

Value at Risk

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Contents

1	Fundamentals	1
1.1	Risk Factor Evolution	3
1.2	Value at Risk of a Single Risk Factor	7
1.3	Approximation for the Risk Factors	12
1.4	The Covariance Matrix	13
1.4.1	Cholesky-Decomposition of the Covariance Matrix	16
1.5	The Variance-Covariance Method	20
1.6	The Delta-Normal Method	25

1.6.1	Value at Risk with respect to a Single Risk Factor	25
1.6.2	The Value at Risk with respect to Several Risk Factors	28
2	Essential Statistical Tools for 2nd Order VaR	33
2.0.3	Moment Generating Functions	33
2.0.4	Characteristic Functions	39
2.0.5	The χ^2 -Distribution	41
3	The Delta-Gamma Method	49
3.1	Decoupling of the Risk Factors	51
3.2	Diagonalization of the Gamma Matrix	52
3.3	The Distribution of the Portfolio-Value Changes	60
3.3.1	Moments of the Portfolio-value Distribution	64
3.3.2	Johnson Transformation	72
3.3.3	Cornish-Fisher Expansion	73
3.3.4	Fourier-Transformation of the Portfolio-Value Distribution . . .	75
3.3.5	Monte Carlo Simulations of the Portfolio-value Distribution . .	78

1 Fundamentals

- The *Value at Risk* (VaR) of the value V of a financial instrument or a portfolio is the upper bound for the *loss* which will not be exceeded with confidence c over a time span δt .

$$c \stackrel{!}{=} \text{cpf}_{\delta V}(\delta V > -\text{VaR}(c)) \quad (1.1)$$

- δV denotes the change of V and $\text{cpf}_{\delta V}$ (instead of P) denotes the *cumulative probability function* of the random variable δV .

– more explicitly

$$c \stackrel{!}{=} 1 - \text{cdf}_{\delta V}(\delta V \leq -\text{VaR}(c)) = 1 - \int_{-\infty}^{-\text{VaR}(c)} \text{pdf}_{\delta V}(x) dx \quad (1.2)$$

– $\text{pdf}_{\delta V}(x)$ denotes the *probability density function* of the random variable δV .

– Therefore the *negative* VaR is the $(1 - c)$ -percentile of the distribution of δV :

$$\text{VaR}(c) = -Q_{1-c}^{\text{cdf}_{\delta V}} = -\text{cdf}_{\delta V}^{-1}(1 - c) \quad (1.3)$$

- The Value at Risk is defined by the probability distribution of δV , *not* by the probability distributions of the risk factors!
- Using the probability distribution of the *risk factor* instead of the probability distribution of the *portfolio* is only possible when V is a *monotonous* function of the risk factor process S .

– Only then we have

$$\begin{aligned}\text{VaR}_V(c) &= V(S) - \min\{V(S = a), V(S = \tilde{a})\} \\ &= \max\{V(S) - V(S = a), V(S) - V(S = \tilde{a})\}\end{aligned}\tag{1.4}$$

– where a and \tilde{a} denote the lower and upper boundary of the risk factor's confidence interval.

1.1 Risk Factor Evolution

- Single risk factor S governed by a *geometric Brownian motion* (GBM)

$$d \ln S(t) = \mu dt + \sigma dW \text{ with } dW \sim N(0, dt)\tag{1.5}$$

- What is the process for S itself?

– Define the stochastic variable $y := \ln(S(t))$. Thus y satisfies

$$dy(t) = \mu dt + \sigma dW$$

– Choose the function f as $f(y, t) = e^y$.

– Ito's Lemma:

$$df(S, t) = \left[\mu \frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 f}{\partial S^2} \right] dt + \frac{\partial f}{\partial S} \sigma dW \quad (1.6)$$

– Since $f(y, t) = S(t)$ we have for the *differential* of S :

$$dS(t) = S(t) \underbrace{\left(\mu + \frac{\sigma^2}{2} \right)}_{\tilde{\mu}} dt + S(t) \sigma dW \quad (1.7)$$

• What is the process for the risk factor over a *finite* time?

– *finite* changes in S (over a finite, positive time span δt) can be derived by solving the SPDE 1.7

– Define a stochastic variable $y = W(t)$ = the value of the Wiener process at time t . This satisfies

$$dy(t) = 0dt + 1dW(t)$$

- Construct a function S of the stochastic variable y by

$$S(y, t) := S_0 \exp(\mu t + \sigma y)$$

where S_0 is an arbitrary factor.

- Ito's lemma gives the process for S induced by the process dy :

$$\begin{aligned} dS &= \left[\underbrace{\frac{\partial S}{\partial y}}_{\sigma S} 0 + \underbrace{\frac{\partial S}{\partial t}}_{\mu S} + \frac{1}{2} \underbrace{\frac{\partial^2 S}{\partial y^2}}_{\sigma^2 S} \right] dt + \underbrace{\frac{\partial S}{\partial y}}_{\sigma S} 1 dW \\ &= \left(\mu + \frac{\sigma^2}{2} \right) S dt + \sigma S dW \end{aligned}$$

- This corresponds exactly to the process in Equation 1.7.
- The process S thus constructed is therefore a solution of the SPDE 1.7.

- Simply making the substitution $t \rightarrow t + \delta t$ we obtain

$$\begin{aligned} S(t + \delta t) &= S_0 \exp(\mu t + \mu \delta t + \sigma y(t + \delta t)) \\ &= \underbrace{S_0 \exp(\mu t)}_{\tilde{S}_0(t)} \exp(\mu \delta t + \sigma W(t + \delta t)) \\ &= \tilde{S}_0(t) \exp(\sigma W(t) + \mu \delta t + \sigma \delta W) \end{aligned}$$

with the notation δW for a change in a Brownian motion after the passing of a finite time interval δt :

$$\delta W := W(t + \delta t) - W(t) \implies \delta W \sim N(0, \delta t) \quad (1.8)$$

- The first term in the exponent refers to (already known) values at time t . It can also be absorbed into the (still arbitrary) pre-factor

$$S(t + \delta t) = \underbrace{\tilde{S}_0(t) e^{\sigma W(t)}}_{\tilde{S}_0(t)} \exp(\mu \delta t + \sigma \delta W)$$

- Initial condition for the solution of the SPDE:

$$S(t + \delta t) \stackrel{\delta t \rightarrow 0}{=} S(t) \implies \tilde{S}_0(t) = S(t)$$

- Thus, we obtain the change in S corresponding to Equation 1.7 or 1.5 over a *finite* time span δt :

$$S(t + \delta t) = S(t) \exp(\mu\delta t + \sigma\delta W) \text{ with } \delta W \sim N(0, \delta t) \quad (1.9)$$

- The finite time span δt can be taken to be arbitrarily long. δt is taken to be the *liquidation period*.

1.2 Value at Risk of a Single Risk Factor

- Consider a portfolio consisting of a single position in N of the same risk factor S .

$$V = NS(t) , \quad \delta V(t) = N\delta S(t) \text{ with } \delta S(t) = S(t + \delta t) - S(t)$$

- The change in S induces a change in V amplified by the *constant* factor N .
- The factor N is the *sensitivity* of V with respect to S .

– δV is a *linear* function of δS .

