# Finite Difference for Pricing Options

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# Program of Talk

- Introducing FD and Discretization
- Application of FD for  BS-Equation
  - Theta Method
  - Computational Issues
- Pricing of arithmetic average option as a special case of an option on a traded account (Vecer, 2001, JoCF, Vol. 4 No. 4)
- Practical Implementation

# Why use Finite Differences ?

- In small number of dimensions, FD may be faster than MC (even than fancy MC)
- In FD, effort and accuracy always scale in the same way, in MC scaling may jump (from $1/n^{1/2}$ to $1/n^{3/2}$, for example)
- Handle early exercise, discrete sampling, and complex boundaries and barriers
- FD are ideally suited for simultaneous solutions of multiple instruments

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# Disadvantages

- The driving factors must be <u>Markovian</u>
- The number of dimensions must be small
  - This is a DATA issue, not a speed issue
- Practical number of max dimensions: 3

# The general Pricing Equation

- The pricing equation is a PIDE: Parabolic partial integro-differential equation in reverse time.
- Parabolic: Information propagates across all states instantaneously.
- Integro: There may be source terms containing integrals, but the integrals should not depend on the path of the underlying processes.
- Reverse time: Information "concentrates" as time goes by (the opposite of Physics)

# The general Pricing Equation

$$\frac{\partial V}{\partial t} + \overbrace{a\frac{\partial V}{\partial S} + b\frac{\partial V}{\partial r} + c\frac{\partial V}{\partial I}}^{\text{Convection}} + \overbrace{d\frac{\partial^2 V}{\partial S^2} + e\rho\frac{\partial^2 V}{\partial S\partial r} + f\frac{\partial^2 V}{\partial r^2}}^{\text{Diffusion}} =$$

Parabolic directions

Hyperbolic direction

$$= \overbrace{rV - \underbrace{\lambda\left(\underbrace{\int \eta(s)V(s)ds}_{\text{Convolution}} - V\right)}_{\text{Jump}}}^{\text{Source}}$$

$$V = V(t, S, r, I)$$

# The BS discretization

– Original BM :

$$dS = (r - q)S\,dt + \sigma S\,dz$$

– PDE $\quad V = V(S,t)$

$$\frac{\partial V}{\partial t} + (r-q)S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV$$

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# The Essential FD discretization

Central in space coordinates:

$$\frac{\partial V}{\partial S} = \frac{V(S+\Delta S) - V(S-\Delta S)}{2\Delta S} + O(\Delta S^2)$$

$$\frac{\partial^2 V}{\partial S^2} = \frac{V(S+\Delta S) - 2V(S) + V(S+\Delta S)}{\Delta S^2} + O(\Delta S^2)$$

# Implementing space discretization

- Construct a grid of equally spaced points

$$\{S_i\} = \{S_0, S_1, S_2, \ldots S_I\}$$

$$S_{i+1} = S_i + \Delta S$$

- Define $V = V(i\Delta S, j\Delta t) = V_i^j$

$$V_i = V(S_i)$$

$$V_{i+1} = V(S_i + \Delta S)$$

$$V_{i-1} = V(S_i - \Delta S)$$

# Implementing space discretization

- Replace $V(S)$, $V(S+\Delta S)$, $V(S-\Delta S)$ with $V(S_i)$, $V(S_i+\Delta S)$ and $V(S_i-\Delta S)$ in

$$\frac{\partial V}{\partial t}+(r-q)S\frac{V(S+\Delta S)-V(S-\Delta S)}{2\Delta S}+\tfrac{1}{2}\sigma^2S^2\frac{V(S+\Delta S)-2V(S)+V(S-\Delta S)}{\Delta S^2}=Vr$$

  - If we focus on interior points, {1,2,…I-1}, this gives us I ODEs, one for each interior grid point.

$$\frac{dV_i}{dt}=-\frac{(r-q)S}{2\Delta S}\left(V_{i+1}-V_{i-1}\right)-\tfrac{1}{2}\frac{\sigma^2S^2}{\Delta S^2}\left(V_{i+1}-2V_i+V_{i-1}\right)+V_ir$$

