A Finite Element Implementation of Generalized Passport Options

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Abstract

Except for special cases, generalized passport options do not have closed-form solutions. Here we show how to derive approximate solutions using a collocation finite element method. We also show that finite elements offer advantages in computing the hedge parameters. This is illustrated with various Asian and passport options as examples.

Contents

1 Introduction

Passport options are a new kind of financial instrument introduced by Bankers Trust in 1997. They are used to protect trading accounts. The basic passport option allows the holder to take the profit from a trading account while any losses are covered by the writer of the option. The maximal amount a trader can go either long or short is limited to the same pre-specified amount. To make passport options cheaper, or to reduce the risk to the writer, certain exotic features such as caps, floors and barriers have been employed. The concept of passport options has been extended to general options on trading accounts where the limits for going short or long do not necessarily have to be equal anymore. This concept of an option on a trading account contains many special cases such as plain vanilla European and American options, passport options, and Asian options.

Passport options can be used to:

- protect the trading account of inexperienced traders;
- to price life insurance claims contingent on the performance of a reference fund [4];
- to develop new commodity hedging strategies [14].

2 The Pricing Model for Passport Options

2.1 Different Approaches

Currently, there are three approaches to pricing passport options which differ in the assumption on interest rates:

- Ahn/Penaud/Wilmott ([1], [2], [30]): There is only one interest rate for borrowing and lending. The underlying does not pay any dividends.
- Andersen/Andreasen/Brotherton-Ratcliffe [3]: There is only one interest rate r for borrowing and lending. The underlying pays a continuous dividend yield γ.
- Hyer/Lipton-Lifschitz/Pugachevsky [16]: There are different rates for borrowing and lending.

We have chosen the approach by Andersen/Andreasen/Brotherton-Ratcliffe because it can be extended easily by different interest rates for borrowing and lending ([3], S. 16). Also, we have adapted the notation of that paper. Different interest rates for borrowing and lending are a rather unusual assumption; even the authors of this approach discuss the special case of equal rates (the so-called *symmetric case*) in much more detail than the general case ([16], S. 129f). The approach by Ahn/Penaud/Wilmott is, therefore, obviously a special case of the model by Andersen/Andreasen/Brotherton-Ratcliffe with $\gamma = 0$.

2.2 European Passport Options

The starting point is the Black-Scholes framework, in which the underlying S follows the following stochastic differential equation; see [3]:

$$\frac{dS(t)}{S(t)} = (r - \gamma)\, dt + \sigma dW(t) \quad . \tag{1}$$

Consider an investor at t_i holding $u(t_i) \in [-1, 1]$ in this underlying. From t_i to t_{i+1} the investor gains $u(t_i)\,[S(t_{i+1}) - S(t_i)]$. Summing up over all periods, the investor's total gain w is

$$w = \sum_{i=0}^{H-1} u(t_i)\,[S(t_{i-1}) - S(t_i)] \quad . \tag{2}$$

Assuming continuous trading, i.e. $\lim_{i\to 0}(t_i - t_{i+1}) = 0$ the gain can be expressed as:

$$w(t) = \int_0^t u(s)dS(s) \tag{3}$$

$$\Longleftrightarrow \; dw(t) = u(t)dS(t) \;\text{ with }\; w(0) = 0 \quad . \tag{4}$$

The European passport option gives the holder the right but not the obligation to receive w in T. In case $w < 0$ the rational investor is not interested in delivery so that the payoff equals

$$[w(T)]^+ \equiv \max\,[0, w(T)] \tag{5}$$

For deriving the pricing equation we will use a similar argument as Black and Scholes: An instantaneously riskless portfolio Π consists of one passport option and $-k$ units of the underlying:

$$\Pi = V - kS \tag{6}$$

Within the time interval $(t, t + dt)$ the value of this portfolio changes by

$$d\Pi = dV - k(dS + \gamma S dt) \tag{7}$$

We assume the existence of an optimal strategy u^* and the derivatives V_{SS}, V_{ww}, and V_{Sw}. We also presuppose that the holder of the option maximizes his revenues without being hindered from taking u^* by hedging necessities or other superimposed circumstances. Then the following holds:

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial w}dw + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS)^2 + \frac{\partial^2 V}{\partial S \partial w}(dS\; dw) + \frac{1}{2}\frac{\partial^2 V}{\partial w^2}(dw)^2 \tag{8}$$

To simplify this equation, two more results are needed. Squaring eq. (1) gives:

$$(dS)^2 = \sigma^2 S^2 dt \tag{9}$$

The profit-maximizing behavior of the holder turns eq. (4) into:

$$dw = u^* dS \tag{10}$$

Putting these results into eq. (8) leads to:

$$dV = \frac{\partial V}{\partial t}dt + \left(\frac{\partial V}{\partial S} + u^* \frac{\partial V}{\partial w}\right) dS + \frac{1}{2}\left(\frac{\partial^2 V}{\partial S^2} + 2u^* \frac{\partial^2 V}{\partial S \partial w} + \frac{\partial^2 V}{\partial w^2}\right) \sigma^2 S^2 dt \tag{11}$$

The parameter k has to be chosen for the portfolio Π to become instantaneously riskless.

$$k = \frac{\partial V}{\partial S} + u^* \frac{\partial V}{\partial w} \tag{12}$$

Because of the absence of arbitrage the riskless portfolio Π has to grow by the same rate as a money market account r.

$$d\Pi = r\Pi dt \tag{13}$$

Combining above results:

