

# Finite Difference for Pricing Options

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# Program of Talk

- Introducing FD and Discretization
- Application of FD for BS-Equation
  - Theta Method
  - Computational Issues
- Pricing of arithmetic average option as a special case of an option on a traded account (Vecer, 2001, JoCF, Vol. 4 No. 4)
- Practical Implementation

# Why use Finite Differences ?

- In small number of dimensions, FD may be faster than MC (even than fancy MC)
- In FD, effort and accuracy always scale in the same way, in MC scaling may jump (from  $1/n^{1/2}$  to  $1/n^{3/2}$ , for example)
- Handle early exercise, discrete sampling, and complex boundaries and barriers
- FD are ideally suited for simultaneous solutions of multiple instruments

# Disadvantages

- The driving factors must be Markovian
- The number of dimensions must be small
  - This is a DATA issue, not a speed issue
- Practical number of max dimensions: 3

# The general Pricing Equation

- The pricing equation is a PIDE: Parabolic partial integro-differential equation in reverse time.
- Parabolic: Information propagates across all states instantaneously.
- Integro: There may be source terms containing integrals, but the integrals should not depend on the path of the underlying processes.
- Reverse time: Information “concentrates” as time goes by (the opposite of Physics)

# The general Pricing Equation

$$\begin{aligned}
 & \text{Convection} && \text{Diffusion} \\
 & \overbrace{\frac{\partial V}{\partial t} + a \frac{\partial V}{\partial S} + b \frac{\partial V}{\partial r} + c \frac{\partial V}{\partial I}} & + & \overbrace{d \frac{\partial^2 V}{\partial S^2} + e \rho \frac{\partial^2 V}{\partial S \partial r} + f \frac{\partial^2 V}{\partial r^2}} = \\
 & \uparrow \quad \nearrow & & \\
 & \text{Parabolic directions} & & \\
 & \uparrow & & \\
 & \text{Hyperbolic direction} & & \text{Source} \\
 & & & \overbrace{= rV - \lambda \left( \underbrace{\int \eta(s) V(s) ds}_{\text{Convolution}} - V \right)} \\
 & & & \underbrace{\hspace{10em}}_{\text{Jump}}
 \end{aligned}$$

$$V = V(t, S, r, I)$$

# The BS discretization

– Original BM :

$$dS = (r - q)S dt + \sigma S dz$$

– PDE  $V = V(S, t)$

$$\frac{\partial V}{\partial t} + (r - q)S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV$$

# The Essential FD discretization

Central in space coordinates:

$$\frac{\partial V}{\partial S} = \frac{V(S + \Delta S) - V(S - \Delta S)}{2\Delta S} + O(\Delta S^2)$$

$$\frac{\partial^2 V}{\partial S^2} = \frac{V(S + \Delta S) - 2V(S) + V(S - \Delta S)}{\Delta S^2} + O(\Delta S^2)$$

# Implementing space discretization

- Construct a grid of equally spaced points

$$\{S_i\} = \{S_0, S_1, S_2, \dots, S_I\}$$

$$S_{i+1} = S_i + \Delta S$$

- Define  $V = V(i\Delta S, j\Delta t) = V_i^j$

$$V_i = V(S_i)$$

$$V_{i+1} = V(S_i + \Delta S)$$

$$V_{i-1} = V(S_i - \Delta S)$$

# Implementing space discretization

- Replace  $V(S), V(S + \Delta S), V(S - \Delta S)$  with  $V(S_i), V(S_i + \Delta S)$  and  $V(S_i - \Delta S)$  in

$$\frac{\partial V}{\partial t} + (r - q)S \frac{V(S + \Delta S) - V(S - \Delta S)}{2\Delta S} + \frac{1}{2} \sigma^2 S^2 \frac{V(S + \Delta S) - 2V(S) + V(S - \Delta S)}{\Delta S^2} = Vr$$

- If we focus on interior points,  $\{1, 2, \dots, I-1\}$ , this gives us  $I$  ODEs, one for each interior grid point.

$$\frac{dV_i}{dt} = -\frac{(r - q)S}{2\Delta S} (V_{i+1} - V_{i-1}) - \frac{1}{2} \frac{\sigma^2 S^2}{\Delta S^2} (V_{i+1} - 2V_i + V_{i-1}) + V_i r$$