# The Finite Difference Problem

- This is a tridiagonal *rectangular* system

$$\frac{dV_i}{dt} = \left( \frac{(r-q)S}{2\Delta S} - \frac{1}{2}\frac{\sigma^2 S^2}{\Delta S^2} \right) V_{i-1} + \left( \frac{\sigma^2 S^2}{\Delta S^2} + r \right) V_i + \left( -\frac{(r-q)S}{2\Delta S} - \frac{1}{2}\frac{\sigma^2 S^2}{\Delta S^2} \right) V_{i+1}$$

- The boundaries are at $S_0$ and $S_I$
  - If we keep $V_0$ and $V_I$ as unknowns, we don't have enough equations.
    - We can clear $V_0$ and $V_I$ using reasonable assumptions: The first and last columns go away and the system becomes square.
    - We can add two additional equations at the top and bottom: This also makes the system square.

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# The Finite Difference Problem

- When the proper assumptions about boundaries are introduced, the system is written as follows:

$$\frac{dV}{dt} = AV$$

  - V is a vector
  - A is the ***discretization matrix***.
- The finite difference problem is the formulation and implementation of a scheme for the <u>time-discretization</u> of this system

# The Finite Difference problem

- Difference between European and Early exercise options:
  - In pricing a European option, we have a Finite Difference problem as described.
  - In pricing options with early exercise (American or Bermudan), we have a Linear Complementarity Problem, of which this formulation is a part.

# Effect of the convolution integral

- The convolution integral produces a source term and does not compromise the feasibility of the solution.
  – It may cause the discretization matrix to become full.
  – There are ways of dealing with the convolution integral, such that the discretization matrix remains sparse (iterative methods.)

# Mechanics of FD

- Pricing equation must be solved in the direction of information flow (from the future to the past)
- Two equivalent alternatives:
  - Reverse time and advance in the positive direction (good for analysis)
  - Keep physical time and advance in the negative direction (better for implementation)

# The Mechanics of FD

- The set of ODEs $\frac{dV}{dt} = AV$ is discretized in time.
  - Define

$$V^j = V(j\Delta t)$$

- The FD implementation leads to

$$A_{lhs} V^j = A_{rhs} V^{j+1}$$

  - $A_{lhs}$ and $A_{rhs}$ are (large and usually sparse) matrices
  - The particulars of these matrices depends on the time solution scheme.

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# The Time Solution Scheme

- Explicit Finite Difference
- Implicit Finite Difference
- Mixed Explicit/Implicit Scheme (CN)

Time Derivative

$$\frac{\partial V}{\partial t} \approx \frac{V(S_i, t_{j+1}) - V(S_i, t_j)}{\Delta t}$$

# The Time Solution Scheme

At an arbitrary node $V = V(S_i, t_j)\; i = 1,\ldots,N,\; j = 0,\ldots,M$ in the grid we introduce the following difference approximations to the terms in the BS-PDE:

$$\frac{\partial V}{\partial S} \approx (1-\Theta)\frac{V(S_{i+1},t_j)-V(S_{i-1},t_j)}{2\Delta S} + \Theta\frac{V(S_{i+1},t_{j+1})-V(S_{i-1},t_{j+1})}{2\Delta S}$$

$$\frac{\partial^2 V}{\partial S^2} \approx (1-\Theta)\frac{V(S_{i+1},t_j)-2V(S_i,t_j)+V(S_{i-1},t_j)}{\Delta S^2} + \Theta\frac{V(S_{i+1},t_{j+1})-2V(S_i,t_{j+1})+V(S_{i-1},t_{j+1})}{\Delta S^2}$$

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# The Time Solution Scheme

The parameter $\theta$ determines the time at which the partial derivatives are evaluated