- The value change δV over the liquidation period δt follows directly from Equation 1.9

$$\begin{aligned}\delta V &= NS(t + \delta t) - NS(t) \\ &= NS(t) [\exp(\mu\delta t + \sigma\delta W) - 1]\end{aligned}\tag{1.10}$$

- The VaR as defined in Equation 1.2 is:

$$\text{cpf}_{\delta V}(\delta V \leq -\text{VaR}) = \text{cpf}_{\delta V}(NS(t) [e^{\mu\delta t + \sigma\delta W} - 1] \leq -\text{VaR})$$

– $\text{cpf}_{\delta V}$ is unknown. But: The only stochastic variable involved is the Brownian motion:

$$\delta W \sim N(0, \delta t) \Rightarrow \delta W \sim X\sqrt{\delta t} \quad \text{with } X \sim N(0, 1)$$

– Rewrite the event that $\delta V \leq -\text{VaR}$ with the purpose of isolating δW .

$$\begin{aligned}\text{cpf}_{\delta V}(\delta V \leq -\text{VaR}) &= \text{cpf}_{\delta V} \left(\delta W \leq \frac{\ln \left(1 - \frac{\text{VaR}}{NS(t)} \right) - \mu \delta t}{\sigma} \right) \\ &= \text{cpf}_{\delta V} \left(\delta W \leq a\sqrt{\delta t} \right)\end{aligned}$$

with the abbreviation

$$a := \frac{\ln \left(1 - \frac{\text{VaR}}{NS(t)} \right) - \mu \delta t}{\sigma \sqrt{\delta t}} \quad (1.11)$$

- The probability that δW is less than or equal to a certain value is dependent on the distribution of δW alone and not on that of δV . We can therefore simply replace $\text{cpf}_{\delta V}$ with $\text{cpf}_{\delta W}$:

$$\begin{aligned}\text{cpf}_{\delta V}(\delta V \leq -\text{VaR}) &= \text{cpf}_{\delta W} \left(\delta W \leq a\sqrt{\delta t} \right) \\ &= \text{cpf}_{\delta W} \left(X\sqrt{\delta t} \leq a\sqrt{\delta t} \right) \\ &= \text{cpf}_{\delta W} (X \leq a)\end{aligned}$$

- The probability that X is smaller than a particular variable is dependent on the distribution of X alone and not on that of δW allowing $\text{cpf}_{\delta W}$ to be simply replaced by $\text{cpf}_X = N(0, 1)$

$$\text{cpf}_{\delta V}(\delta V \leq -\text{VaR}) = \text{cpf}_X(X \leq a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^a \exp(-x^2/2) dx$$

- The Value at Risk with respect to a confidence c follows now from the requirement

$$c \stackrel{!}{=} 1 - \text{cpf}_{\delta V}(\delta V \leq -\text{VaR}) = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^a \exp(-x^2/2) dx$$

- a in 1.11 is thus the $(1 - c)$ -percentile of the standard normal distribution

$$a = Q_{1-c}^{N(0,1)}$$

- Solving 1.11 for VaR finally yields the value at risk of a long position in N risk factors:

$$\text{VaR}(c) = NS(t) \left[1 - \exp\left(\mu\delta t + Q_{1-c}^{N(0,1)} \sigma\sqrt{\delta t}\right) \right] \quad (1.12)$$

– Examples of percentiles of the standard normal distribution:

$$\begin{aligned}c = 95\% = 0,95 &\Rightarrow a = Q_{1-c}^{N(0,1)} \approx -1,65 \\c = 99\% = 0,99 &\Rightarrow a = Q_{1-c}^{N(0,1)} \approx -2,326\end{aligned}\tag{1.13}$$

• Analogously: the VaR of a *short* position in $-N$ risk factors S is

$$\text{VaR}(c) = -NS(t) \left[1 - \exp \left(\mu\delta t - Q_{1-c}^{N(0,1)} \sigma \sqrt{\delta t} \right) \right]\tag{1.14}$$

• These two VaR's are *not* equal, due to

- the drift μ
- *log*normally distributed price changes. The lognormal distribution is not symmetric.

1.3 Approximation for the Risk Factors

- For short liquidation periods (e.g., $\delta t = 10$ days = 0,0274 years): $\exp(x) \approx 1+x$ and/or $\mu \approx 0$:

$$\delta S(t) \approx \begin{cases} S(t) [e^{\sigma \delta W} - 1] & \mu \text{ neglected} \\ S(t) [\mu \delta t + \sigma \delta W] & \text{linear proxy for exp} \\ S(t) \sigma \delta W & \mu \text{ neglected and linear proxy} \end{cases} \quad (1.15)$$

- That means for the Value at Risk of a *long* position in N risk factors S

$$\text{VaR}_{\text{long}}(c) \approx \begin{cases} NS(t) \left[1 - \exp\left(Q_{1-c}^{N(0,1)} \sigma \sqrt{\delta t}\right) \right] & \mu \approx 0 \\ NS(t) \left[-\mu \delta t - Q_{1-c}^{N(0,1)} \sigma \sqrt{\delta t} \right] & \text{exp linear} \\ -NS(t) Q_{1-c}^{N(0,1)} \sigma \sqrt{\delta t} & \mu \approx 0, \text{exp linear} \end{cases} \quad (1.16)$$

- VaR of the *short* position: the only differences are signs of N and $Q_{1-c}^{N(0,1)}$

- Long and Short VaR equal *only if*

- linear approximation *and*
- drift is neglected *and*
- portfolios value changes are (approximately) linear function of underlying risk factor (Delta-Normal approximation)
- Only in this case: square root of time law and the linear scaling with percentiles:

$$\text{VaR}(c', \delta t') \approx \frac{Q_{1-c'}}{Q_{1-c}} \sqrt{\frac{\delta t'}{\delta t}} \text{VaR}(c, \delta t) \quad (1.17)$$

1.4 The Covariance Matrix

- In general n (often hundreds of) risk factors modeled by random walks obeying the *coupled* stochastic differential equations

$$d \ln S_i(t) = \mu_i dt + dZ_i \quad \text{for } i = 1, 2, \dots, n \quad (1.18)$$

- Here dZ_i are *correlated* drift-free Brownian motions with *covariance*

$$\begin{aligned} \text{cov}[dZ_i, dZ_j] &= d\Sigma_{ij}, \quad \text{E}[dZ_i] = 0 \\ \text{with } d\Sigma_{ij} &= \rho_{ij} \sigma_i \sqrt{\delta t} \sigma_j \sqrt{\delta t} = \sigma_i \rho_{ij} \sigma_j dt \end{aligned} \quad (1.19)$$

- $\sigma_i = \text{volatility}$ of Z_i , i.e. of risk factor S_i
- $\rho_{ij} = \text{correlation}$ between Z_i and Z_j , i.e. between S_i and S_j
- $d\Sigma_{ij}$ is called *covariance matrix*.

- Notation:

$$\mathbf{X} \sim \mathbf{N}(\mathbf{R}, \mathbf{V}) \iff$$

X_i normally distributed with
 $\text{cov}[X_i, X_j] = V_{ij}$, $\mathbf{E}[X_i] = R_i$

i.e.

$$d\mathbf{Z} = \begin{pmatrix} dZ_1 \\ dZ_2 \\ \vdots \\ dZ_n \end{pmatrix} \sim \mathbf{N}(\mathbf{0}, d\Sigma)$$

- After a finite time δt :

- Solutions as in 1.9 to the stochastic differential equations 1.18:

$$S_i(t + \delta t) = S_i(t) \exp(\mu_i \delta t + \delta Z_i) \text{ with } \delta \mathbf{Z} \sim N(\mathbf{0}, \delta \Sigma) \quad (1.20)$$

- with the covariance matrix

$$\delta \Sigma = \begin{pmatrix} \delta \Sigma_{11} & \delta \Sigma_{12} & \cdots & \cdots & \delta \Sigma_{1n} \\ \delta \Sigma_{21} & \ddots & & \ddots & \delta \Sigma_{2n} \\ \vdots & & \delta \Sigma_{ij} & & \vdots \\ \vdots & \ddots & & \ddots & \vdots \\ \delta \Sigma_{n1} & \delta \Sigma_{n2} & \cdots & \cdots & \delta \Sigma_{nn} \end{pmatrix} \text{ where } \delta \Sigma_{ij} = \sigma_i \rho_{i,j} \sigma_j \delta t$$

for $i, j = 1, 2, \dots, n$ (1.21)

- Here, δt is the reference time interval for the change in the risk factors,
 - * i.e. $\delta t = 1$ day for a daily changes in the risk factors,
 - * $\delta t = 25$ days for a monthly change, etc.