# The Finite Difference Problem

- This is a tridiagonal *rectangular* system

$$\frac{dV_i}{dt} = \left( \frac{(r-q)S}{2\Delta S} - \frac{1}{2} \frac{\sigma^2 S^2}{\Delta S^2} \right) V_{i-1} + \left( \frac{\sigma^2 S^2}{\Delta S^2} + r \right) V_i + \left( -\frac{(r-q)S}{2\Delta S} - \frac{1}{2} \frac{\sigma^2 S^2}{\Delta S^2} \right) V_{i+1}$$

- The boundaries are at  $S_0$  and  $S_I$ 
  - If we keep  $V_0$  and  $V_I$  as unknowns, we don't have enough equations.
    - We can clear  $V_0$  and  $V_I$  using reasonable assumptions: The first and last columns go away and the system becomes square.
    - We can add two additional equations at the top and bottom: This also makes the system square.

# The Finite Difference Problem

- When the proper assumptions about boundaries are introduced, the system is written as follows:

$$\frac{dV}{dt} = AV$$

- V is a vector
  - A is the *discretization matrix*.
- The finite difference problem is the formulation and implementation of a scheme for the time-discretization of this system

# The Finite Difference problem

- Difference between European and Early exercise options:
  - In pricing a European option, we have a Finite Difference problem as described.
  - In pricing options with early exercise (American or Bermudan), we have a Linear Complementarity Problem, of which this formulation is a part.

# Effect of the convolution integral

- The convolution integral produces a source term and does not compromise the feasibility of the solution.
  - It may cause the discretization matrix to become full.
  - There are ways of dealing with the convolution integral, such that the discretization matrix remains sparse (iterative methods.)

# Mechanics of FD

- Pricing equation must be solved in the direction of information flow (from the future to the past)
- Two equivalent alternatives:
  - Reverse time and advance in the positive direction (good for analysis)
  - Keep physical time and advance in the negative direction (better for implementation)

# The Mechanics of FD

- The set of ODEs  $\frac{dV}{dt} = AV$  is discretized in time.

- Define

$$V^j = V(j\Delta t)$$

- The FD implementation leads to

$$A_{lhs} V^j = A_{rhs} V^{j+1}$$

- $A_{lhs}$  and  $A_{rhs}$  are (large and usually sparse) matrices
- The particulars of these matrices depends on the time solution scheme.

# The Time Solution Scheme

- Explicit Finite Difference
- Implicit Finite Difference
- Mixed Explicit/Implicit Scheme (CN)

## Time Derivative

$$\frac{\partial V}{\partial t} \approx \frac{V(S_i, t_{j+1}) - V(S_i, t_j)}{\Delta t}$$

# The Time Solution Scheme

At an arbitrary node  $V = V(S_i, t_j)$   $i = 1, K, N, j = 0, K, M$  in the grid we introduce the following difference approximations to the terms in the BS-PDE:

$$\frac{\partial V}{\partial S} \approx (1 - \Theta) \frac{V(S_{i+1}, t_j) - V(S_{i-1}, t_j)}{2\Delta S} + \Theta \frac{V(S_{i+1}, t_{j+1}) - V(S_{i-1}, t_{j+1})}{2\Delta S}$$

$$\begin{aligned} \frac{\partial^2 V}{\partial S^2} \approx & (1 - \Theta) \frac{V(S_{i+1}, t_j) - 2V(S_i, t_j) + V(S_{i-1}, t_j)}{\Delta S^2} \\ & + \Theta \frac{V(S_{i+1}, t_{j+1}) - 2V(S_i, t_{j+1}) + V(S_{i-1}, t_{j+1})}{\Delta S^2} \end{aligned}$$

# The Time Solution Scheme

The parameter  $\theta$  determines the time at which the partial derivatives are evaluated