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\left(\frac{\partial^2 V}{\partial S^2} + 2u^* \frac{\partial^2 V}{\partial S \partial w} + (u^*)^2 \frac{\partial^2 V}{\partial w^2}\right) + (r-\gamma)S\left(\frac{\partial V}{\partial S} + u^* \frac{\partial V}{\partial w}\right) = rV \tag{14}$$

with the following final condition:

$$V(T,S,w) = w^+ \quad , \tag{15}$$

the change of variables $x \equiv w/S$, reduces the dimensions of the problem by one. Using this substitution generates the following PDE:

$$\frac{\partial v}{\partial t} + (u^* - x)(r-\gamma)\frac{\partial v}{\partial x} + \frac{1}{2}(u^* - x)^2 \sigma^2 \frac{\partial^2 v}{\partial x^2} = \gamma v \tag{16}$$

with

$$u^* = \operatorname{sign}\left((r-\gamma)\frac{\partial v}{\partial x} - x\sigma^2 \frac{\partial^2 v}{\partial x^2}\right) \tag{17}$$

where the payoff function $v(x, t = T)$ has to be monotonically increasing and convex in x.[1] Equivalent formulations of eq. 17 are:

$$\frac{\partial v}{\partial t} - x(r-\gamma)\frac{\partial v}{\partial x} + \frac{1}{2}(1+x^2)\sigma^2 \frac{\partial^2 v}{\partial x^2} + u^*\left((r-\gamma)\frac{\partial v}{\partial x} - x\sigma^2 \frac{\partial^2 v}{\partial x^2}\right) = \gamma v \tag{18}$$

and:

$$\frac{\partial v}{\partial t} - x(r-\gamma)\frac{\partial v}{\partial x} + \frac{1}{2}(1+x^2)\sigma^2 \frac{\partial^2 v}{\partial x^2} + \left|(r-\gamma)\frac{\partial v}{\partial x} - x\sigma^2 \frac{\partial^2 v}{\partial x^2}\right| = \gamma v \tag{19}$$

with:

$$v(T,x) = v_T(x) \tag{20}$$

as a final condition. Only for $r = \gamma$ a closed-form solution is known (compare sec. 4.1.1). For $r \neq \gamma$ we have to resort to numerical techniques.

Eq. (19) is the PDE for the passport option value, given a strategy u^*. It is also possible to view this problem from a different perspective by looking for an equation that defines the optimal strategy μ^*. Via the principles of dynamic programming, a PDE, called the *Hamilton-Jacobi-Bellmann equation* (HJB), can be derived which defines this optimal strategy [18]:

$$(r-\gamma)SV_S + rwV_w - rV - \frac{\sigma^2 S^2}{2} \max_{-1 \le \mu^* \le 1}\left[V_{SS} + 2\mu^* V_{Sw} + (\mu^*)^2 V_{ww}\right] = V_t \tag{21}$$

[1] Non-convex Payoffs are discussed below in sec. 2.3.

This equation has to be used as a basis for numerical computations when u^* and μ^* are not known. Such problems arise when caps or barriers on either w and/or S are introduced. A transformation as $x \equiv w/S$ is then not possible anymore.

The *Hedge Ratio* k is slightly different to the Black-Scholes framework. The basic idea is that a portfolio Π consisting of a long call C and a short position in k shares S

$$\Pi = C - kS \tag{22}$$

is riskfree for an infinitesimal amount of time.[2] The hedge parameter in the Black-Scholes model is

$$k = \frac{\partial C}{\partial S} = \Delta \tag{23}$$

Portfolios containing passport options can be immunized against infinitesimal changes in the share via eq. (22). For k we have (compare ([3], S. 33f)):

$$k = v + (u^* - x)\frac{\partial v}{\partial x} \tag{24}$$

This implies that numerical difficulties arising in computing Δ are also present in computing k. Finite elements provide approximate solutions to the entire domain consisting of simple algebraic functions. Whenever $u^*(S)$ changes its sign, $k(x)$ shows a jump.

2.3 Non-Convex Payoffs

2.4 A Correction

According to ([3], Proposition 5) the result of the previous section can be generalized to non-convex payoffs by changing the control to $u^* \in [-1,1]$:

$$u^* = \begin{cases} \psi & \text{if } \psi(x,t) \in [-1,1] \text{ and } \frac{\partial^2 v}{\partial x^2} < 0 \\ \text{sign}(\psi(x,t)) & \text{otherwise} \end{cases} \tag{25}$$

with

$$\psi(x,t) = x - \frac{r-\gamma}{\sigma^2}\frac{\frac{\partial v}{\partial x}}{\frac{\partial^2 v}{\partial x^2}} \tag{26}$$

With the help of a simple counterexample it can be shown that this proposition is wrong. Convex payoffs are supposed to be a special case of eq. (25). This is not the case. We consider the special case of $r = \gamma$ which can easily be extended to $r \neq \gamma$. The unique optimal control u^* for convex payoffs is according to eq. (17) (see also [3], Proposition 2):

$$u^*(x,t) = \text{sign}\left((r-\gamma)\frac{\partial v}{\partial x} - x\sigma^2\frac{\partial^2 v}{\partial x^2}\right) \tag{27}$$

Inserting $r = \gamma$ simplifies the expression:

$$u^*(x,t) = \text{sign}\left(-x\sigma^2\frac{\partial^2 v}{\partial x^2}\right) \tag{28}$$

Convexity of v in x implies $\frac{\partial^2 v}{\partial x^2} \geq 0$. Together with $\sigma > 0$ it has to hold in $t \in [0,T[$

$$\text{sign}\left(-x\sigma^2\frac{\partial^2 v}{\partial x^2}\right) = \text{sign}(-x) \tag{29}$$

Inserting $r = \gamma$ into the general payoff eq. (25) gives:

$$u^* = \text{sign}(\psi) = \text{sign}(x) \tag{30}$$

Therefore, the convex payoff is not a special case of the general payoff. That shows that ([3], Proposition 5) is wrong.

[2] To be exact: Black and Scholes [5] consider a portfolio *short* in the call and *long* in the share. In order to keep conformity to [3] we have changed the positions.

2.5 General Payoffs

We will present the general control u^* for arbitrary payoffs in this section first.[3] Then we will deduce the special controls for convex and concave payoffs.