- Proxy for $\delta S_i(t) = S_i(t + \delta t) - S_i(t)$ over a short time δt :

$$\delta S_i(t) \approx \begin{cases} S_i(t) [e^{\delta Z_i} - 1] & \text{drift neglected} \\ S(t) [\mu_i \delta t + \delta Z_i] & \text{linear proxy for exp} \\ S(t) \delta Z_i & \text{drift neglected and linear proxy} \end{cases} \quad (1.22)$$

– Usually:

$$\delta S_i(t) \approx S_i(t) \delta Z_i \text{ with } \delta \mathbf{Z} \sim \mathbf{N}(\mathbf{0}, \delta \Sigma) \quad (1.23)$$

1.4.1 Cholesky-Decomposition of the Covariance Matrix

- The symbol $\mathbf{1}$ denotes the *identity matrix*

$$\mathbf{1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \iff (\mathbf{1})_{ij} = \delta_{ij} \quad (1.24)$$

- The matrix \mathbf{A} denotes the “square root” of the covariance matrix

$$\mathbf{A}\mathbf{A}^T = \delta \Sigma \quad (1.25)$$

– \mathbf{A}^T denotes the *transpose* of the matrix \mathbf{A} :

$$(\mathbf{A}^T)_{ij} = A_{ji}$$

• \mathbf{A} transforms uncorrelated random variables into correlated ones (with covariance $\delta\Sigma$):

– Let $X_i, i = 1, \dots, n$ be uncorrelated:

$$\mathbf{X} = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} \text{ with } \mathbf{X} \sim \mathbf{N}(\mathbf{0}, \mathbf{1}) \quad (1.26)$$

* more explicitly

$$X_i \sim \mathbf{N}(0, 1) \quad \forall i = 1, \dots, n, \quad \text{cov}[X_i, X_j] = \delta_{ij} \quad \forall i, j = 1, \dots, n$$

– Applying the matrix \mathbf{A} to \mathbf{X} generates new random variables \mathbf{Y} :

$$\mathbf{Y} = \mathbf{A}\mathbf{X} \iff Y_i = \sum_k A_{ik} X_k$$

- Since this is a linear transformation, the Y_i are also normally distributed.
- Covariances of the new variables:

$$\begin{aligned}
 \text{cov}[Y_i, Y_j] &= \text{cov}\left[\sum_k A_{ik} X_k, \sum_m A_{jm} X_m\right] \\
 &= \sum_k A_{ik} \sum_m A_{jm} \underbrace{\text{cov}[X_k, X_m]}_{\delta_{km}} \\
 &= \sum_k A_{ik} A_{jk} = \left(\mathbf{A}\mathbf{A}^T\right)_{ij} = \delta\Sigma_{ij}
 \end{aligned}$$

- The expectations of these random variables are

$$\mathbb{E}[Y_i] = \mathbb{E}\left[\sum_k A_{ik} X_k\right] = \sum_k A_{ik} \underbrace{\mathbb{E}[X_k]}_0 = 0$$

- Therefore:

$$\mathbf{A}\mathbf{X} = \mathbf{Y} \sim \mathcal{N}(\mathbf{0}, \delta\Sigma) \tag{1.27}$$

- The inverse \mathbf{A}^{-1} transforms correlated random variables into uncorrelated ones.

- Let $Y_i, i = 1, \dots, n$ be correlated multivariate normally distributed random variables with covariance $\delta\Sigma$.
- Then the transformed variables \mathbf{X} are uncorrelated and standard normally distributed:

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{Y} \sim N(\mathbf{0}, \mathbf{1}) \quad (1.28)$$

- Explicit construction of matrix \mathbf{A} iteratively via *Cholesky decomposition*:

$$A_{ji} = \begin{cases} 0 & \text{for } j < i \\ \sqrt{\delta\Sigma_{ii} - \sum_{k=1}^{i-1} A_{ik}^2} & \text{for } j = i \\ \frac{1}{A_{ii}} \left(\delta\Sigma_{ji} - \sum_{k=1}^{i-1} A_{ik} A_{jk} \right) & \text{for } j > i \end{cases} \quad (1.29)$$

- with $\delta\Sigma_{ji}$ as given in Equation 1.21:

$$\delta\Sigma_{ji} = \begin{cases} \sigma_i^2 \delta t & \text{for } j = i \\ \rho_{ij} \sigma_i \sigma_j \delta t & \text{for } j \neq i \end{cases}$$

- Begin with $i = 1, j = 1$ and proceed by solving for A_{j1} for all j .
- Subsequently set $i = 2$ and solve for A_{j2} for all j . Repeat the procedure.

1.5 The Variance-Covariance Method

- Vector of risk factors:

$$\mathbf{S}(t) = \begin{pmatrix} S_1(t) \\ \vdots \\ S_n(t) \end{pmatrix}$$

- Linear approximations of the risk factor evolutions throughout (see 1.15):

$$\delta S_i(t) \approx S_i(t) [\mu_i \delta t + \delta Z_i] \approx S_i(t) \delta Z_i \quad (1.30)$$

- Main idea: Expand portfolio value V in its Taylor series.

- Portfolio value change $\delta V(\mathbf{S})$ up to second order

$$\begin{aligned}
\delta V(\mathbf{S}(t)) &= V(\mathbf{S}(t) + \delta \mathbf{S}(t)) - V(\mathbf{S}(t)) \\
&\approx \sum_i^n \frac{\partial V}{\partial S_i} \delta S_i(t) + \frac{1}{2} \sum_{i,j}^n \delta S_i(t) \frac{\partial^2 V}{\partial S_i \partial S_j} \delta S_j(t) \\
&= \sum_i^n \Delta_i \delta S_i(t) + \frac{1}{2} \sum_{i,j}^n \delta S_i(t) \Gamma_{ij} \delta S_j(t) \\
&\approx \sum_i^n \tilde{\Delta}_i [\mu_i \delta t + \delta Z_i] + \frac{1}{2} \sum_{i,j}^n [\mu_i \delta t + \delta Z_i] \tilde{\Gamma}_{ij} [\mu_j \delta t + \delta Z_j] \\
&\approx \sum_i^n \tilde{\Delta}_i \delta Z_i + \frac{1}{2} \sum_{i,j}^n \delta Z_i \tilde{\Gamma}_{ij} \delta Z_j \tag{1.31}
\end{aligned}$$

- First “ \approx ”: broken off the Taylor series for *portfolio value* V after 2nd order terms,
- Second “ \approx ”: linear approximation of the *risk factors* S_i
- Third “ \approx ”: drift neglected.

- Second order proxy for δV is called *Delta-Gamma approximation*
- First order proxy for δV is called *Delta approximation*
- Abbreviations Δ_i and Γ_{ij} denote the risk factor *sensitivities* of V :

$$\Delta_i := \frac{\partial V}{\partial S_i}, \quad \Gamma_{ij} := \frac{\partial^2 V}{\partial S_i \partial S_j}, \quad i, j = 1, \dots, n$$

- Γ_{ij} (sometimes called *Hessian matrix*) contains also *mixed* partial derivatives.
- The sensitivities usually appear in connection with the *current levels* $S_i(t)$ and $S_j(t)$:

$$\tilde{\Delta}_i := S_i(t) \frac{\partial V}{\partial S_i}, \quad \tilde{\Gamma}_{ij} := S_i(t) S_j(t) \frac{\partial^2 V}{\partial S_i \partial S_j} \quad (1.32)$$

- Delta-Gamma Proxy for δV in vector form:

$$\begin{aligned} \delta V(\mathbf{S}(t)) &= \begin{pmatrix} \tilde{\Delta}_1 & \cdots & \tilde{\Delta}_n \end{pmatrix} \begin{pmatrix} \delta Z_1 \\ \vdots \\ \delta Z_n \end{pmatrix} \\ &+ \frac{1}{2} \begin{pmatrix} \delta Z_1 & \cdots & \delta Z_n \end{pmatrix} \begin{pmatrix} \tilde{\Gamma}_{1,1} & \cdots & \tilde{\Gamma}_{1,n} \\ \vdots & \ddots & \vdots \\ \tilde{\Gamma}_{n,1} & \cdots & \tilde{\Gamma}_{n,n} \end{pmatrix} \begin{pmatrix} \delta Z_1 \\ \vdots \\ \delta Z_n \end{pmatrix} \\ &= \tilde{\Delta}^T \delta \mathbf{Z} + \frac{1}{2} \delta \mathbf{Z}^T \tilde{\Gamma} \delta \mathbf{Z} \end{aligned}$$

- Taylor-approximations for a *straddle* (a portfolio made up of a call and a put) are presented in Figure 1.1.