$\theta = 0$ implicit FD

$\theta = 1$ explicit FD

$\theta = 0.5$ CN

Plugging in and collecting terms we end up

# The Solution Scheme for BS

with $S_i = i\Delta S$ and the notation $V_i^j = V(S_i, t_j)$ we reach at

$$
\begin{aligned}
&-\left[(1-\Theta)\cdot 0.5\cdot\left(\sigma^2 i^2-(r-q)i\right)\Delta t\right]V_{i-1}^j\\
&+\left[1+r(1-\Theta)\Delta t+(1-\Theta)\sigma^2 i^2\Delta t\right]V_i^j\\
&-\left[(1-\Theta)\cdot 0.5\cdot\left(\sigma^2 i^2+(r-q)i\right)\Delta t\right]V_{i+1}^j =\\
&\quad\left[\Theta\cdot 0.5\cdot\left(\sigma^2 i^2-(r-q)i\right)\Delta t\right]V_{i-1}^{j+1}\\
&-\left[\Theta\Delta t\sigma^2 i^2-1+r\Theta\Delta t\right]V_i^{j+1}\\
&+\left[\Theta\cdot 0.5\cdot\left(\sigma^2 i^2+(r-q)i\right)\Delta t\right]V_{i+1}^{j+1}
\end{aligned}
$$

# Solving Sparse Systems

- Time advancement leads to a sequence of linear systems

$$AV^{j+1} = b$$

- $A$ is very large (in 2D it has order of $10^8$ elements)
- Solution approaches:
  - Direct solvers
  - Iterative solvers

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# Solving Sparse Systems

- Definition: A direct solver reaches the solution in a finite number of computational steps.
- Naive direct solvers:
  – In 1D with tridiagonal structure, the LU algorithm is extremely efficient.
  – In several dimensions there are efficient direct solvers if you use fractional steps or factored schemes. In this case, the solution is a sequence of tridiagonal problems.

# Solving Sparse Systems

- Sophisticated direct solvers:
  - Strategy: Factor matrix *A* into upper and lower diagonal product (LU decomposition). This can now be solved directly by upward and downward sweeps.
  - Problem with the strategy: In order for strategy to work, L and U must be sparse. Naive decomposition will fill up the U and L!