$\theta = 0$  implicit FD

$\theta = 1$  explicit FD

$\theta = 0.5$  CN

Plugging in and collecting terms we end up

# The Solution Scheme for BS

with  $s_i = i\Delta s$  and the notation  $V_i^j = V(s_i, t_j)$  we reach at

$$\begin{aligned} & -\left[(1-\Theta) \cdot 0.5 \cdot (\sigma^2 i^2 - (r-q)i)\Delta t\right] V_{i-1}^j \\ & + \left[1 + r(1-\Theta)\Delta t + (1-\Theta)\sigma^2 i^2 \Delta t\right] V_i^j \\ & - \left[(1-\Theta) \cdot 0.5 \cdot (\sigma^2 i^2 + (r-q)i)\Delta t\right] V_{i+1}^j = \\ & \left[\Theta \cdot 0.5 \cdot (\sigma^2 i^2 - (r-q)i)\Delta t\right] V_{i-1}^{j+1} \\ & - \left[\Theta\Delta t\sigma^2 i^2 - 1 + r\Theta\Delta t\right] V_i^{j+1} \\ & + \left[\Theta \cdot 0.5 \cdot (\sigma^2 i^2 + (r-q)i)\Delta t\right] V_{i+1}^{j+1} \end{aligned}$$

# Solving Sparse Systems

- Time advancement leads to a sequence of linear systems

$$AV^{j+1} = b$$

- $A$  is very large (in 2D it has order of  $10^8$  elements)
- Solution approaches:
  - Direct solvers
  - Iterative solvers

# Solving Sparse Systems

- Definition: A direct solver reaches the solution in a finite number of computational steps.
- Naive direct solvers:
  - In 1D with tridiagonal structure, the LU algorithm is extremely efficient.
  - In several dimensions there are efficient direct solvers if you use fractional steps or factored schemes. In this case, the solution is a sequence of tridiagonal problems.

# Solving Sparse Systems

- Sophisticated direct solvers:
  - Strategy: Factor matrix  $A$  into upper and lower diagonal product (LU decomposition). This can now be solved directly by upward and downward sweeps.
  - Problem with the strategy: In order for strategy to work,  $L$  and  $U$  must be sparse. Naive decomposition will fill up the  $U$  and  $L$ !