The general control is:

$$u^* = \begin{cases} \psi & \text{if } \psi(x,t) \in [-1,1] \text{ and } \frac{\partial^2 v}{\partial x^2} < 0 \\ \operatorname{sign}(-\psi \frac{\partial^2 v}{\partial x^2}) & \text{otherwise} \end{cases} \tag{31}$$

For strictly concave payoff functions ($\frac{\partial^2 v}{\partial x^2} < 0$) eq. (31) simplifies to:

$$u^* = \begin{cases} \psi & \text{if } \psi(x,t) \in [-1,1] \\ \operatorname{sign}(\psi) & \text{otherwise} \end{cases} \tag{32}$$

For (strictly and simply) convex payoff functions the following control function holds:

$$u^* = \operatorname{sign}\left(-\psi \frac{\partial^2 v}{\partial x^2}\right) \tag{33}$$

$$= \operatorname{sign}\left(\left[x - \frac{r-\gamma}{\sigma^2} \frac{\frac{\partial v}{\partial x}}{\frac{\partial^2 v}{\partial x^2}}\right] \frac{\partial^2 v}{\partial x^2}\right) \tag{34}$$

$$= \operatorname{sign}\left(\frac{1}{\sigma^2}\left[(r-\gamma)\frac{\partial v}{\partial x} + \sigma^2 x \frac{\partial^2 v}{\partial x^2}\right]\right) \tag{35}$$

$$= \operatorname{sign}\left((r-\gamma)\frac{\partial v}{\partial x} + \sigma^2 x \frac{\partial^2 v}{\partial x^2}\right) \tag{36}$$

since $\sigma^2 \geq 0$. This is eq. (18) from [3].

2.6 Integrating Early Exercise

Early exercise of the option can be integrated with a penalty function p. This function p ensures that in areas of the (t,S) space where early exercise is optimal, the pricing eq. (19) is forced to take on the intrinsic value of the option while it vanishes on the rest of the domain. For details of this technique and various specifications of p see [8] and [19].

$$\gamma v = \frac{\partial v}{\partial t} - x(r-\gamma)\frac{\partial v}{\partial x} + \frac{1}{2}(1+x^2)\sigma^2 \frac{\partial^2 v}{\partial x^2} + \left|(r-\gamma)\frac{\partial v}{\partial x} - x\sigma^2 \frac{\partial^2 v}{\partial x^2}\right| + p \tag{37}$$

$$p = c_{penalty}\left\{\min\left[v - \max\left(S,0\right),0\right]^2\right\} \tag{38}$$

According to [8], $c_{penalty}$ depends on the type of the element; according to our experience, it suffices to choose $c_{penalty}$ sufficiently large such as $c_{penalty} = 10^8$. Computing the value of American passport options can be greatly simplified when all interest rates in the model are identical and there are no dividends. Since you can always emulate early exercise with an European option by entering a zero position in the stock, the American option is worth the same.

2.7 Extending Passport Options to General Options on Trading Accounts

The defining property of an option on a trading account is that the control is not restricted to the closed interval $[-1,1]$ anymore ([23], [24]). Besides, we generalize eq. (4) to

$$dw(t) = u^*(t)dS(t) + \bar{r}\left[w - u^*(t)S(t)\right]dt \tag{39}$$

$$w(0) = w_0 \tag{40}$$

[3] For the results of this section I want to thank Leif Anderson.

The initial wealth is represented by w_0, and $\bar{r}$ is the interest rate corresponding to reinvesting the cash position $w - u^*(t)S(t)$ (possibly different from the risk-neutral interest rate r).

$$rSV_S - rV - \max_{a \le u^* \le b} \left([\bar{r}w + u^*(r - \bar{r})] + \frac{\sigma^2 S^2}{2} \left[V_{SS} + 2u^* V_{Sw} + (u^*)^2 V_{ww} \right] \right) = V_t \tag{41}$$

The plain vanilla passport contract is defined by the payoff function

$$V(T, S, w) = w^+ \tag{42}$$

The optimal control is known to be

$$u^* = \alpha - \operatorname{sign}\left(\frac{w}{S} - \alpha\right)\beta \tag{43}$$

$$\alpha = \frac{b - a}{2} \tag{44}$$

$$\beta = \frac{b + a}{2} \tag{45}$$

$$\tag{46}$$

Using the substitution $x \equiv w/S$ this HJB equation becomes [28]:

$$0 = v_t + \max_{a \le u^* \le b} \left[(r - \bar{r})(u^* - x) + \frac{\sigma^2}{2}(u^* - z)^2 u_{xx} \right] \tag{47}$$

$$u(T, x) = x^+ \tag{48}$$

Again the change of the variable has reduced the dimensionality of the problem by one.

It can be shown in several ways that Asian options are a special case of the general option on a trading account. Two approaches leading to simple PDEs have been explored by [28] and [29].

3 A Numerical Solution with Finite Elements

3.1 One Spatial Variable

Most passport option models have to be solved numerically since they are non-linear parabolic PDEs For this reason we have chosen a collocation finite element method. First, we consider the stationary problem $L(u(x)) = f$, L being a non-linear differential operator. Time is integrated in a later step. We look for an approximate solution $\tilde{u}$ for the following problem:

$$L(u) = f, \;\; u(x_{min}) = u_{min}, \;\; u(x_{max}) = u_{max} \tag{49}$$

An approximate solution $\tilde{u}$ is of the following form:

$$\tilde{u} = \sum_{k=0}^{N+1} a_k \phi_k(x) \tag{50}$$

The task is to find values for a_i which make $\tilde{u}$ the "best" approximation. While the Galerkin finite element methods normally used in derivative pricing (cf. [17], [9], [25]) determine the a_i by solving

$$\int_{x_{min}}^{x_{max}} \left(L(\tilde{u}) - f\right) \phi_k \, dx = 0, \; k = 1, \ldots, N \tag{51}$$

collocation finite element methods take the following approach:

$$\left(L(\tilde{u}) - f\right)(\xi_k) = 0, \; k = 1, \ldots, N \tag{52}$$

The collocation points for each element $[x_i; x_{i+1}]$ in the method used here are

$$\xi_{2i+1,2i+2} \;=\; x_i + \left(\frac{1}{2} \pm \frac{1}{2\sqrt{3}}\right)(x_{i+1} - x_i) \tag{53}$$