# Example for Stock Option

European Put Option

S= 50 $\qquad S_{MAX} = S_N = 2 \cdot X$

X = 50 $\qquad S_{MIN} = S_0 = 0$

r = 0.1

σ = 0.4

T =5/12

Analytical Value: 4.076

# Example for Stock Option

A) Approach with Matrix Inversion

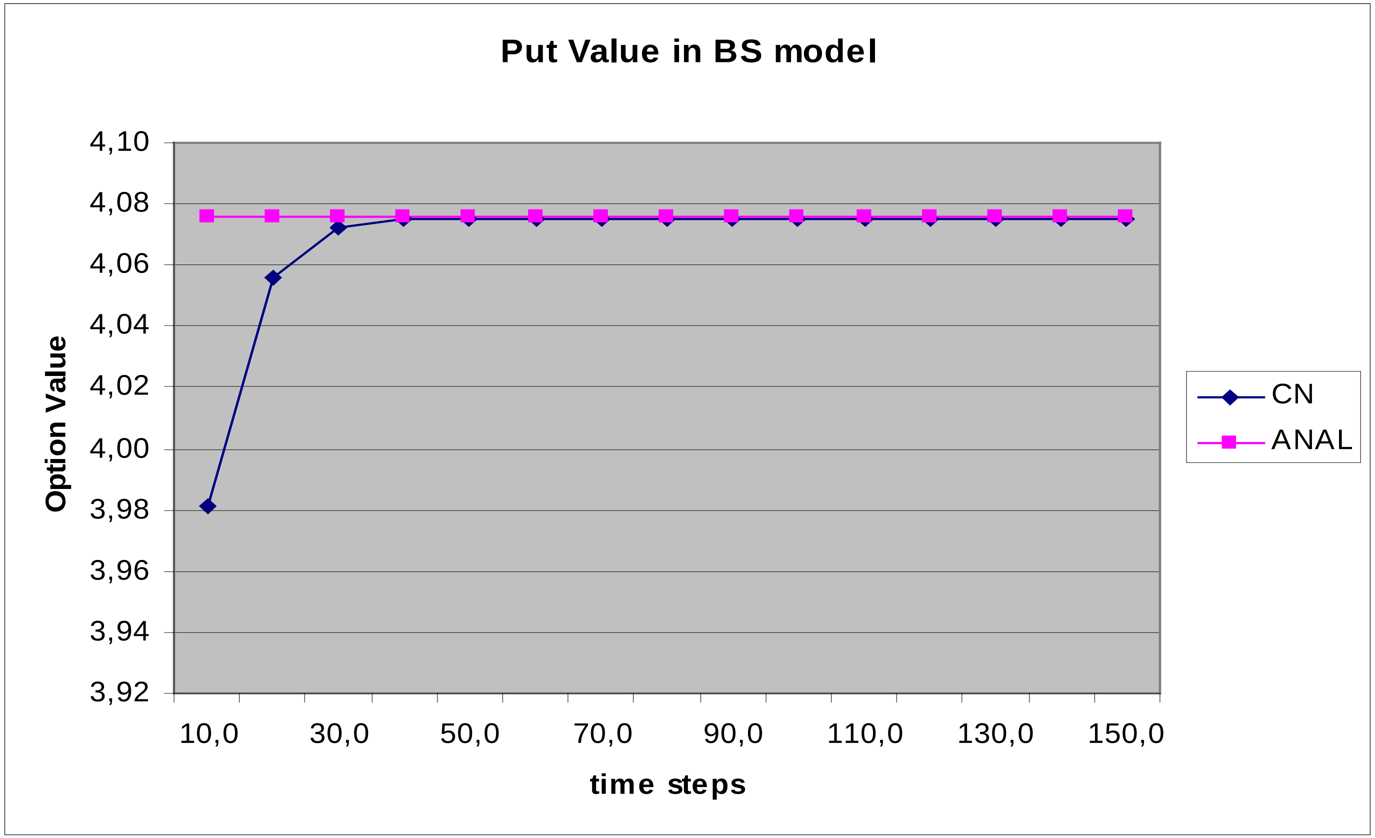
see sheet finite_sem1.xls N = 20  M =10

B) Approach with LUAlgorithm

see sheet finite_sem2.xls N = 200 M= variable

Delta, Gamma, Theta are evaluated by Cubic Spline Interpolation

# Stock Put Option



# Example for Interest Rate Option

Term structure model in continous time

HW Model short rate model

$$dr = (\phi(t) - ar)dt + \sigma dz$$

yields a PDE with $P = P(r_i, t_j)$ for the zero bond

$$\frac{\partial P}{\partial t} + \left(\phi(t) - ar\right)\frac{\partial P}{\partial r} + \frac{1}{2}\sigma^2 \frac{\partial^2 P}{\partial r^2} = rP$$

# Example for Interest Rate Option

For this model closed form bond prices are given

$$P(t,T) = A(t,T) \cdot e^{-r(t)B(t,T)}$$

$$B(t,T) = \frac{1}{a}\left(1 - e^{-a(T-t)}\right)$$

$$\ln A(t,T) = \ln \frac{P(0,T)}{P(0,t)} + B(t,T) \cdot f(0,t)$$

$$- \frac{1}{4a^3}\sigma^2\left(e^{-aT} - e^{-at}\right)^2\left(e^{2at} - 1\right)$$

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# The Solution Scheme for HW

with $\phi(t) = \phi^j$ and the notation $\alpha = \Delta t / \Delta r^2$ we reach

$$
\begin{aligned}
&\left[\alpha(1-\Theta)\cdot 0.5\cdot\left(\sigma^2 - (\phi^j - ar_i)\Delta r\right)\right]P_{i-1}^j \\
&-\left[1 + r_i\Delta t(1-\Theta) + \alpha(1-\Theta)\sigma^2\right]P_i^j \\
&+\left[\alpha(1-\Theta)\cdot 0.5\cdot\left(\sigma^2 + (\phi^j - ar_i)\Delta r\right)\right]P_{i+1}^j = \\
&-\left[\alpha\cdot\Theta\cdot 0.5\cdot\left(\sigma^2 - (\phi^{j+1} - ar_i)\Delta r\right)\right]P_{i-1}^{j+1} \\
&+\left[\alpha\cdot\Theta\sigma^2 - 1 + r_i\Delta t\Theta\right]P_i^{j+1} \\
&-\left[\alpha\cdot\Theta\cdot 0.5\cdot\left(\sigma^2 + (\phi^{j+1} - ar_i)\Delta r\right)\Delta t\right]P_{i+1}^{j+1}
\end{aligned}
$$