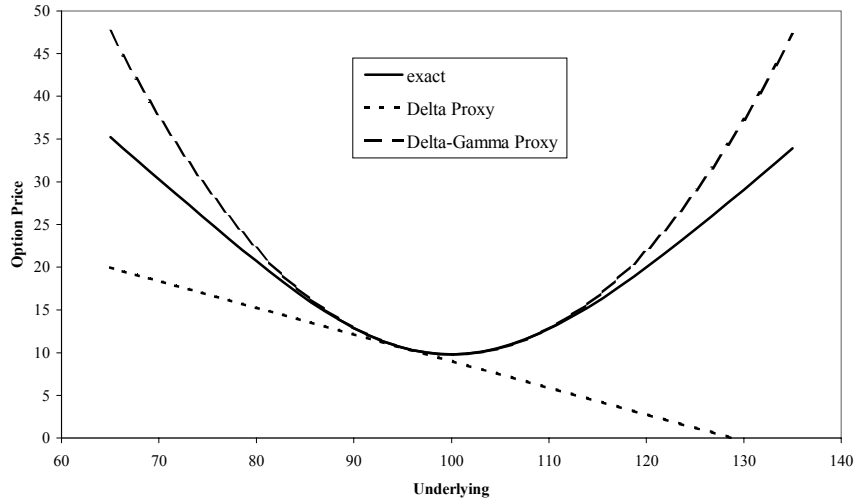


Figure 1.1: Black-Scholes price of a straddle (strike = 100, time to maturity = 1 year) on an underlying S (vol 25%, dividend yield 6%, repo rate 3%). The dashed line is the Delta-Gamma proxy, the dotted line is the simple Delta proxy. The Taylor expansion was done around $S = 95$.

1.6 The Delta-Normal Method

- Taylor series 1.31 of the portfolio value only up to the *linear* term:

$$\begin{aligned}\delta V(\mathbf{S}(t)) &\approx \sum_i^n \frac{\partial V}{\partial S_i} \delta S_i(t) \\ &\approx \sum_i^n \tilde{\Delta}_i [\mu_i \delta t + \delta Z_i] \approx \sum_i^n \tilde{\Delta}_i \delta Z_i = \tilde{\Delta}^T \delta \mathbf{Z}\end{aligned}\tag{1.33}$$

1.6.1 Value at Risk with respect to a Single Risk Factor

- For a single risk factor with the portfolio sensitivity $\Delta := \partial V / \partial S$:

$$\delta V(S(t)) \approx \Delta \delta S(t)\tag{1.34}$$

- A portfolio with a linear sensitivity Δ can be intuitively interpreted as a portfolio consisting of Δ risk factors.

– The value at risk is thus given either

- * by Equation 1.12 with the correspondence $\Delta \hat{=} N$ for $\Delta > 0$ or
- * by Equation 1.14 with the correspondence $\Delta \hat{=} -N$ for $\Delta < 0$

- In any case: V is linear and in consequence, a monotonous function of S . Therefore, in the sense of Equation 1.4

$$\text{VaR}_V(c) \approx \max\left\{\tilde{\Delta} \left[1 - \exp\left(\mu\delta t + Q_{1-c}\sigma\sqrt{\delta t}\right)\right], \right. \\ \left. \tilde{\Delta} \left[1 - \exp\left(\mu\delta t - Q_{1-c}\sigma\sqrt{\delta t}\right)\right]\right\} \quad (1.35)$$

- As usual: $\tilde{\Delta} := S(t)\Delta$ and Q_{1-c} is the $(1-c)$ -percentile of the standard normal distribution.

- The maximum function in Equation 1.4 effects the correct choice for the VaR.
 - For $\Delta > 0$, the *lower bound* of the confidence interval of the risk factor is relevant
 - For $\Delta < 0$, the *upper bound* of the confidence interval of the risk factor is relevant.

- Now: Approximate the *risk factor* change with Equation 1.30:

$$\begin{aligned}
\text{VaR}_V(c) &\approx \max\{\tilde{\Delta} [-\mu\delta t - Q_{1-c}\sigma\sqrt{\delta t}], \tilde{\Delta} [-\mu\delta t + Q_{1-c}\sigma\sqrt{\delta t}]\} \\
&= \max\{-\tilde{\Delta}Q_{1-c}\sigma\sqrt{\delta t}, +\tilde{\Delta}Q_{1-c}\sigma\sqrt{\delta t}\} - \tilde{\Delta}\mu\delta t \\
&= |\tilde{\Delta}Q_{1-c}\sigma\sqrt{\delta t}| - \tilde{\Delta}\mu\delta t
\end{aligned} \tag{1.36}$$

- In this proxy, the maximum function produces precisely the absolute value of the risk which is caused by the volatility of the risk factor.
- A positive drift μ of the risk factor
 - * reduces the portfolio risk when $\tilde{\Delta} > 0$
 - * *increases* the portfolio risk when $\tilde{\Delta} < 0$
- We were able to deduce information about the *unknown* distribution of the *portfolio's value* V from the *known* distribution of the *risk factor* S !

1.6.2 The Value at Risk with respect to Several Risk Factors

- Avoidance of the *unknown* distribution of V is only possible within the roughest approximation in 1.30:

$$\delta S_i(t) \approx S_i(t) \delta Z_i$$

- Squaring both sides of 1.36 with $\mu = 0$ yields:

$$\begin{aligned} \text{VaR}_V^2(c) &\approx \tilde{\Delta}^2 (Q_{1-c})^2 \sigma^2 \delta t \\ &= \Delta^2 (Q_{1-c})^2 S(t)^2 \sigma^2 \delta t \\ &= \Delta^2 (Q_{1-c})^2 S(t)^2 \text{var} [\delta Z] \\ &= \Delta^2 (Q_{1-c})^2 \text{var} [\delta S(t)] \end{aligned}$$

- On the other hand, the variance of V is

$$\text{var} [\delta V] \approx \text{var} [\Delta \delta S(t)] = \Delta^2 \text{var} [\delta S(t)]$$

- In this approximation the (square of the) *Value at Risk* of the portfolio can be expressed in terms of the *variance* of the portfolio:

$$\text{VaR}_V^2(c) \approx (Q_{1-c})^2 \text{var} [\delta V]$$

- Now, only the *variance* of the portfolio's value needs to be determined *not* its entire *distribution* nor its *percentiles*!
- This is also true for several risk factors in the approximation 1.23, i.e.

$$\delta V \approx \sum_{i=1}^n \tilde{\Delta}_i \delta Z_i \implies \text{VaR}_V(c) \approx Q_{1-c} \sqrt{\text{var} [\delta V]}$$

- The variance of a sum of random variables is simply the sum of the covariances

of these random variables:

$$\begin{aligned}
 \text{var} [\delta V] &\approx \sum_{i,j=1}^n \tilde{\Delta}_i \tilde{\Delta}_j \text{cov} [\delta Z_i, \delta Z_j] \\
 &= \sum_{i,j=1}^n \tilde{\Delta}_i \delta \Sigma_{ij} \tilde{\Delta}_j \\
 &= \delta t \sum_{i,j=1}^n \tilde{\Delta}_i \sigma_i \rho_{ij} \sigma_j \tilde{\Delta}_j
 \end{aligned} \tag{1.37}$$

- Thus the Value at Risk is

$$\begin{aligned}
 \text{VaR}_V(c) &\approx Q_{1-c} \sqrt{\text{var} [\delta V]} \\
 &= Q_{1-c} \sqrt{\tilde{\Delta} \delta \Sigma \tilde{\Delta}} \\
 &= Q_{1-c} \sqrt{\delta t} \sqrt{\sum_{i,j=1}^n \tilde{\Delta}_i \sigma_i \rho_{ij} \sigma_j \tilde{\Delta}_j}
 \end{aligned} \tag{1.38}$$

- This is the central equation for the Delta-Normal method.

- The mean return can be introduced into approximation 1.38 after the fact analogously to the single risk factor case:

$$\text{VaR}_V(c) \approx Q_{1-c} \sqrt{\delta t} \sqrt{\sum_{i,j=1}^n \tilde{\Delta}_i \sigma_i \rho_{ij} \sigma_j \tilde{\Delta}_j} - \delta t \sum_i \tilde{\Delta}_i \mu_i \quad (1.39)$$

- Summary of the Delta-Normal approach to the calculation of the Value at Risk
 - Calculate of the sensitivities Δ_i of the portfolio with respect to all risk factors S_i .
 - Multiply the covariance matrix $\delta\Sigma$ with the portfolio sensitivities Δ_i and the current risk factor values S_i to obtain the portfolio variance as in 1.37.
 - * The covariance matrix's elements are products of the risk factor volatilities and correlations as in 1.21.
 - Multiply the portfolio variance as in Equation 1.38 with the liquidation period and the square of the percentile corresponding to the desired confidence interval (for example, 2.326 for 99% confidence).
 - The square root of the thus obtained number is the Value at Risk of the entire portfolio, neglecting the effect of the risk factor drifts.