# Example for Stock Option

## European Put Option

$$S = 50$$

$$S_{MAX} = S_N = 2 \cdot X$$

$$X = 50$$

$$S_{MIN} = S_0 = 0$$

$$r = 0.1$$

$$\sigma = 0.4$$

$$T = 5/12$$

Analytical Value: 4.076

# Example for Stock Option

A) Approach with Matrix Inversion

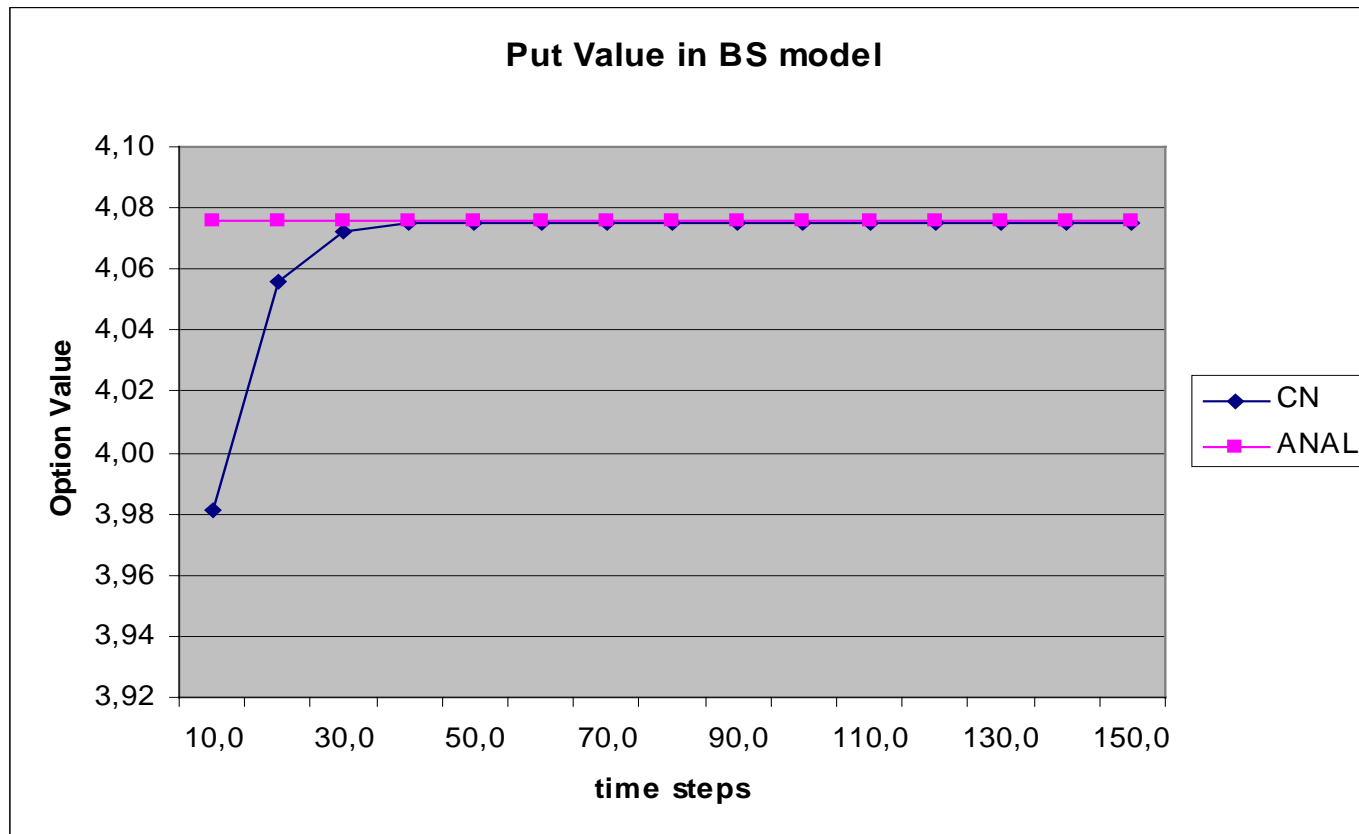
see sheet finite\_sem1.xls  $N = 20$   $M = 10$

B) Approach with LUAlgorithm

see sheet finite\_sem2.xls  $N = 200$   $M =$   
variable

Delta, Gamma, Theta are evaluated by Cubic  
Spline Interpolation

# Stock Put Option



# Example for Interest Rate Option

Term structure model in continuous time

HW Model short rate model

$$dr = (\phi(t) - ar)dt + \sigma dz$$

yields a PDE with  $P = P(r_i, t_j)$  for the zero bond

$$\frac{\partial P}{\partial t} + (\phi(t) - ar) \frac{\partial P}{\partial r} + \frac{1}{2} \sigma^2 \frac{\partial^2 P}{\partial r^2} = rP$$

# Example for Interest Rate Option

For this model closed form bond prices are given

$$P(t, T) = A(t, T) \cdot e^{-r(t)B(t, T)}$$

$$B(t, T) = \frac{1}{a} \left( 1 - e^{-a(T-t)} \right)$$

$$\ln A(t, T) = \ln \frac{P(0, T)}{P(0, t)} + B(t, T) \cdot f(0, t) - \frac{1}{4a^3} \sigma^2 \left( e^{-aT} - e^{-at} \right)^2 \left( e^{2at} - 1 \right)$$

## The Solution Scheme for HW

with  $\phi(t) = \phi^j$  and the notation  $\alpha = \Delta t / \Delta r^2$  we reach

$$\begin{aligned}
 & \left[ \alpha(1-\Theta) \cdot 0.5 \cdot (\sigma^2 - (\phi^j - ar_i)\Delta r) \right] P_{i-1}^j \\
 & - \left[ 1 + r_i \Delta t (1-\Theta) + \alpha(1-\Theta) \sigma^2 \right] P_i^j \\
 & + \left[ \alpha(1-\Theta) \cdot 0.5 \cdot (\sigma^2 + (\phi^j - ar_i)\Delta r) \right] P_{i+1}^j = \\
 & - \left[ \alpha \cdot \Theta \cdot 0.5 \cdot (\sigma^2 - (\phi^{j+1} - ar_i)\Delta r) \right] P_{i-1}^{j+1} \\
 & + \left[ \alpha \cdot \Theta \sigma^2 - 1 + r_i \Delta t \Theta \right] P_i^{j+1} \\
 & - \left[ \alpha \cdot \Theta \cdot 0.5 \cdot (\sigma^2 + (\phi^{j+1} - ar_i)\Delta r) \Delta t \right] P_{i+1}^{j+1}
 \end{aligned}$$