# Example for Interest Rate Option

With $\phi(t)$ known to be

$$\phi(t) = \frac{\partial f(0,t)}{\partial t} + af(0,t) + \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right)$$

In this formulation we have now time dependent coefficients at the grid nodes

Payoff of a Put Option on a zero bond

is $MAX\left[X - P(t,T),0\right]$ (Caplet)

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# Example for Interest Rate Option

Let the curve be

$$r(t) = 0.08 - 0.05 \cdot e^{-0.18t}$$

Option Maturity 5.5 years

Bond Maturity 6 years

Strike = ATM Forward Bond = 0.9605

a= 0.15 Notional = 100, space steps = 200

σ = 0.01 Analytical Value = 0,2160793

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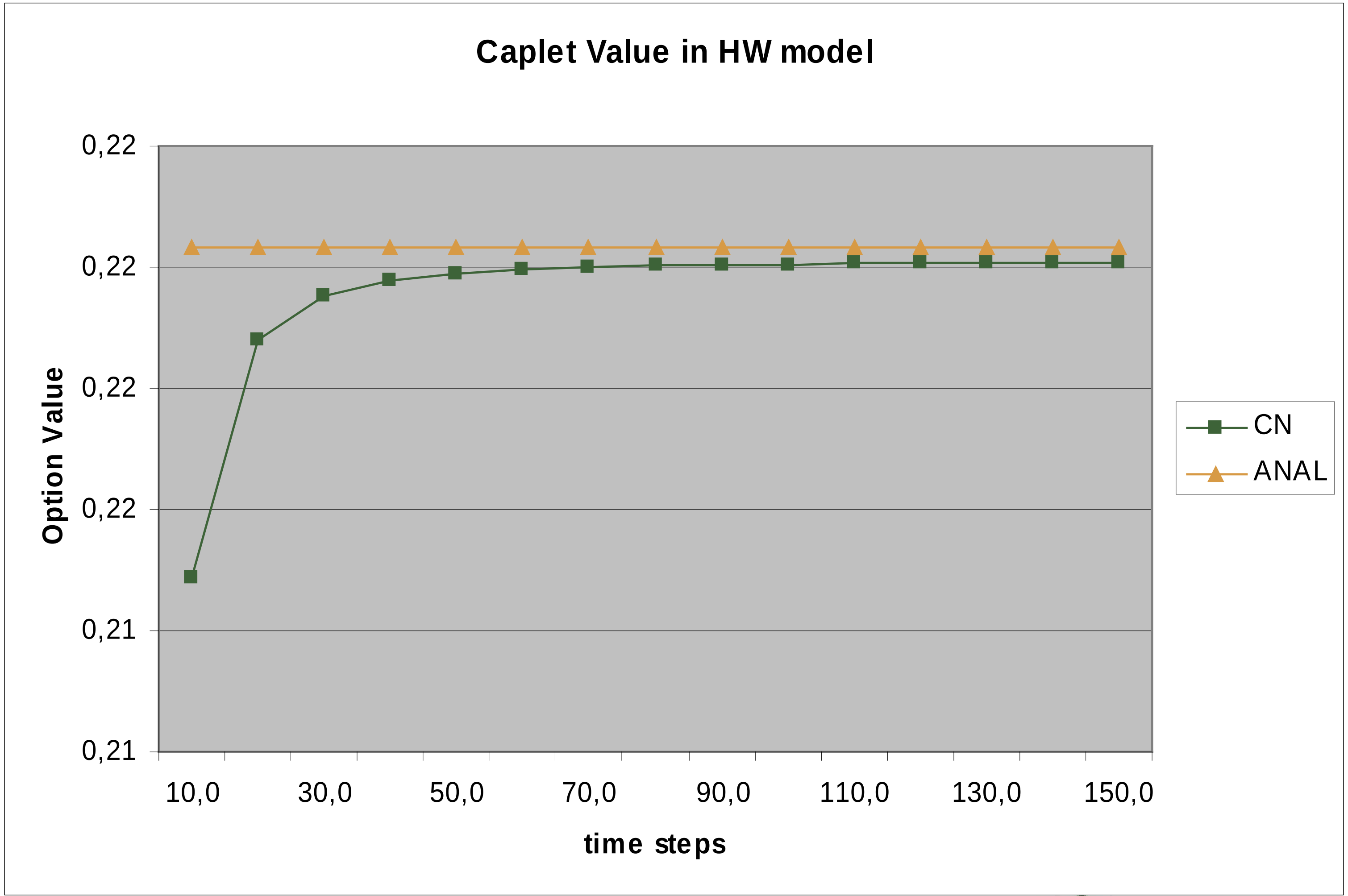
# Interest Rate Option