The approximate function is defined by

$$\tilde{u} \;=\; \sum_{k=0}^{N+1} a_k \phi_k \tag{54}$$

$$\;=\; a_0 H_0 + a_1 S_0 + a_2 H_1 + a_3 S_1 + \ldots + a_N H_{\frac{N}{2}} + a_{N+1} S_{\frac{N}{2}} \tag{55}$$

for

$$\begin{aligned} H_k(x) &= 3\left(\frac{x - x_{k-1}}{x_k - x_{k-1}}\right)^2 - 2\left(\frac{x - x_{k-1}}{x_k - x_{k-1}}\right)^3 \;\text{ for }\; x_{k-1} \le x \le x_k \\ &= 3\left(\frac{x_{k+1} - x}{x_{k+1} - x_k}\right)^2 - 2\left(\frac{x_{k+1} - x}{x_{k+1} - x_k}\right)^3 \;\text{ for }\; x_k \le x \le x_{k+1} \\ &= 0 \text{ elsewhere} \end{aligned} \tag{56}$$

$$\begin{aligned} S_k(x) &= -\frac{(x - x_{k-1})^2}{(x_k - x_{k-1})} + \frac{(x - x_{k-1})^3}{(x_k - x_{k-1})^2} \;\text{ for }\; x_{k-1} \le x \le x_k \\ &= \frac{(x_{k+1} - x)^2}{(x_{k+1} - x_k)} - \frac{(x_{k+1} - x)^3}{(x_{k+1} - x_k)^2} \;\text{ for }\; x_k \le x \le x_{k+1} \\ &= 0 \text{ elsewhere} \end{aligned} \tag{57}$$

The integration of the boundary conditions is achieved by

$$a_0 \;=\; u(x_{min}) \tag{58}$$

$$a_N \;=\; u(x_{max}) \tag{59}$$

Finding the N parameters $a_1, \ldots, a_{N-1}, a_{N+1}$ results in a system of non-linear equations:

$$\begin{aligned} f_1(a_1, \ldots, a_{N-1}, a_{N+1}) &= 0 \\ &\vdots \\ f_N(a_1, \ldots, a_{N-1}, a_{N+1}) &= 0 \end{aligned} \tag{60}$$

This system of non-linear equations is solved via Newton's method. The integration of time is similar to the Galerkin finite element methods. Spatial variables are discretized with finite elements while time is treated with finite differences. This can be visualised as the non-linear elliptic operator $L(u)$ evolving through time. Each equation corresponds to a collocation point. The dynamic counterpart to eq. (60) is given by

$$\dot{\boldsymbol{a}} = \begin{pmatrix} \dot{a}_1 \\ \vdots \\ \dot{a}_{N-1} \\ \dot{a}_{N+1} \end{pmatrix} = \begin{pmatrix} g_1\left(a_1, \ldots, a_{N-1}, a_{N+1}\right) \\ \vdots \\ g_N\left(a_1, \ldots, a_{N-1}, a_{N+1}\right) \end{pmatrix} \tag{61}$$

This stiff system of non-linear *ordinary* differential equations can be solved with various time-stepping procedures. We have choosen a second order Adams-Moulton and a first order backward difference implementation. The initial conditions to eq. (61) are given by a discretization of the final condition belonging to the PDEs.

Approximating the solution of a PDE using finite element methods (Galerkin or collocation), using piecewise polynomials of degree n, the approximation to the solution itself should be of the order $O(h^{n+1})$; the approximation to the m-th derivative should be $O(h^{n+1-m})$, i.e. one is losing one order of precision for each derivative. Why is this? Let us assume a smooth function $f(x)$ in the interval $0 \leq x \leq h$. We want to approximate this function using a Taylor approximant on degree n:

$$T(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \ldots + f^{(n)}(0)\frac{x^n}{n!} \tag{62}$$

By the Taylor series remainder theorem, the error is

$$f(x) - T(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1} \tag{63}$$

$$0 \leq c \leq x \tag{64}$$

which is $O(h^{n+1})$ when $0 < x < h$. If the first derivative of $T(x)$ is taken, it can be noted that it is exactly the Taylor polynomial approximant of degree $(n-1)$ to $f'(x)$, and thus $f'(x) - T'(x)$ is of $O(h^n)$. Similarly, $f''(x) - T''(x)$ is of $O(h^{n-1})$, etc. Though it requires a little more work to prove other cases, similar error bounds on $f'(x) - T'(x)$, $f''(x) - T''(x)$, etc. can be derived when $T(x)$ is the Lagrange polynomial interpolant, Hermite polynomial interpolant or spline interpolant of degree n.

3.2 Two Spatial Variables

We again consider the stationary problem $L(u(x,y)) = f$, L being a non-linear differential operator. We search for an approximate solution $\tilde{u}(x,y)$ for the following problem:

$$L(u) = f \, , \tag{65}$$

$$u(x_{min}) = u_{min_x}, \; u(x_{max}) = u_{max_x} \, , \tag{66}$$

$$u(y_{min}) = u_{min_y}, \; u(y_{max}) = u_{max_y} \, , \tag{67}$$

That is, we consider a rectangular domain with Dirichlet conditions. This method can easily be generalized for non-rectangular domains and Neumann and mixed boundary conditions [21]. An approximate solution $\tilde{u}$ can take the following form:

$$\tilde{u} = \sum_{k=1}^{N} a_k \phi_k(x,y) \, , \tag{68}$$

Again, the task is to find values for a_i which make $\tilde{u}$ the "best" approximation. While the Galerkin finite element methods sometimes used in two-asset derivative pricing (cf. [9], [25]) determine the a_i's by solving

$$\int_{x_{min}}^{x_{max}} \int_{y_{min}}^{y_{max}} \left(L(\tilde{u}) - f\right) \phi_k \, dy \, dx = 0, \; k = 1, \ldots, N \, , \tag{69}$$

collocation finite element methods take the following approach. They enforce that at certain points in the domain, the so-called *collocation points*, the PDE is exactly satisfied.