- The effect of the mean returns can be taken into consideration as in Equation 1.39.

2 Essential Statistical Tools for 2nd Order VaR

2.0.3 Moment Generating Functions

- The *moment generating function* (in short *MGF*) of a random variable x with density function $\text{pdf}(x)$ is defined as the expectation of e^{sx} for an arbitrary real value s

$$G_x(s) = \mathbb{E}[e^{sx}] = \int_{-\infty}^{\infty} e^{sx} \text{pdf}(x) dx \quad (2.1)$$

- The MGF has very useful property: If two random variables x and y are *independent* then

$$G_{x+y}(s) = G_x(s) G_y(s) \quad (2.2)$$

- The *distribution* of a *sum* of random variables is generally very difficult to determine (as we will see below).
- The *MGF* of such a sum, in contrast, is simply the product of the MGFs of each of the *individual* distributions!
- The random variables in the sum can be governed by completely *different* distributions, as long as they are all statistically *independent*.
- Equation 2.2 is quite simple to prove (independence is needed in the second to last step):

$$G_{x+y}(s) \equiv \mathbb{E}[e^{s(x+y)}] = \mathbb{E}[e^{sx} e^{sy}] = \mathbb{E}[e^{sx}] \mathbb{E}[e^{sy}] = G_x(s) G_y(s)$$

- Similarly, for all non-stochastic values a, b and random variables x we have

$$G_{ax+b}(s) = e^{sb} G_x(as) \quad (2.3)$$

– The proof is also quite simple:

$$G_{ax+b}(s) = \mathbb{E}[e^{s(ax+b)}] = e^{bs} \mathbb{E}[e^{(as)x}] = e^{bs} G_x(as)$$

- The most famous property (which gave the function its name), however, is that differentiating the MGF with respect to s at $s = 0$ yields all moments of the distribution:

$$\left. \frac{\partial^n G_x(s)}{\partial s^n} \right|_{s=0} = \mathbb{E}[x^n] \quad (2.4)$$

– This can be shown by Taylor-expanding e^{sx} and then differentiating with respect to s .

- The *central* moments of a random number x with expectation $\mu = \mathbb{E}[x]$ are the moments of the random number $\tilde{x} := x - \mu$.

– Equation 2.3 yields for the moments of \tilde{x}

$$G_{\tilde{x}}(s) = G_{x-\mu}(s) = e^{-s\mu} G_x(s) \quad (2.5)$$

- Thus, the central moments of x can also be calculated directly from the MGF of x :

$$\mathbb{E}[(x - \mathbb{E}[x])^n] = \frac{\partial^n}{\partial s^n} \exp(-s\mathbb{E}[x]) G_x(s) \Big|_{s=0} \quad (2.6)$$

- The general procedure for calculating central moments is therefore:
 - * first calculate the expectation using Equation 2.4.
 - * Then insert the result into Equation 2.6 for the central moments.
- For many distributions an *explicit analytical expression* for the MGF can be obtained from the integral representation 2.1.
- The moments can then be calculated by simply differentiating this analytical expression.

Example: The Normal Distribution

- The MGF of the standard normal distribution is, by Definition 2.1

$$\begin{aligned}
 G_{N(0,1)}(s) &= \mathbb{E}[e^{sx}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{sx} e^{-\frac{x^2}{2}} dx \\
 &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left\{-\frac{x^2 - 2sx}{2}\right\} dx \\
 &= \exp\left(\frac{1}{2}s^2\right) \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left\{-\frac{(x-s)^2}{2}\right\} dx}_1
 \end{aligned}$$

- We completed the square in the last step.
- The remaining integral is the probability that a $N(s, 1)$ -distributed random variable will take on *any* value.
- The moment generating function of the *standard* normal distribution is thus simply

$$G_{N(0,1)}(s) = \exp\left(\frac{1}{2}s^2\right) \tag{2.7}$$

- The moment generating function of the normal distribution with expectation μ and variance σ now follows from 2.3

$$G_{N(\mu, \sigma^2)}(s) = e^{\mu s} G_{N(0,1)}(\sigma s) = \exp\left(\mu s + \frac{1}{2}\sigma^2 s^2\right) \quad (2.8)$$

- From this explicit analytical expression, all *moments* can be calculated using Equation 2.4.

- The MGF for the *central* moments is even simpler in this case:

$$\begin{aligned} \mathbb{E}[(x - \mathbb{E}[x])^n] &= \frac{\partial^n}{\partial s^n} \exp(-s\mu) \exp\left(\mu s + \frac{1}{2}\sigma^2 s^2\right) \Big|_{s=0} \\ &= \frac{\partial^n}{\partial s^n} \exp\left(\frac{1}{2}\sigma^2 s^2\right) \Big|_{s=0} \end{aligned}$$

- From this, the first few moments are calculated as:

$$\begin{aligned} \mathbb{E}[x] &= \mu \\ \mathbb{E}[(x - \mathbb{E}[x])^2] &= \sigma^2 \\ \mathbb{E}[(x - \mathbb{E}[x])^4] &= 3\sigma^4 \\ \mathbb{E}[(x - \mathbb{E}[x])^n] &= 0 \text{ for all odd } n > 2 \end{aligned} \quad (2.9)$$

- From these moments the *skewness* and the *curtosis* of the normal distribution follow directly:

$$\begin{aligned}\text{Skewness} &:= \frac{\mathbb{E}[(x - \mathbb{E}[x])^3]}{\mathbb{E}[(x - \mathbb{E}[x])^2]^{3/2}} = 0 \\ \text{Curtosis} &:= \frac{\mathbb{E}[(x - \mathbb{E}[x])^4]}{\mathbb{E}[(x - \mathbb{E}[x])^2]^2} = 3\end{aligned}\tag{2.10}$$

2.0.4 Characteristic Functions

- Similar to MGF, the *characteristic function* (in short CF) of a random variable x is defined as the expectation of e^{isx} :

$$\Phi_x(s) := \mathbb{E}[e^{isx}] = \int_{-\infty}^{\infty} e^{isx} \text{pdf}(x) dx\tag{2.11}$$

- Here i denotes the imaginary number satisfying $i^2 = -1$.
- The CF is just the *Fourier transformation* of the pdf.

- The advantage of the CF is that its inverse, the *inverse Fourier transformation* always exists:

$$\text{pdf}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-isx} \Phi_x(s) ds \quad (2.12)$$

- The validity of Equation 2.12 can be shown quite easily:

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-isx} \Phi_x(s) ds &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-isx} \int_{-\infty}^{\infty} e^{isx'} \text{pdf}(x') dx' ds \\ &= \int_{-\infty}^{\infty} \underbrace{\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-is(x-x')} ds}_{\delta(x-x')} \text{pdf}(x') dx' \\ &= \text{pdf}(x) \end{aligned}$$

– where the *Dirac delta function* was used.

- Thus, if Φ_x is known, then the *distribution* can be computed *directly* (and not only its moments as was the case with the MGF).

- Analogously to the MGF, Equation 2.2 holds for the characteristic function as well, i.e. for independent random variables x and y :

$$\Phi_{x+y}(s) = \Phi_x(s) \Phi_y(s) \quad (2.13)$$

- Likewise, for non-stochastic values a, b and a random variable x

$$\Phi_{ax+b}(s) = e^{ibs} \Phi_x(as) \quad (2.14)$$

- The CF can (usually) be obtained by simply substituting is for s in the corresponding MGF.

– For instance, the CF for the normal distribution is:

$$\Phi_{N(\mu, \sigma^2)}(s) = e^{i\mu s} \Phi_{N(0,1)}(\sigma s) = \exp\left(i\mu s - \frac{1}{2}\sigma^2 s^2\right) \quad (2.15)$$

2.0.5 The χ^2 -Distribution

- A sum of normally distributed random variables is itself normally distributed.