# Arithmetic Asian Option

- Asian Options are a Subclass of the general case of the option on a traded account

- We are interested in valuing Discrete Arithmetic Asian Options

# Arithmetic Asian Option

Option on a trading account with trading strategy $q_t$ and asset $S_t$ is modelled:

$$dX_t^q = q_t\, dS_t + \mu\left(X_t^q - q_t\, S_t\right)dt;\quad X_0^q = X_0$$

$$dS_t = S_t(rdt + \sigma dw_t)$$

$$q_t \in [\alpha_t, \beta_t]$$

In case of Passport Option $\alpha_t = -1$, $\beta_t = 1$

In case of <u>Asian Option</u> $\alpha_t = \beta_t$

In case of European Call Option $\alpha_t = 1$, $\beta_t = 1$

# Arithmetic Asian Option

The risk-neutral value is the at time t with payoff at Time T $MAX[X_T^q, 0]$ is

$$V^{[\alpha,\beta]}(t,S,X) = \underset{q\in[\alpha,\beta]}{MAX}\ e^{-r(T-t)} E\left[ MAX[X_T^q, 0] \middle| \Im_t \right]$$

Change of variable

$$Z_t^q = \frac{X_q^t}{S_t}$$

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# Arithmetic Asian Option

Using Ito's lemma and Girsanov's theorem we get a SDE

$$dZ_t^q = (q_t - Z_t^q)(r-\mu)dt + (q_t - Z_t^q)\sigma d\widetilde{w}_t$$

Introducing $u(t, Z_t) = \underset{q\in[\alpha,\beta]}{MAX}\, \widetilde{E}\left[MAX[Z_T^q, 0] \middle| \Im_t\right]$

This leads to the PDE

$$\frac{\partial u}{\partial t} + \underset{q\in[\alpha,\beta]}{MAX}\left[(r-\mu)(q-z)\frac{\partial u}{\partial z} + \frac{1}{2}(q-z)^2\sigma^2\frac{\partial^2 u}{\partial z^2}\right] = 0$$

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# Arithmetic Asian Option

<u>Asian Options</u>

with $d(tS_t) = tdS_t + S_t dt$ we get $\frac{1}{T}\int_0^T S_t dt = \int_0^T \left(1 - \frac{t}{T}\right) dS_t + S_0$

So the fixed strike call payoff $MAX[\overline{S} - K, 0]$

is modelled by the strategy $q_t = 1 - t/T$

and initial value $X_0 = S_0 - K$

where the traded account is evolving as

$$dX = q_t dS_t$$

# Arithmetic Asian Option

Diskrete Asian Options with n average points

The strategy is approximated by a step function

$$q_t = 1 - \frac{1}{n} INT\left[ n \frac{t}{T} \right]$$

So we get for the discrete fixed strike call

$$MAX\left( \frac{1}{n} \sum_{k=1}^{n} S_{(k/n)T} - K, 0 \right)$$

The discrete fixed strike put is similar.

# Arithmetic Asian Option

This leads to the PDE for Asian Options

$$\frac{\partial u}{\partial t} + r(q_t - z)\frac{\partial u}{\partial z} + \frac{1}{2}(q_t - z)^2 \sigma^2 \frac{\partial^2 u}{\partial z^2} = 0$$

with boundary condition: $u(T,z) = MAX[z,0]$

Again this has to be solved with a numerical method (FD).