# Example for Interest Rate Option

With  $\phi(t)$  known to be

$$\phi(t) = \frac{\partial f(0, t)}{\partial t} + af(0, t) + \frac{\sigma^2}{2a} (1 - e^{-2at})$$

In this formulation we have now time dependent coefficients at the grid nodes

Payoff of a Put Option on a zero bond

is  $MAX[X - P(t, T), 0]$  (Caplet)

# Example for Interest Rate Option

Let the curve be

$$r(t) = 0.08 - 0.05 \cdot e^{-0.18t}$$

Option Maturity 5.5 years

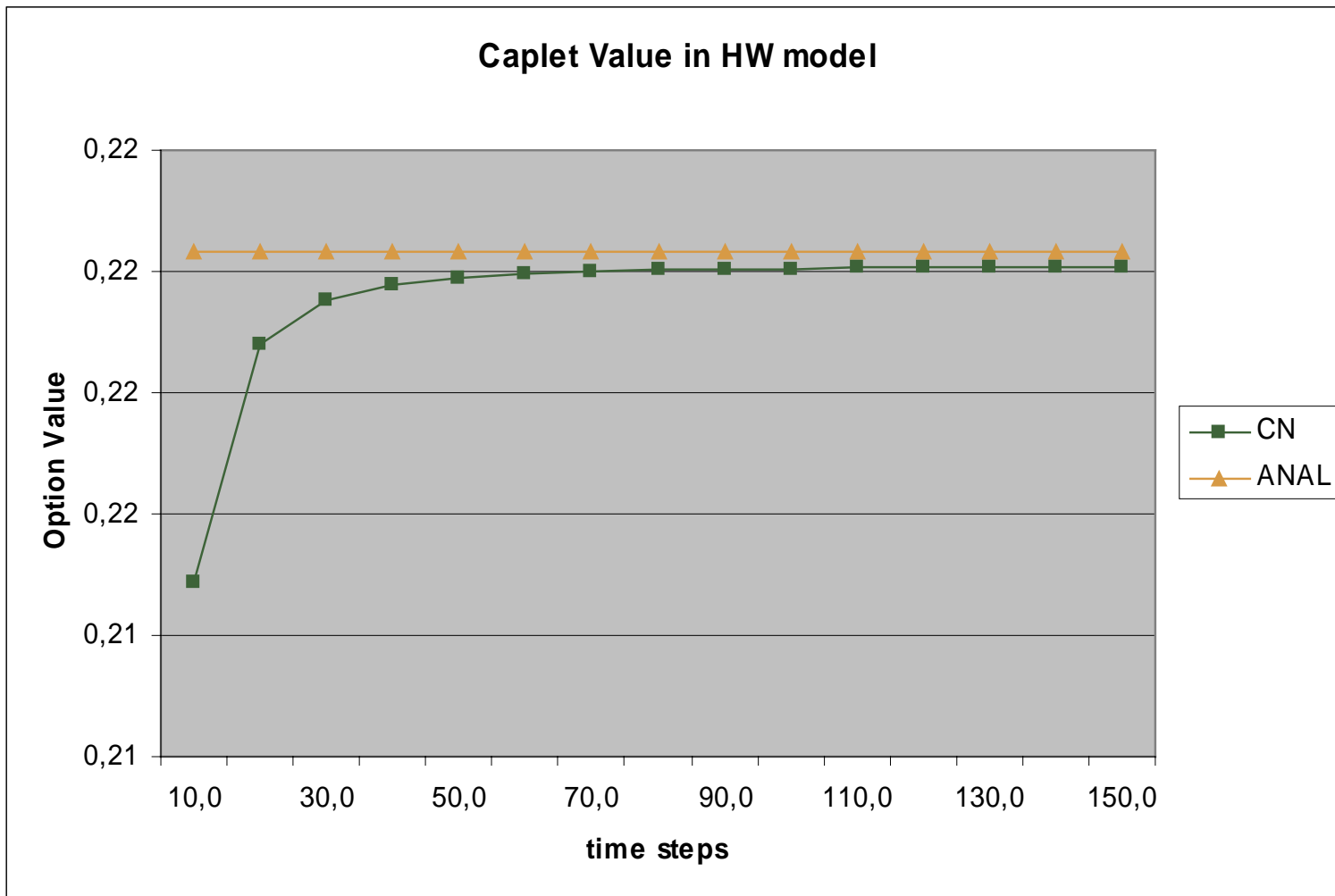
Bond Maturity 6 years

Strike = ATM Forward Bond = 0.9605

$\alpha = 0.15$  Notional = 100, space steps = 200

$\sigma = 0.01$  Analytical Value = 0,2160793

# Interest Rate Option



# Arithmetic Asian Option

- Asian Options are a Subclass of the general case of the option on a traded account
- We are interested in valuing  
Discrete Arithmetic Asian Options

# Arithmetic Asian Option

Option on a trading account with trading strategy  $q_t$  and asset  $S_t$  is modelled:

$$dX_t^q = q_t dS_t + \mu(X_t^q - q_t S_t)dt; \quad X_0^q = X_0$$

$$dS_t = S_t(rdt + \sigma dw_t)$$

$$q_t \in [\alpha_t, \beta_t]$$

In case of Passport Option  $\alpha_t = -1, \beta_t = 1$

In case of Asian Option  $\alpha_t = \beta_t$

In case of European Call Option  $\alpha_t = 1, \beta_t = 1$

# Arithmetic Asian Option

The risk-neutral value is the at time  $t$  with payoff at Time  $T$   $MAX[X_T^q, 0]$  is

$$V^{[\alpha, \beta]}(t, S, X) = \underset{q \in [\alpha, \beta]}{MAX} e^{-r(T-t)} E[ MAX[X_T^q, 0] | \mathcal{F}_t ]$$

Change of variable

$$Z_t^q = \frac{X_t^q}{S_t}$$