- In the Delta-Gamma method we will also need to take sums of the *squares* of random variables.
- The sum of n *squared, independent, standard normally distributed random variables*, $x_i, (i = 1, \dots, n)$ has a distribution known as the χ^2 -*distribution with n degrees of freedom*

$$x_i \sim N(0, 1), \quad i = 1, \dots, n, \quad x_i \text{ iid} \implies \sum_{i=1}^n x_i^2 =: y \sim \chi^2(n) \quad (2.16)$$

- Motivation for the name “degree of freedom”:
 - A $\chi^2(n)$ -distributed variable can be thought of as being “made up” of n *independent* (standard normal) random variables.
- We only need the case $n = 1$
 - The *square* of a standard normal random variable x is governed by the χ^2 -*distribution with one degree of freedom*.

$$x \sim N(0, 1) \implies x^2 =: y \sim \chi^2(1) \quad (2.17)$$

- The moment generating function of $\chi^2(1)$ is

$$G_{\chi^2(1)}(s) = \mathbb{E}[e^{sx^2}]_{N(0,1)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{sx^2} e^{-x^2/2} dx \quad (2.18)$$

- This integral can be solved analytically:

$$G_{\chi^2(1)}(s) = \frac{1}{\sqrt{1-2s}} \quad (2.19)$$

- Since the random variables in Definition 2.16 are all *independent*, Equation 2.2 directly gives the MGF for a χ^2 -distribution with n degrees of freedom

$$G_{\chi^2(n)}(s) = \frac{1}{(1-2s)^{n/2}} \text{ for } n = 1, 2, \dots \quad (2.20)$$

- From this, the *moments* can be derived by differentiating with respect to s :

$$\mathbb{E}[x^k]_{\chi^2(n)} = \prod_{i=0}^{k-1} (n+2i) \quad (2.21)$$

– For example, the expectation and variance are given by

$$\mathbb{E}[x]_{\chi^2(n)} = n, \quad \text{var}[x]_{\chi^2(n)} = 2n$$

- An explicit expression for the density function of $\chi^2(1)$ is derived in [12]:

$$\text{pdf}_{\chi^2(1)}(x) = \frac{1}{\sqrt{2\pi}} x^{-1/2} e^{-x/2} \text{ with } x \in [0, \infty[\quad (2.22)$$

- It is also shown in [12] that the $\chi^2(n)$ equals the gamma distribution with parameters $\lambda = 1/2$ and $t = n/2$:

$$\text{pdf}_{\chi^2(n)}(x) = \frac{1}{\Gamma(n/2)} \left(\frac{1}{2}\right)^{n/2} x^{\frac{n}{2}-1} e^{-x/2} \text{ with } x \in [0, \infty[, \quad n = 1, 2, \dots \quad (2.23)$$

– The gamma functions appearing here are:

$$\Gamma(n/2) = \begin{cases} (n/2 - 1)! & \text{for even values of } n \\ (n/2 - 1)(n/2 - 2)(n/2 - 3) \cdots (1/2)\sqrt{\pi} & \text{for odd values of } n \end{cases}$$

- The characteristic function of the χ^2 -distribution is obtained by replacing s with is in the MGF:

$$\Phi_{\chi^2(n)}(s) = \frac{1}{(1 - 2is)^{n/2}} \text{ for } n = 1, 2, \dots \quad (2.24)$$

The Non-Central χ^2 -Distribution

- The χ^2 -distribution described above is the distribution of a sum of n squared independent *standard* normal random numbers $x_i, (i = 1, \dots, n)$.
- Now: a slight but often needed generalization: the x_i have expectations $\mu_i \neq 0$.
- The distribution of a sum of n squared random numbers of this type is called the *non-central* χ^2 -distribution with n degrees of freedom and with *non-central parameter* θ , where θ denotes the sum of the squared expectations μ_i :

$$\begin{aligned} x_i &\sim N(\mu_i, 1) \quad , \quad i = 1, \dots, n \quad , \quad x_i \text{ iid} \\ &\implies \\ \sum_{i=1}^n x_i^2 &=: y \sim \chi^2(n, \theta) \quad \text{with} \quad \theta = \sum_{i=1}^n \mu_i^2 \end{aligned} \quad (2.25)$$

- The square of a single random number $x \sim \mathcal{N}(0, \mu)$ has the non-central χ^2 -distribution with one degree of freedom:

$$x \sim \mathcal{N}(0, \mu) \implies x^2 =: y \sim \chi^2(1, \mu^2)$$

- The MGF of the non-central χ^2 -distribution is:

$$\begin{aligned} G_{\chi^2(1, \mu^2)}(s) &= \mathbb{E}[e^{sx^2}]_{\mathcal{N}(\mu, 1)} \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{sx^2} e^{-(x-\mu)^2/2} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp \left\{ sx^2 - \frac{1}{2} (x - \mu)^2 \right\} dx \end{aligned}$$

- By completing the square in the exp-function this integral can be solved analytically to yield:

$$G_{\chi^2(1, \mu^2)}(s) = \frac{1}{\sqrt{1-2s}} \exp \left\{ \frac{s\mu^2}{1-2s} \right\} \quad (2.26)$$

- Equation 2.2 now immediately yields the MGF for a non-central χ^2 -distribution with n degrees of freedom:

$$G_{\chi^2(n,\theta)}(s) = \frac{1}{(1-2s)^{n/2}} \exp\left\{\frac{s\theta}{1-2s}\right\} \quad \text{with} \quad \theta = \sum_{j=1}^n \mu_j^2 \quad (2.27)$$

- The characteristic function, Equation 2.11, of the non-central χ^2 -distribution follows again by replacing s by is in Equation 2.27.

$$\Phi_{\chi^2(n,\theta)}(s) = \frac{1}{(1-2is)^{n/2}} \exp\left\{\frac{is\theta}{1-2is}\right\} \quad \text{mit} \quad \theta = \sum_{j=1}^n \mu_j^2 \quad (2.28)$$

3 The Delta-Gamma Method

- Taylor expansion 1.31 of the portfolio value up to second order with proxy 1.30 for the risk factors:

$$\delta V(\mathbf{S}(t)) = \tilde{\Delta}^T \delta \mathbf{Z} + \frac{1}{2} \delta \mathbf{Z}^T \tilde{\Gamma} \delta \mathbf{Z} \quad (3.1)$$

– where

$$\delta \mathbf{Z} \sim N(\mathbf{0}, \delta \Sigma) \iff \text{cov}[\delta Z_i, \delta Z_j] = \sigma_i \rho_{ij} \sigma_j \delta t, \quad \mathbf{E}[\delta Z_i] = 0$$

- This is *not* the sum over the contribution of each risk factor:

$$\sum_i^n \tilde{\Delta}_i \delta Z_i + \frac{1}{2} \sum_{i,j}^n \delta Z_i \tilde{\Gamma}_{ij} \delta Z_j \neq \sum_i^n (\text{Contribution of the } i\text{-th Risk Factor})$$

- The contributions of the δZ_j are *coupled* by the Gamma matrix $\tilde{\Gamma}$.
- The δZ_j themselves are *correlated* through the covariance matrix $\delta\Sigma$.
- The VaR can not be obtained from the distribution of the individual *risk factors*. The (unknown) distribution of the *portfolio value* δV itself must be determined.
- Three steps
 - Cholesky decomposition of the covariance matrix $\delta\Sigma$ to transform the δZ_j into *independent* random variables.
 - Diagonalisation of the Gamma matrix $\tilde{\Gamma}$ to *decouple* the contributions of the δZ_j .
 - Determination of the *distribution* of δV .

3.1 Decoupling of the Risk Factors

- Via Cholesky decomposition a matrix \mathbf{A} can be constructed satisfying the properties 1.25 and 1.28, i.e.

$$\mathbf{A}\mathbf{A}^T = \delta\Sigma \implies \mathbf{A}^{-1}\delta\mathbf{Z} \sim N(\mathbf{0}, \mathbf{1})$$

- Introduce identity matrices into Equation 3.1 and replace them with $\mathbf{A}\mathbf{A}^{-1}$ or $(\mathbf{A}^T)^{-1}\mathbf{A}^T$:

$$\begin{aligned} \delta V(\mathbf{S}(t)) &= \tilde{\Delta}^T \mathbf{1} \delta\mathbf{Z} + \frac{1}{2} \delta\mathbf{Z}^T \mathbf{1} \tilde{\Gamma} \mathbf{1} \delta\mathbf{Z} \\ &= \tilde{\Delta}^T \mathbf{A}\mathbf{A}^{-1} \delta\mathbf{Z} + \frac{1}{2} \delta\mathbf{Z}^T (\mathbf{A}^T)^{-1} \mathbf{A}^T \tilde{\Gamma} \mathbf{A}\mathbf{A}^{-1} \delta\mathbf{Z} \\ &= \tilde{\Delta}^T \mathbf{A}\mathbf{A}^{-1} \delta\mathbf{Z} + \frac{1}{2} \delta\mathbf{Z}^T (\mathbf{A}^{-1})^T \mathbf{A}^T \tilde{\Gamma} \mathbf{A}\mathbf{A}^{-1} \delta\mathbf{Z} \\ &= \tilde{\Delta}^T \mathbf{A} (\mathbf{A}^{-1} \delta\mathbf{Z}) + \frac{1}{2} (\mathbf{A}^{-1} \delta\mathbf{Z})^T \mathbf{A}^T \tilde{\Gamma} \mathbf{A} (\mathbf{A}^{-1} \delta\mathbf{Z}) \end{aligned}$$

- $\delta\mathbf{Z}$ only appears in combination with \mathbf{A}^{-1} .