$$\left(L(\tilde{u}) - f\right)(\xi_k) = 0, \; k = 1, \ldots, N \tag{70}$$

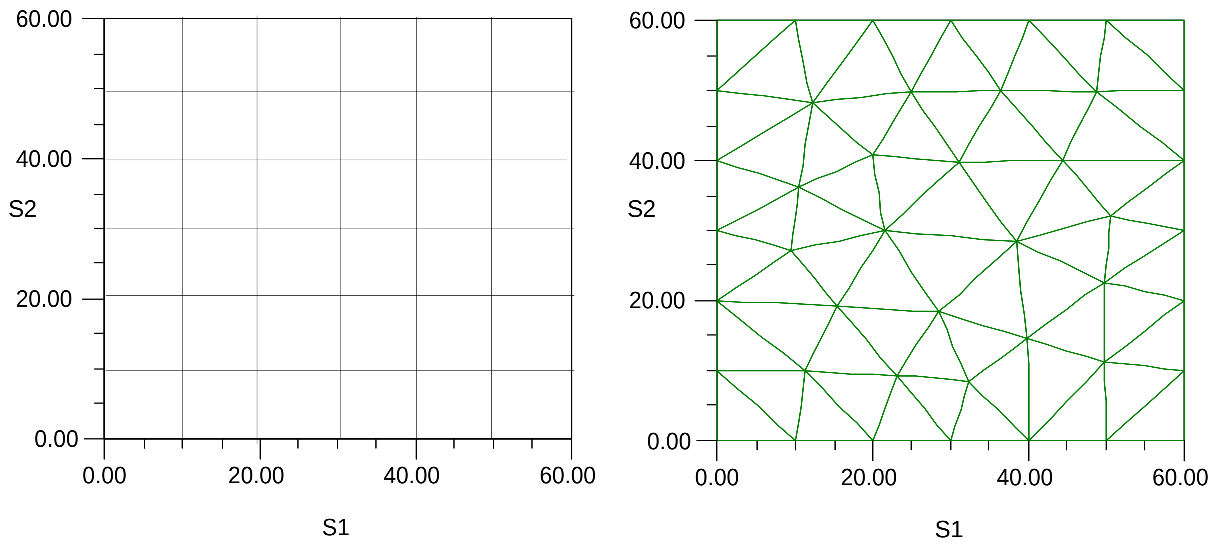

Figure 1: Structured and Unstructured Grids

The domain is divided into disjoint elements, the *finite elements*. These finite elements are usually either rectangular or triangular. The latter type of element can form either a structured or an unstructured grid. Here we employ rectangular elements which are not necessarily of the same size.
The four collocation points for each element $[x_i; x_{i+1}] \times [y_i; y_{i+1}]$ in the method used here are

$$\left(x_i + \left(\frac{1}{2} \pm \frac{1}{2\sqrt{3}}\right)(x_{i+1} - x_i); y_i + \left(\frac{1}{2} \pm \frac{1}{2\sqrt{3}}\right)(y_{i+1} - y_i)\right) \tag{71}$$

Let s_i denote the number of gridlines in direction i. The approximate solution is defined by a linear combination of $4 s_x s_y$ basis functions

$$H_i(x)H_j(y),\ H_i(x)S_j(y),\ S_i(x)H_j(y),\ S_i(x)S_j(y) \tag{72}$$

with

$$\begin{aligned} H_k(x) &= 3\left(\frac{x - x_{k-1}}{x_k - x_{k-1}}\right)^2 - 2\left(\frac{x - x_{k-1}}{x_k - x_{k-1}}\right)^3 \quad \text{for } x_{k-1} \le x \le x_k \\ &= 3\left(\frac{x_{k+1} - x}{x_{k+1} - x_k}\right)^2 - 2\left(\frac{x_{k+1} - x}{x_{k+1} - x_k}\right)^3 \quad \text{for } x_k \le x \le x_{k+1} \\ &= 0 \text{ elsewhere} \end{aligned} \tag{73}$$

$$\begin{aligned} S_k(x) &= -\frac{(x - x_{k-1})^2}{(x_k - x_{k-1})} + \frac{(x - x_{k-1})^3}{(x_k - x_{k-1})^2} \quad \text{for } x_{k-1} \le x \le x_k \\ &= \frac{(x_{k+1} - x)^2}{(x_{k+1} - x_k)} - \frac{(x_{k+1} - x)^3}{(x_{k+1} - x_k)^2} \quad \text{for } x_k \le x \le x_{k+1} \\ &= 0 \text{ elsewhere} \end{aligned} \tag{74}$$

$$\begin{aligned} H_k(y) &= 3\left(\frac{y - y_{k-1}}{y_k - y_{k-1}}\right)^2 - 2\left(\frac{y - y_{k-1}}{y_k - y_{k-1}}\right)^3 \quad \text{for } y_{k-1} \le y \le y_k \\ &= 3\left(\frac{y_{k+1} - y}{y_{k+1} - y_k}\right)^2 - 2\left(\frac{y_{k+1} - y}{y_{k+1} - y_k}\right)^3 \quad \text{for } y_k \le y \le y_{k+1} \\ &= 0 \text{ elsewhere} \end{aligned} \tag{75}$$

$$S_k(y) = -\frac{(y - y_{k-1})^2}{(y_k - y_{k-1})} + \frac{(y - y_{k-1})^3}{(y_k - y_{k-1})^2} \quad \text{for } y_{k-1} \le y \le y_k$$

$$
\begin{aligned}
&= \frac{(y_{k+1}-y)^2}{(y_{k+1}-y_k)} - \frac{(y_{k+1}-y)^3}{(y_{k+1}-y_k)^2} \ \text{ for } \ y_k \le y \le y_{k+1} \\
&= 0 \ \text{ elsewhere}
\end{aligned} \tag{76}
$$

The approximate solution $\tilde{u}(x,y)$ has the form

$$
\tilde{u}(x,y) = \sum_{i=1}^{s_x}\sum_{j=1}^{s_y} \left(A_{ij}H_i(x)H_j(y) + B_{ij}H_i(x)S_j(y) + C_{ij}S_i(x)H_j(y) + D_{ij}S_i(x)S_j(y)\right) \tag{77}
$$

so the $N = 4s_xs_y$ unknowns are A_{11}, B_{11}, C_{11}, D_{11}, A_{12}, B_{12}, C_{12}, D_{12}, $\ldots$, $A_{s_xs_y}$, $B_{s_xs_y}$, $C_{s_xs_y}$, $D_{s_xs_y}$ which are relabeled to $a_1,\ldots,a_N$ for notational convenience. The approximating function $\tilde{u}(x,y)$ has a continuous mixed derivative $\tilde{u}_{xy}$ since the mixed derivative of each of the basis functions is continuous. To be succesful in this setting, the key feature of continuous mixed derivatives has to hold. This feature is not common to most FE methods in use today. The basis functions normally used by Galerkin methods do not even have continuous $\tilde{u}_x$ or $\tilde{u}_y$.

