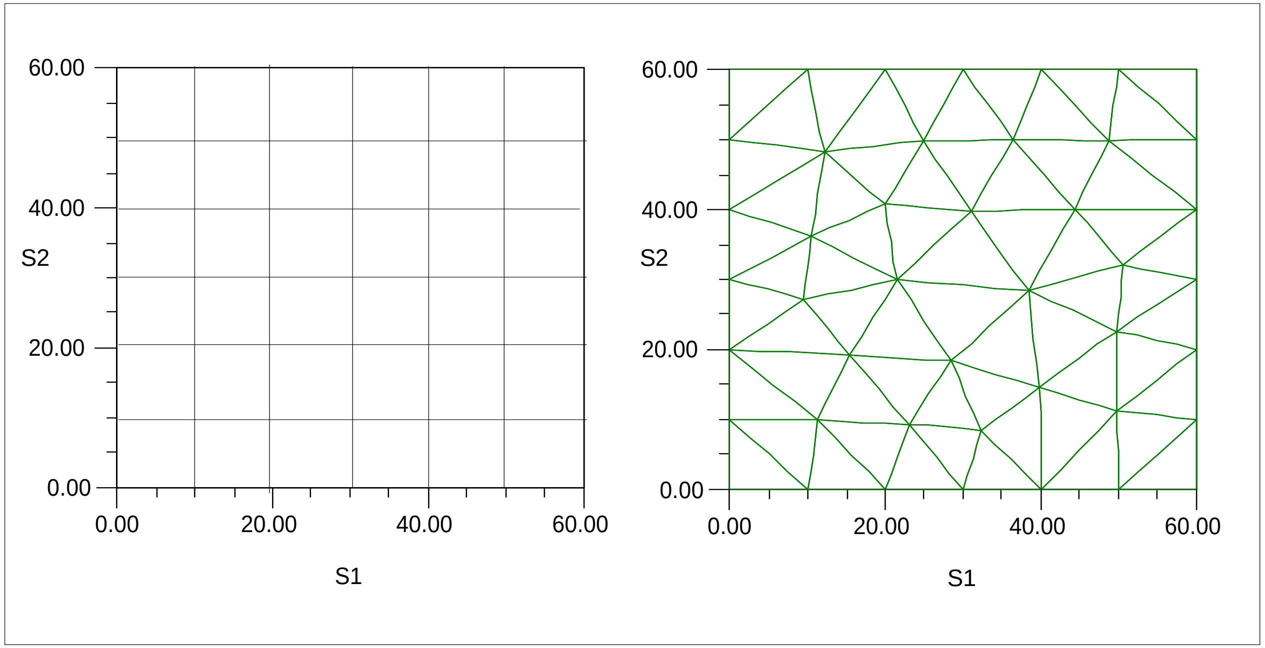
collocation finite element methods take the following approach. They enforce that at certain points in the domain, the so-called *collocation points*, the PDE is exactly satisfied.

$$\left(L(\tilde{u}) - f\right)(\xi_k) = 0, \; k = 1, \ldots, N \tag{29}$$

The domain is divided into disjoint elements, the *finite elements*. These finite elements are usually either rectangular or triangular. The latter type of element can form either a structured or an unstructured grid. Here we employ rectangular elements which are not necessarily of the same size. The four collocation points for each element $[x_i; x_{i+1}] \times [y_i; y_{i+1}]$ in the method used here are

$$\left(x_i + \left(\frac{1}{2} \pm \frac{1}{2\sqrt{3}} \right) (x_{i+1} - x_i); y_i + \left(\frac{1}{2} \pm \frac{1}{2\sqrt{3}} \right) (y_{i+1} - y_i) \right) \tag{30}$$

2D Collocation FE (continued)



structured vs. unstructured grids

2D Collocation FE (continued)

Let s_i denote the number of gridlines in direction i. The approximate solution is defined by a linear combination of $4s_xs_y$ basis functions

$$H_i(x)H_j(y),\ H_i(x)S_j(y),\ S_i(x)H_j(y),\ S_i(x)S_j(y) \tag{31}$$

The approximate solution $\tilde{u}(x,y)$ has the form

$$\tilde{u}(x,y) = \sum_{i=1}^{s_x}\sum_{j=1}^{s_y}\left(A_{ij}H_i(x)H_j(y) + B_{ij}H_i(x)S_j(y) + C_{ij}S_i(x)H_j(y) + D_{ij}S_i(x)S_j(y)\right) \tag{32}$$

2D Collocation FE (continued)

... with $N = 4s_xs_y$ unknowns A_{11}, B_{11}, C_{11}, D_{11}, A_{12}, B_{12}, C_{12}, D_{12}, ..., $A_{s_xs_y}$, $B_{s_xs_y}$, $C_{s_xs_y}$, $D_{s_xs_y}$ which are relabeled to $a_1, \ldots, a_N$ for notational convenience. The approximating function $\tilde{u}(x, y)$ has a continuous mixed derivative $\tilde{u}_{xy}$ since the mixed derivative of each of the basis functions is continuous.

The approximate solution is required to satisfy the PDE exactly at the four collocation points in each of the $(s_x-1)(s_y-1)$ subrectangles, and to satisfy the boundary conditions at certain points. The number of boundary collocation points plus the number of interior collocations points $4(s_x-1)(s_y-1)$ is equal to the number of basis functions $4s_xs_y$, which is equal to the number of unknowns N. Finding the N parameters $a_1, \ldots, a_N$ results in a system of non-linear equations:

$$\begin{aligned} f_1(a_1, \ldots, a_N) &= 0 \\ &\vdots \\ f_N(a_1, \ldots, a_N) &= 0 \end{aligned} \tag{33}$$

2D Collocation FE (continued)

The dynamic counterpart to eq. (33) is given by

$$\dot{\boldsymbol{a}} = \begin{pmatrix} \dot{a}_1 \\ \vdots \\ \dot{a}_N \end{pmatrix} = \begin{pmatrix} g_1\left(a_1, \ldots, a_N\right) \\ \vdots \\ g_N\left(a_1, \ldots, a_N\right) \end{pmatrix} \tag{34}$$

which has similar properties as the its 1D counterpart.