- $\mathbf{A}^{-1}\delta\mathbf{Z}$ are *iid* random variables! Thus, the first goal has been accomplished:

$$\begin{aligned}\delta V(\mathbf{S}(t)) &= \tilde{\Delta}^T \mathbf{A} \delta \mathbf{Y} + \frac{1}{2} \delta \mathbf{Y}^T \mathbf{M} \delta \mathbf{Y} \\ \text{with } \delta \mathbf{Y} &:= \mathbf{A}^{-1} \delta \mathbf{Z} \sim N(\mathbf{0}, \mathbf{1}), \text{ iid} \\ \text{and } \mathbf{M} &:= \mathbf{A}^T \tilde{\Gamma} \mathbf{A}\end{aligned}\tag{3.2}$$

- Because $\tilde{\Gamma}$ is by definition a symmetric matrix, i.e. $\tilde{\Gamma}_{ij} = \tilde{\Gamma}_{ji}$, the newly defined matrix \mathbf{M} is symmetric as well.
- The δY_i are *independent, identically distributed (iid) standard normal* random variables.

3.2 Diagonalization of the Gamma Matrix

- Diagonalize the transformed Gamma matrix \mathbf{M} .
 - Standard procedure in *linear algebra* (see [25], for example).

– We sketch the procedure here since it entails essential elements of the practical VaR computations in the Delta-Gamma method.

- The *eigenvectors* \mathbf{e}^i of a matrix \mathbf{M} are the vectors which are mapped by \mathbf{M} to the same vector multiplied by a number (called *scalar* in algebra):

$$\begin{aligned} \mathbf{M}\mathbf{e}^i &= \lambda_i\mathbf{e}^i \Leftrightarrow \\ (\mathbf{M} - \lambda_i\mathbf{1})\mathbf{e}^i &= \mathbf{0} \end{aligned} \tag{3.3}$$

- These scalars, λ_i , are called *eigenvalues* of the matrix.
- This has a non-trivial solution ($\mathbf{e}^i \neq \mathbf{0}$) iff the matrix $(\mathbf{M} - \lambda_i\mathbf{1})$ is *singular*, i.e. the *determinant* must be zero:

$$\det(\mathbf{M} - \lambda_i\mathbf{1}) = 0 \tag{3.4}$$

- The solutions of this Equation 3.4 are the *eigenvalues* λ_i .
- Once they have been determined, they can be substituted into Equation 3.3 to determine the *eigenvectors* \mathbf{e}^i .

- The eigenvectors are yet only defined up to a multiplicative scalar since if \mathbf{e}^i solves Equation 3.3 then $c\mathbf{e}^i$ does as well. Demand that the eigenvectors have norm 1:

$$(\mathbf{e}^i)^T \mathbf{e}^i = 1 \quad (3.5)$$

- A *symmetric* $n \times n$ matrix has n linearly independent *orthogonal* eigenvectors.
- Together with the normalization the eigenvectors are *orthonormal*:

$$(\mathbf{e}^i)^T \mathbf{e}^j = \sum_k e_k^i e_k^j = \delta_{ij} \quad (3.6)$$

- Notation: e_k^i is the k th component of the i th eigenvector.

- Now construct a matrix \mathbf{O} whose columns are the *eigenvectors* of \mathbf{M} :

$$\mathbf{O} = \left(\begin{array}{cccc} \mathbf{e}^1 & \mathbf{e}^2 & \dots & \mathbf{e}^n \end{array} \right) = \left(\begin{array}{cccc} e_1^1 & e_1^2 & \dots & e_1^n \\ e_2^1 & e_2^2 & & \vdots \\ \vdots & & \ddots & \vdots \\ e_n^1 & \dots & \dots & e_n^n \end{array} \right) \Rightarrow O_{ij} = e_i^j \quad (3.7)$$

– As can be immediately verified

$$\mathbf{O}^T \mathbf{O} = \mathbf{1} \quad (3.8)$$

– from which it follows that

$$\mathbf{O}^T = \mathbf{O}^{-1} \Rightarrow \mathbf{O} \mathbf{O}^T = \mathbf{1} \quad (3.9)$$

– Equation 3.8 characterizes *orthonormal transformations*. (graphically: rotations)

- The *eigenvalues* can be used to construct a diagonal matrix:

$$\lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & \lambda_n \end{pmatrix} \quad (3.10)$$

- Now Equation 3.3 can be written as

$$\mathbf{M} \mathbf{O} = \mathbf{O} \lambda \quad (3.11)$$

- Multiplying both sides of this equation *on the left* with the matrix \mathbf{O}^T yields (with Equation 3.8)

$$\mathbf{O}^T \mathbf{M} \mathbf{O} = \mathbf{O}^T \mathbf{O} \lambda = \lambda \quad (3.12)$$

– This is the desired diagonalization of \mathbf{M} , since λ is a diagonal matrix:

- Now introduce the diagonalized matrix $\mathbf{O}^T \mathbf{M} \mathbf{O}$ into Equation 3.2 by inserting identity matrices and subsequently replacing them by $\mathbf{O} \mathbf{O}^T$ (see Equation 3.9):

$$\begin{aligned} \delta V(\mathbf{S}(t)) &= \tilde{\Delta}^T \mathbf{A} \mathbf{1} \delta \mathbf{Y} + \frac{1}{2} \delta \mathbf{Y}^T \mathbf{1} \mathbf{M} \mathbf{1} \delta \mathbf{Y} \\ &= \tilde{\Delta}^T \mathbf{A} \mathbf{O} \mathbf{O}^T \delta \mathbf{Y} + \frac{1}{2} \delta \mathbf{Y}^T \underbrace{\mathbf{O} \mathbf{O}^T \mathbf{M} \mathbf{O} \mathbf{O}^T}_{\lambda} \delta \mathbf{Y} \\ &= \tilde{\Delta}^T \mathbf{A} \mathbf{O} (\mathbf{O}^T \delta \mathbf{Y}) + \frac{1}{2} (\mathbf{O}^T \delta \mathbf{Y})^T \lambda (\mathbf{O}^T \delta \mathbf{Y}) \end{aligned}$$

- The parentheses emphasize that $\delta \mathbf{Y}$ appears only in combination with \mathbf{O}^T . In

consequence, we can write

$$\begin{aligned}\delta V(\mathbf{S}(t)) &= \tilde{\Delta}^T \mathbf{A} \mathbf{O} \delta \mathbf{X} + \frac{1}{2} \delta \mathbf{X}^T \lambda \delta \mathbf{X} \\ \text{with } \delta \mathbf{X} &:= \mathbf{O}^T \delta \mathbf{Y} = \mathbf{O}^T \mathbf{A}^{-1} \delta \mathbf{Z} \\ \text{and } \lambda &:= \mathbf{O}^T \mathbf{M} \mathbf{O} = \mathbf{O}^T \mathbf{A}^T \tilde{\Gamma} \mathbf{A} \mathbf{O}\end{aligned}\tag{3.13}$$

- The δY_i were *iid*, standard normally distributed.
- It is easy to show that the covariances remain invariant under the transformation \mathbf{O}^T . Since matrix multiplication is a *linear* transformation the new random variables δX_i remain normally distributed, i.e.:

$$\delta \mathbf{X} \sim \mathbf{N}(\mathbf{0}, \mathbf{1}), \text{ iid}$$

- With the “transformed sensitivity vector“ $\mathbf{L} := \mathbf{O}^T \mathbf{A}^T \tilde{\Delta}$ the portfolio-change has the simple form

$$\delta V(\mathbf{S}(t)) = \mathbf{L}^T \delta \mathbf{X} + \frac{1}{2} \delta \mathbf{X}^T \lambda \delta \mathbf{X} \text{ with } \delta \mathbf{X} \sim \mathbf{N}(\mathbf{0}, \mathbf{1})\tag{3.14}$$

– expressed component-wise

$$\delta V(\mathbf{S}(t)) = \sum_i \left[L_i \delta X_i + \frac{1}{2} \lambda_i \delta X_i^2 \right] = \sum_i \delta V_i$$

$$\text{with } \delta V_i = L_i \delta X_i + \frac{1}{2} \lambda_i \delta X_i^2, \quad i = 1, \dots, n \quad (3.15)$$

– Now the value change is indeed a *decoupled* sum of the individual contributions of *iid* random variables!