The approximate solution is required to satisfy the PDE exactly at the four collocation points in each of the $(s_x-1)(s_y-1)$ subrectangles, and to satisfy the boundary conditions at certain points. The number of boundary collocation points plus the number of interior collocations points $4(s_x-1)(s_y-1)$ is equal to the number of basis functions $4s_xs_y$, which is equal to the number of unknowns N. Finding the N parameters $a_1,\ldots,a_N$ results in a system of non-linear equations:

$$
\begin{aligned}
f_1(a_1,\ldots,a_N) &= 0 \\
&\vdots \\
f_N(a_1,\ldots,a_N) &= 0
\end{aligned} \tag{78}
$$

This system of non-linear equations is solved via Newton's method. The integration of time is similar to the Galerkin finite element method. Spatial variables are discretized with finite elements while time is treated with finite differences. This can be visualised as the non-linear elliptic operator $L(u)$ evolving through time. Each equation corresponds to a collocation point. The dynamic counterpart to eq. (78) is given by

$$
\dot{\boldsymbol{a}} = \begin{pmatrix} \dot{a}_1 \\ \vdots \\ \dot{a}_N \end{pmatrix} = \begin{pmatrix} g_1(a_1,\ldots,a_N) \\ \vdots \\ g_N(a_1,\ldots,a_N) \end{pmatrix} \tag{79}
$$

This stiff system of non-linear *ordinary* differential equations can be solved with various time-stepping procedures. We have chosen the same routines as for the problem with only one spatial variable. The initial conditions to eq. (79) are given by a discretization of the final condition of the PDEs of ch. 2. All computations have been performed with PDE2D, a general purpose finite element solver described in [22].

4 Applications

4.1 Plain Vanilla Passport Options

4.1.1 European Passport Options

A Linear Model For $r = \gamma$ eq. (16) reduces to:

$$\frac{\partial v}{\partial t} + \frac{1}{2}(1+|x|)^2\sigma^2\frac{\partial^2 v}{\partial x^2} = \gamma v \tag{80}$$

For the payoff function $v(T) = \max(0, X) \equiv X^+$ an analytical solution has been found [3]:

$$v(t,x) = e^{-\gamma(T-t)}\left[x^+ + N(d) - (1+|x|)\,N\left(d - \sigma\sqrt{T-t}\right) + \Omega\right] \tag{81}$$

with

$$d = \frac{-\ln(1+|x|) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \tag{82}$$

$$\begin{aligned}\Omega &= \frac{1}{2}\left[d\sigma\sqrt{T-t} - 1\right]N(d) \\ &\quad + \frac{1}{2}(1+|x|)\,N\left(d - \sigma\sqrt{T-t}\right) \\ &\quad + \frac{1}{2}\sigma\sqrt{T-t}N'(d)\end{aligned} \tag{83}$$

N and N' denote the cumulative function and the density function of the normal distribution respectively. This analytical solution is to be used as a benchmark. We will repeat the example from ([3], Table 1). Some of our results do not agree with the results reported there because some of the results in [3] are erroneous.[4] Eq. (81) is sensitive against the accuracy of the approximation of the normal distribution. We show the results using approximations which are accurate to four and seven digits, respectively; compare ([15], S. 243f).

Parameter	Value
Spot S	100.0
Dividend yield γ	0.0
Interest rate r	0.0
Volatility σ	0.3
Time to maturity	1 Year

w	Result from [3]	Correct Solution: $N(\cdot)$ accurate to ...	
		... 4 digits	... 7 digits
100	100,1566	100,15580	100,15660
50	51,6456	51,58188	51,58181
20	25,9063	25,88776	25,88757
10	18,8846	18,87984	18,88084
0	13,1381	13,13906	13,13810
-10	8,8808	8,87984	8,88084
-20	5,8876	5,88776	5,88757
-50	1,5893	1,58188	1,58181
-100	0,1566	0,15576	0,15660

For the numerical solution of the PDE (80), besides the initial condition given by eq. (15), two boundary conditions are necessary which turn the unbounded domain of the PDE $-\infty < x < \infty$ into a compact one. Andersen et al. suggest ([3], S. 23f)

$$v(t, -e^{h\sigma\sqrt{T}}) = 0 \tag{84}$$

$$\frac{\partial v(t, e^{h\sigma\sqrt{T}})}{\partial x} = e^{-r(T-t)} \tag{85}$$

$$h = 4 \tag{86}$$

[4] Andersen et al. compare a Crank-Nicolson FD Schema with the analytical solution in ([3], Tab. 3). Seemingly, the numerical values do not converge to the analytical solution esp. for high values of w. However, the numerical algorithm does converge, but not to the *wrong* analytical solution.

We solve this problem several times using different numbers of elements and time steps to observe the convergence behavior.