Example 1 (continued): Passport Call

$$\frac{\partial v}{\partial t} + (u^* - x)(r-\gamma)\frac{\partial v}{\partial x} + \frac{1}{2}(u^* - x)^2\sigma^2\frac{\partial^2 v}{\partial x^2} = \gamma v \tag{35}$$

$$u^* = \text{sign}\left((r-\gamma)\frac{\partial v}{\partial x} - x\sigma^2\frac{\partial^2 v}{\partial x^2}\right) \tag{36}$$

$$v(T,x) = x^+ \tag{37}$$

$$v(t, -e^{h\sigma\sqrt{T-t}}) = 0 \tag{38}$$

$$\frac{\partial v(t, e^{h\sigma\sqrt{T-t}})}{\partial x} = e^{-r(T-t)} \tag{39}$$

$$h = 4 \tag{40}$$

numerical details:

- 400 equidistant nodes
- 800 time steps (Crank-Nicolson)

Example 1 (continued): Passport Call

- data of the problem

Parameter	Value
Spot S	100.0
Dividend yield γ	5.0
Interest rate r	4.5
Volatility σ	0.3
Time to maturity	2 Years

- results

w	FD	FE (Galerkin)	FE (Coll.)	Hedge Ratio $k_{Galerkin}$	$k_{Collocation}$
20	28.2277	28.2249	28.2295	-0,4674	-0.4679
10	22.3741	22.3734	22.3760	-0,3724	-0.3729
0	17.4323	17.4423	17.4438	-	-
-10	13.5100	13.5113	13.5135	0.5180	0.5176
-20	10.4261	10.4293	10.4320	0,4302	0.4300

Example 2 (continued): Discrete Asian Call

$$\begin{aligned} v_t + r(q-x)v_z &+ \frac{1}{2}(q-x)^2\sigma^2 v_{xx} = 0 && (41)\\ v(T,x) &= x^+ && (42)\\ u(z_{min} = -1) &= 0 && (43)\\ \frac{\partial u(z_{max}=1)}{\partial z} &= 1 && (44)\\ q &= 1 - \frac{1}{n}\left[n\frac{t}{T}\right] && (45) \end{aligned}$$

numerical details:

- 200 equidistant nodes
- 1000 time steps (adaptive second Order Adams-Moulton sheme)

Example 2 (continued): Discrete Asian Call

- data of the problem

Parameter	Symbol	Value
Interest rate	r	0.09
Price of Underlying in $t = 0$	S_0	100
Time to Maturity	T	15 days
Equidistant Samplings		14

- results

	Premium			Delta		
Strike	Approximation	MC	FE	Approximation	MC	FE
95	5.202	5.194	5.203	0.979	0.974	0.975
100	1.139	1.135	1.144	0.536	0.534	0.523
105	0.038	0.038	0.039	0.037	0.030	0.036

Example 3: Knock-out Passport Call

$$(r-\gamma)SV_S + rwV_w - rV \quad -\frac{\sigma^2 S^2}{2} \quad \max_{a \le \mu^* \le b}\left[V_{SS} + 2\mu^* V_{Sw} + (\mu^*)^2 V_{ww}\right] = V_t \tag{46}$$

$$-a = b \tag{47}$$

$$V(S,w,T) = \max(0,w) \tag{48}$$

$$V(0,w,t) = \max(0,w) \tag{49}$$

$$\frac{\partial^2 V(200,w,t)}{\partial S^2} = 0 \tag{50}$$

$$V(S,H,t) = 0 \tag{51}$$

$$V(S,200,t) = w \tag{52}$$

Example 3 (continued): Knock-out Passport Call

numerical details

- rectangular mesh of 19 nodes in S and 42 nodes in w
- adaptive second order Adams Moulton method
- max-operator: one has to solve the following optimization problem for each node in each time step:

$$\max_{a\leq\mu^*\leq b}(V_{SS}+2\mu^*V_{Sw}+(\mu^*)^2V_{ww}) \tag{53}$$

This is just a quadratic polynomial in μ^*, so that the existence of a maximum is guaranteed. The maximum has to occur at either $\mu^*=a$, $\mu^*=b$ or $\mu^*=-\frac{V_{Sw}}{V_{ww}}$ (if $a<\mu^*<b$), the latter point being the vertex of the quadratic polynomial. All three values are calculated and the maximum is taken.

Example 3 (continued): Knock-out Passport Call

- data of the problem

Parameter	Value
Spot S	100.0
Dividend yield γ	5.0
Interest rate r	4.5
Volatility σ	0.3
Time to maturity	2 Years

- results

w	Location of barrier		
	$w = -20$	$w = -30$	$w = -40$
100	100.0000	100.1408	100.1602
50	51.4286	51.5643	51.5798
20	25.3353	25.7934	25.8769
10	18.0489	18.7373	18.8666
0	11.7753	12.9275	13.1203
-10	6.0061	8.3566	8.8363
-20	0.0000	4.3371	5.6642
-50	0.0000	0.0000	0.0000
-100	0.0000	0.0000	0.0000
Time Steps	64	60	104

Conclusions

- FE allow the exact computation of the option's premium
- some of the Greeks come as a by-product
- off-the-shelf software can be used (here: PDE2D)
- much quicker than brute force methods such as MC