# The Asian Solution Scheme

At an arbitrary node $u = u(z_i, t_j)\ i = 1,\ldots, M,\ j = 0,\ldots, N$ in the grid we introduce the following difference approximations ($\Theta$=1 implicit, $\Theta$=0 explicit) :

$$\frac{\partial u}{\partial z} \approx \Theta \frac{u(z_{i+1}, t_j) - u(z_{i-1}, t_j)}{2\Delta z} + (1-\Theta)\frac{u(z_{i+1}, t_{j+1}) - u(z_{i-1}, t_{j+1})}{2\Delta z}$$

$$\frac{\partial^2 u}{\partial z^2} \approx \Theta \frac{u(z_{i+1}, t_j) - 2u(z_i, t_j) + u(z_{i-1}, t_j)}{\Delta z^2}$$

$$+ (1-\Theta)\frac{z(z_{i+1}, t_{j+1}) - 2u(z_i, t_{j+1}) + u(z_{i-1}, t_{j+1})}{\Delta z^2}$$

# The Solution Scheme Asian Options

with $q(t) = q^j$ and $u(z_i, t_j) = u_i^j$, $k = \Delta z^2 / \Delta t$

$t_j = j\Delta t, z_i = z_0 + i\Delta z, z_0 = -1; z_M = 1$

$$
\begin{aligned}
&\Theta\left[\sigma^2 (q^j - z_i)^2 - r(q^j - z_i)\Delta z\right] u_{i-1}^j \\
&- 2\left[\sigma^2 \Theta (q^j - z_i)^2 + k\right] u_i^j \\
&+ \Theta\left[\sigma^2 (q^j - z_i)^2 + r(q^j - z_i)\Delta z\right] u_{i+1}^j = \\
&-(1-\Theta)\left[\sigma^2 (q^j - z_i)^2 - r(q^j - z_i)\Delta z\right] u_{i-1}^{j+1} \\
&+ 2\left[(1-\Theta)\sigma^2 (q^j - z_i)^2 - k\right] u_i^{j+1} \\
&-(1-\Theta)\left[\sigma^2 (q^j - z_i)^2 + r(q^j - z_i)\Delta\right] u_{i+1}^{j+1}
\end{aligned}
$$

# The Solution Scheme Asian Options

Payoff $u_i^N = MAX[z_i, 0]$

Boundary conditions and strategy:

Call: $u_0^j = 0,\ u_M^j = 2u_{M-1}^j - u_{M-2}^j$ $\quad q_t = 1 - \frac{1}{n} INT\left[n\frac{t}{T}\right]$

Put: $u_0^j = 2u_{M-1}^j - u_{M-2}^j, u_M^j = 0$ $\quad q_t = \frac{1}{n} INT\left[n\frac{t}{T}\right] - 1$

Solution is given by $Optvalue = S \cdot u(z_0, 0)$

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# Results for Asian Options FD

Calculations with N= 200, M = 200, MC = 10000

S=100,vola =0.2, r= 0.05, T=1y, n = 10

| Strike | MC | FD-Call-CN | MC | FD-Put-CN |
|---|---|---|---|---|
| 90,0 | 12,98 | **12,98** | 0,81 | **0,81** |
| 92,5 | 11,04 | **11,05** | 1,25 | **1,25** |
| 95,0 | 9,26 | **9,27** | 1,85 | **1,85** |
| 97,5 | 7,65 | **7,66** | 2,62 | **2,62** |
| 100,0 | 6,23 | **6,23** | 3,57 | **3,57** |
| 102,5 | 4,99 | **4,99** | 4,71 | **4,71** |
| 105,0 | 3,94 | **3,94** | 6,04 | **6,04** |
| 107,5 | 3,06 | **3,07** | 7,54 | **7,54** |
| 110,0 | 2,35 | **2,35** | 9,20 | **9,20** |

# Summary

- FD is flexible and fast
- FD is straightforward to implement
- Convergence properties are known

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