		Time Steps							
		25	50	100	200	400	800	1600	3200
Spatial Steps	25	13.0824	13.1120	13.1268	13.1342	13.1379	13.1398	13.1407	13.1412
	50	13.1860	13.2150	13.2290	13.2369	13.2405	13.2423	13.2433	13.2437
	100	13.1047	13.1341	13.1488	13.1561	13.1598	13.1617	13.1626	13.1631
	200	13.0853	13.1148	13.1295	13.1369	13.1406	13.1425	13.1433	13.1439
	400	13.0806	13.1101	13.1249	13.1322	13.1359	13.1378	13.1387	13.1392
	800	13.0795	13.1089	13.1236	13.1311	13.1348	13.1366	13.1376	13.1380
	1600	13.0791	13.1086	13.1234	13.1308	13.1345	13.1363	13.1373	13.1377

For the following computations we employ 400 spatial steps (equaling 399 elements) and 800 time steps. For the time integration a first order backward difference method is used:

w	Analytical Solution	Numerical Solution
100	100.15660	100.1575
50	51.58181	51.5833
20	25.88757	25.8878
10	18.88084	18.8881
0	13.13810	13.1378
-10	8.88084	8.8806
-20	5.88757	5.8878
-50	1.58181	1.5833
-100	0.15660	0.1575

A Non-Linear Model Here we solve eq. (18) with eq. (20). The data are taken from [3].

Parameter	Value
Spot S	100.0
Dividend yield γ	5.0
Interest rate r	4.5
Volatility σ	0.3
Time to maturity	2 Years

This problem has been solved already with Finite Differences by [3] and a Galerkin Finite Element method by [26]. For $w = 0$ no result can be given because of the jump there.

w	FD [3]	FE [26]	FE	Hedge Ratio k [26]	k_{FE}
20	28.2277	28.2249	28.2295	-0,4674	-0.4679
10	22.3741	22.3734	22.3760	-0,3724	-0.3729
0	17.4323	17.4423	17.4438	-	-
-10	13.5100	13.5113	13.5135	0.5180	0.5176
-20	10.4261	10.4293	10.4320	0,4302	0.4300

4.1.2 American Passport Options

Here we introduce early exercise to the non-symmetric example from above. Again, this problem has already been solved with Finite Differences by [3] and a Galerkin Finite Element method by [26].

w	FD [3]	FE [26]	FE	Hedge Ratio k [26]	k_{FE}
20	29.1764	29.2110	29.2135	-0,5042	-0.5041
10	23.0050	23.0272	23.0298	-0,3974	-0.3871
0	17,8418	17.8648	17.8668	-	-
-10	13,7776	13.7873	13.7902	0.5330	0.5331
-20	10,6031	10.6124	10.6150	0,4406	0.4406

4.2 Relative Exotics

So-called *relative exotics* in the world of passport options have the exotic feature on x; i.e. the cap, floor and/or barrier(s) is/are applied to the quotient of wealth and stock $x = \frac{w}{S}$. *Absolute exotics* bear the exotic feature(s) on w and/or S individually. These contracts are discussed further below.

As a numerical example we add a knock-out barrier at $x = 20$ to the non-symmetric contract from above. Note that by introducing a rebate of $R = 20$ the payoff function does not lose its convexity.

w	FE
20	20
10	16.5737
0	13.4722
-10	10.8090
-20	8.5884

4.3 Asian Options

4.3.1 Vecer's First Approach

It can be shown that Asian options can be interpreted as options on a trading account as well. For a derivation see [28].

$$v_t + r(q-x)v_z + \frac{1}{2}(q-x)^2\sigma^2 v_{xx} = 0 \tag{87}$$

with the final condition

$$v(T,x) = x^+ \tag{88}$$

The function q depends on the contract at hand. This approach can be simplified leading to an even simpler PDE [29].

Continuous Sampling For this type of sampling q is a continuous function:

$$q = 1 - \frac{t}{T} \tag{89}$$

In order to be able to tackle the final-value problem eq. (87) and (88) numerically, it has to be converted into a final-boundary-value problem, i. e. the infinite domain has to be cut off by replacing $-\infty$ and ∞ by taking finite values for z. Also, these points have to be prescribed with values for u or its derivative. Following [28] we choose

$$u(z_{min} = -1) \quad = \quad 0 \tag{90}$$

$$\frac{\partial u(z_{max} = 1)}{\partial z} \quad = \quad 1 \tag{91}$$

Additionally, to be comparable to [28] and [10] we take the same number of space and time points. The spatial variable is discretized with 200 elements of equal length with cubic Hermite basis functions. Time is integrated by a Crank-Nicholson scheme with 400 steps.

Parameter	Symbol	Value
Interest rate	r	0.15
Price of Underlying in $t = 0$	S_0	100
Time to Maturity	T	1

Parameter		Method					
σ	K	FE (Coll.)	FD [28]	[10]	Monte Carlo [11]	Lower Bound [20]	Upper
0.05	95	11.094	11.094	11.094	11.094	11.094	11.114
	100	6.794	6.795	6.793	6.795	6.794	6.810
	105	2.746	2,744	2,744	2,745	2,744	2,761
0.10	90	15,399	15.399	15.399	15.399	15.399	15.445
	100	7.028	7.029	7.030	7.028	7.028	7.066
	110	1.415	1.415	1.410	1.418	1.413	1.451
0.20	90	15.642	15.643	15.643	15.642	15.641	15.748
	100	8.409	8.412	8.409	8.409	8.408	8.515
	110	3.556	3.560	3.554	3.556	3.554	3.661
0.30	90	16.513	16.516	16.514	16.516	16.512	16.732
	100	10.210	10.215	10.210	10.210	10.208	10.429
	110	5.731	5.736	5.729	5.731	5.728	5.948

Discrete Sampling This kind of sampling calls for a different q:

$$q = 1 - \frac{1}{n}\left[n\frac{t}{T}\right] \tag{92}$$

with $[\cdot]$ denoting the integer part function. Note that this formula allows for non-equidistant time-steps between the fixings.