# Arithmetic Asian Option

Using Ito's lemma and Girsanov's theorem  
we get a SDE

$$dZ_t^q = (q_t - Z_t^q)(r - \mu)dt + (q_t - Z_t^q)\sigma d\tilde{w}_t$$

Introducing  $u(t, Z_t) = \text{MAX}_{q \in [\alpha, \beta]} \tilde{E}[\text{MAX}[Z_T^q, 0] | \mathfrak{F}_t]$

This leads to the PDE

$$\frac{\partial u}{\partial t} + \text{MAX}_{q \in [\alpha, \beta]} \left[ (r - \mu)(q - z) \frac{\partial u}{\partial z} + \frac{1}{2} (q - z)^2 \sigma^2 \frac{\partial^2 u}{\partial z^2} \right] = 0$$

# Arithmetic Asian Option

## Asian Options

with  $d(tS_t) = tdS_t + S_t dt$  we get  $\frac{1}{T} \int_0^T S_t dt = \int_0^T \left(1 - \frac{t}{T}\right) dS_t + S_0$

So the fixed strike call payoff  $\text{MAX}[\bar{S} - K, 0]$

is modelled by the strategy  $q_t = 1 - t/T$

and initial value  $X_0 = S_0 - K$

where the traded account is evolving as

$$dX = q_t dS_t$$

# Arithmetic Asian Option

## Diskrete Asian Options with n average points

The strategy is approximated by a step function

$$q_t = 1 - \frac{1}{n} INT \left[ n \frac{t}{T} \right]$$

So we get for the discrete fixed strike call

$$MAX \left( \frac{1}{n} \sum_{k=1}^n S_{(k/n)T} - K, 0 \right)$$

The discrete fixed strike put is similar.

# Arithmetic Asian Option

This leads to the PDE for Asian Options

$$\frac{\partial u}{\partial t} + r(q_t - z) \frac{\partial u}{\partial z} + \frac{1}{2} (q_t - z)^2 \sigma^2 \frac{\partial^2 u}{\partial z^2} = 0$$

with boundary condition:  $u(T, z) = \text{MAX}[z, 0]$

Again this has to be solved with a numerical method (FD).