Parameter		Method					
σ	K	FE	MC	FE	MC	FE	MC
		3 Fixings		10 Fixings		50 Fixings	
0.05	95	13.435	13.450	11.792	11.780	11.233	11.222
	100	9.134	9.145	7.492	7.495	6.934	6.933
	105	4.943	4.937	3.384	3.388	2.872	2.861
0.10	90	17.742	17.767	16.097	16.081	15.538	15.538
	100	9.377	9.385	7.729	7.737	7.168	7.150
	110	3.026	3.009	1.852	1.853	1.499	1.487
0.20	90	18.166	18.146	16.393	16.364	15.791	15.748
	100	11.019	10.976	9.189	9.202	8.564	8.557
	110	5.772	5.762	4.196	4.199	3.682	3.673
0.30	90	19.429	19.522	17.383	17.392	16.687	16.622
	100	13.237	13.325	11.116	11.081	10.391	10.327
	110	8.512	8.502	6.547	6.540	5.892	5.851

The details of the numerical implementation are the same as in the previous section. The Monte Carlo results are achieved in a most simple fashion as described in [13]. We use 100,000 sample paths for all examples so that these results are rather crude estimates. Obviously, the computational burden of solving the PDE is much smaller than generating the Monte Carlo paths.

Sensitivy Analysis: Delta With another example from the literature [27] we will show that finite elements can be used to compute Δ *directly* from the numerical solution for the option premium.

Parameter	Symbol	Value
Interest rate	r	0.09
Price of Underlying in $t = 0$	S_0	100
Time to Maturity	T	15 days
Equidistant Samplings		14

For the only run necessary for the finite element computation we use 199 elements of equal length; time integration is performed with an adaptive second order Adams-Moulton scheme using 1000 time steps. For details of this time integration technique, see [21], [6].

	Premium			Delta		
Strike	Approximation	MC	FE	Approximation	MC	FE
95	5.202	5.194	5.203	0.979	0.974	0.975
100	1.139	1.135	1.144	0.536	0.534	0.523
105	0.038	0.038	0.039	0.037	0.030	0.036

4.3.2 Vecer's Second Approach

- reduces the pricing equation to the heat equation
- the q function is less easy to interpret
- redo the same examples
- compare the methods (list!)

4.4 Absolute Exotics

4.4.1 The Algorithm

Absolute exotics refer to passport options with a cap, floor or barrier put on either the current value of the trading account $w(t)$ or the underlying S. Putting a barrier or cap on $w(t)$ seems natural because beyond some limit the interest in further hedging should be small. Besides, it makes the option cheaper [2]. For this kind of option, the HJB equation has to be solved directly since no control is known *a priori*. In this section, we outline the idea of the algorithm which involves some basic ideas from (static) optimization.[5] Then we

[5]For the fundamental idea of this algorithm I want to thank Granville Sewell.

apply this algorithm to a problem which has already been solved above.

Starting point is the HJB equation as in eq. (21):

$$(r-\gamma)SV_S + rwV_w - rV - \frac{\sigma^2 S^2}{2} \max_{-1\le\mu^*\le 1}\left[V_{SS} + 2\mu^* V_{Sw} + (\mu^*)^2 V_{ww}\right] = V_t \tag{93}$$

We slightly generalize this problem by allowing the control to vary within $[a, b]$. Then, one has to solve the following optimization problem for each node in each time step:

$$\max_{a\le\mu^*\le b}(V_{SS} + 2\mu^* V_{Sw} + (\mu^*)^2 V_{ww}) \tag{94}$$

This is just a quadratic polynomial in μ^*, so that the existence of a maximum is guaranteed. The maximum has to occur at either $\mu^* = a$, $\mu^* = b$ or $\mu^* = -\frac{V_{Sw}}{V_{ww}}$ (if $a < \mu^* < b$), the latter point being the vertex of the quadratic polynomial. All three values are calculated and the maximum is taken. Obviously, in case we do not have a constant solution or a bang-bang solution,[6] accurate approximations of the second derivatives are needed.

As a first test, we recompute the symmetric problem from sec. 4.1.1. This problem cannot test all features of the proposed algorithm since it is known that its optimal control is of the bang-bang type. The routine should, therefore, never be forced to compute second derivatives. Since this problem has two spatial dimensions, we also provide a numerical solution based on eq. (14) so that we have two numerical solutions based on two spatial variables. For both problems the following boundary conditions have been applied:

$$V(0,w,t) = \max(0,w) \tag{95}$$
$$\frac{\partial^2 V(200,w,t)}{\partial S^2} = 0 \tag{96}$$
$$V(S,-200,t) = 0 \tag{97}$$
$$V(S,200,t) = w \tag{98}$$

With a rectangular mesh of 19 elements in S and 42 in w and an adaptive second order Adams-Moulton method for the integration of time, the following results are produced:

w	Analytical Solution	Numerical Solution	
		eq. (14)	eq. (21)
100	100.1566	100.1566	100.1581
50	51.5818	51.5804	51.5835
20	25.8876	25.8878	25.8904
10	18.8808	18.8820	18.8840
0	13.1381	13.1397	13.1414
-10	8.8808	8.8820	8.8840
-20	5.8876	5.8878	5.8904
-50	1.5818	1.5804	1.5836
-100	0.1566	0.1566	0.1581
Time Steps	-	31	99

The algorithm which computes the optimal control implicitly gives slightly less accurate results for the same mesh while using about three times more time steps.

4.4.2 Knock-Out Barriers

Knock-out barriers can be applied to both stock and/or the account. In the second case the rationale is that when the loss becomes too large the coverage is off; in the first case, the usual rationale for cheapening the option is applied. As an example, we take the problem from above and introduce a knock-out barrier on w. With a recangular mesh of 19 elements in S and 42 in w and an adaptive second order Adams Moulton method for the integration of time, the following results were produced:

[6] Controls switching back and forth between their possible extremes are called *bang-bang controls*; compare [7].

w	Location of barrier		
	$w = -20$	$w = -30$	$w = -40$
100	100.0000	100.1408	100.1602
50	51.4286	51.5643	51.5798
20	25.3353	25.7934	25.8769
10	18.0489	18.7373	18.8666
0	11.7753	12.9275	13.1203
-10	6.0061	8.3566	8.8363
-20	0.0000	4.3371	5.6642
-50	0.0000	0.0000	0.0000
-100	0.0000	0.0000	0.0000
Time Steps	64	60	104

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