# The Asian Solution Scheme

At an arbitrary node  $u = u(z_i, t_j)$   $i = 1, K, M, j = 0, K, N$   
in the grid we introduce the following difference  
approximations ( $\Theta=1$  implicit,  $\Theta=0$  explicit) :

$$\frac{\partial u}{\partial z} \approx \Theta \frac{u(z_{i+1}, t_j) - u(z_{i-1}, t_j)}{2\Delta z} + (1 - \Theta) \frac{u(z_{i+1}, t_{j+1}) - u(z_{i-1}, t_{j+1})}{2\Delta z}$$

$$\frac{\partial^2 u}{\partial z^2} \approx \Theta \frac{u(z_{i+1}, t_j) - 2u(z_i, t_j) + u(z_{i-1}, t_j)}{\Delta z^2} \\ + (1 - \Theta) \frac{u(z_{i+1}, t_{j+1}) - 2u(z_i, t_{j+1}) + u(z_{i-1}, t_{j+1})}{\Delta z^2}$$

# The Solution Scheme Asian Options

with  $q(t) = q^j$  and  $u(z_i, t_j) = u_i^j$ ,  $k = \Delta z^2 / \Delta t$

$$t_j = j\Delta t, z_i = z_0 + i\Delta z, z_0 = -1; z_M = 1$$

$$\begin{aligned} & \Theta \left[ \sigma^2 (q^j - z_i)^2 - r(q^j - z_i)\Delta z \right] u_{i-1}^j \\ & - 2 \left[ \sigma^2 \Theta (q^j - z_i)^2 + k \right] u_i^j \\ & + \Theta \left[ \sigma^2 (q^j - z_i)^2 + r(q^j - z_i)\Delta z \right] u_{i+1}^j = \\ & - (1 - \Theta) \left[ \sigma^2 (q^j - z_i)^2 - r(q^j - z_i)\Delta z \right] u_{i-1}^{j+1} \\ & + 2 \left[ (1 - \Theta) \sigma^2 (q^j - z_i)^2 - k \right] u_i^{j+1} \\ & - (1 - \Theta) \left[ \sigma^2 (q^j - z_i)^2 + r(q^j - z_i)\Delta z \right] u_{i+1}^{j+1} \end{aligned}$$

# The Solution Scheme Asian Options

Payoff  $u_i^N = \text{MAX}[z_i, 0]$

Boundary conditions and strategy:

Call:  $u_0^j = 0, u_M^j = 2u_{M-1}^j - u_{M-2}^j \quad q_t = 1 - \frac{1}{n} \text{INT} \left[ n \frac{t}{T} \right]$

Put:  $u_0^j = 2u_{M-1}^j - u_{M-2}^j, u_M^j = 0 \quad q_t = \frac{1}{n} \text{INT} \left[ n \frac{t}{T} \right] - 1$

Solution is given by  $\text{Optvalue} = S \cdot u(z_0, 0)$

# Results for Asian Options FD

Calculations with  $N=200$ ,  $M=200$ ,  $MC=10000$   
 $S=100$ ,  $\text{vola}=0.2$ ,  $r=0.05$ ,  $T=1y$ ,  $n=10$

Strike	MC	FD-Call-CN	MC	FD-Put-CN
90,0	12,98	<b>12,98</b>	0,81	<b>0,81</b>
92,5	11,04	<b>11,05</b>	1,25	<b>1,25</b>
95,0	9,26	<b>9,27</b>	1,85	<b>1,85</b>
97,5	7,65	<b>7,66</b>	2,62	<b>2,62</b>
100,0	6,23	<b>6,23</b>	3,57	<b>3,57</b>
102,5	4,99	<b>4,99</b>	4,71	<b>4,71</b>
105,0	3,94	<b>3,94</b>	6,04	<b>6,04</b>
107,5	3,06	<b>3,07</b>	7,54	<b>7,54</b>
110,0	2,35	<b>2,35</b>	9,20	<b>9,20</b>

# Summary

- FD is flexible and fast
- FD is straightforward to implement
- Convergence properties are known

# Literature

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