- At this stage, we collect all transformations involved :

$$\delta \mathbf{X} := \mathbf{O}^T \mathbf{A}^{-1} \delta \mathbf{Z}, \quad \lambda := \mathbf{O}^T \mathbf{A}^T \tilde{\Gamma} \mathbf{A} \mathbf{O}, \quad \mathbf{L} := \mathbf{O}^T \mathbf{A}^T \tilde{\Delta}$$

– Because $\mathbf{O}^T \mathbf{A}^{-1} = \mathbf{O}^{-1} \mathbf{A}^{-1} = (\mathbf{A} \mathbf{O})^{-1}$ (see Equ. 3.9) all these transformations can be represented by a single matrix

$$\mathbf{D} := \mathbf{A} \mathbf{O} \quad (3.16)$$

- With this matrix, the transformations become simply

$$\delta \mathbf{X} := \mathbf{D}^{-1} \delta \mathbf{Z}, \quad \lambda := \mathbf{D}^T \tilde{\Gamma} \mathbf{D}, \quad \mathbf{L} := \mathbf{D}^T \tilde{\Delta} \quad (3.17)$$

- The matrix \mathbf{D}

- *directly* diagonalizes the Gamma matrix $\tilde{\Gamma}$ (as opposed to \mathbf{O} , which diagonalizes the matrix \mathbf{M})
- and is *also* a “square root” of the covariance matrix:

$$\mathbf{D}\mathbf{D}^T = \mathbf{A}\mathbf{O}(\mathbf{A}\mathbf{O})^T = \mathbf{A}\mathbf{O}\mathbf{O}^T\mathbf{A}^T = \mathbf{A}\mathbf{1}\mathbf{A}^T = \mathbf{A}\mathbf{A}^T = \delta\Sigma \quad (3.18)$$

- The matrix \mathbf{D} satisfies *both* tasks, namely the decoupling of the Gamma matrix *and* the transformation of the correlated random variables into uncorrelated ones.
- With the matrix \mathbf{D} the equivalence of Equations 3.14 and original 3.1 is immediate:

$$\begin{aligned} \delta V(\mathbf{S}(t)) &= \left(\mathbf{D}^T \tilde{\Delta}\right)^T \mathbf{D}^{-1} \delta \mathbf{Z} + \frac{1}{2} (\mathbf{D}^{-1} \delta \mathbf{Z})^T \mathbf{D}^T \tilde{\Gamma} \mathbf{D} \mathbf{D}^{-1} \delta \mathbf{Z} \\ &= \tilde{\Delta}^T \underbrace{\mathbf{D} \mathbf{D}^{-1}}_{\mathbf{1}} \delta \mathbf{Z} + \frac{1}{2} \delta \mathbf{Z}^T \underbrace{(\mathbf{D}^T)^{-1} \mathbf{D}^T}_{\mathbf{1}} \tilde{\Gamma} \underbrace{\mathbf{D} \mathbf{D}^{-1}}_{\mathbf{1}} \delta \mathbf{Z} \end{aligned}$$

3.3 The Distribution of the Portfolio-Value Changes

- δV is *not* simply the sum of normally distributed random variables. It also includes the *square* of normally distributed random variables.
 - The square of a standard normally distributed random variable is χ^2 -distributed with one degree of freedom.
 - This we can write 3.15 as

$$\delta V(\mathbf{S}(t)) = \sum_{i=1}^n L_i \delta X_i + \frac{1}{2} \sum_{i=1}^n \lambda_i \tilde{X}_i \quad \text{where } \delta X_i \sim N(0, 1), \quad \tilde{X}_i \sim \chi^2(1)$$

- The \tilde{X}_i are not independent from the δX_i since obviously:

$$\tilde{X}_i = (\delta X_i)^2 \quad \forall i$$

- We now re-write δV so that *every* term is statistically independent of *every other* term.
- δX_i is independent of every other term in δV if the corresponding eigenvalue λ_i is zero.

- We emphasize this by the index set J which contains only the indices of non-zero eigenvalues:

$$J = \{1, \dots, n \mid \lambda_j \neq 0\} \quad (3.19)$$

- With this index set we can write

$$\begin{aligned} \delta V(\mathbf{S}(t)) &= \sum_{i \notin J} L_i \delta X_i + \sum_{j \in J} L_j \delta X_j + \frac{1}{2} \sum_{j \in J} \lambda_j \delta X_j^2 \\ &= \sum_{i \notin J} L_i \delta X_i + \sum_{j \in J} \left[L_j \delta X_j + \frac{1}{2} \lambda_j \delta X_j^2 \right] \end{aligned} \quad (3.20)$$

- The first sum is a sum of normally distributed random variables.
 - It is again a normally distributed random variable which we denote by u_0 .
 - The expectation of this random variable is

$$\mathbb{E}[u_0] = \mathbb{E} \left[\sum_{i \notin J} L_i \delta X_i \right] = \sum_{i=1}^n L_i \underbrace{\mathbb{E}[\delta X_i]}_0 = 0$$

– Its variance is

$$\begin{aligned} \text{var}[u_0] &= \text{var} \left[\sum_{i \notin J} L_i \delta X_i \right] = \sum_{i, j \notin J} \text{cov}[L_i \delta X_i, L_j \delta X_j] \\ &= \sum_{i, j \notin J} L_i L_j \underbrace{\text{cov}[\delta X_i, \delta X_j]}_{\delta_{ij}} = \sum_{i \notin J} L_i^2 \end{aligned}$$

– Thus

$$u_0 := \sum_{i \notin J} L_i \delta X_i \sim N(0, \sum_{i \notin J} L_i^2)$$

• Consider now the sums over $j \in J$ in Equation 3.20.

– To combine the *dependent* random numbers δX_j and δX_j^2 into one single random number, we complete the square:

$$\frac{1}{2} \lambda_j \delta X_j^2 + L_j \delta X_j = \frac{1}{2} \lambda_j \left(\delta X_j^2 + 2 \frac{L_j}{\lambda_j} \delta X_j \right) = \frac{1}{2} \lambda_j \left(\delta X_j + \frac{L_j}{\lambda_j} \right)^2 - \frac{L_j^2}{2\lambda_j}$$

- Since δX_j is a standard normal random variable we have

$$\delta X_j \sim N(0, 1) \quad \implies \quad \delta X_j + \frac{L_j}{\lambda_j} \sim N\left(\frac{L_j}{\lambda_j}, 1\right)$$

- Therefore, $u_j := (\delta X_j + L_j/\lambda_j)^2$ has a *non-central* χ^2 -distribution (see 2.25) with one degree of freedom and non-central parameter L_j^2/λ_j^2 :

$$\left(\delta X_j + \frac{L_j}{\lambda_j}\right)^2 =: u_j \sim \chi^2\left(1, \frac{L_j^2}{\lambda_j^2}\right) \quad \forall j \in J$$

- In summary, δV has now become a sum of
 - non-central χ^2 -distributed random variables u_j
 - plus a normally distributed random variable u_0
 - plus a constant,

– and all the random variables are *independent of each other*:

$$\delta V(\mathbf{S}(t)) = u_0 + \frac{1}{2} \sum_{j \in J} \lambda_j u_j - \underbrace{\frac{1}{2} \sum_{j \in J} L_j^2 / \lambda_j}_{\text{constant}} \quad (3.21)$$

$$\text{with } u_0 \sim N(0, \sum_{i \notin J} L_i^2) \quad , \quad u_j \sim \chi^2(1, (L_j / \lambda_j)^2) \quad , \quad j \in J$$

- The problem now consists in determining the distribution of the sum of *differently* distributed independent random variables.
 - Remember: the Value at Risk at a specified confidence c is computed from the percentiles of the distribution of δV .

3.3.1 Moments of the Portfolio-value Distribution

- We first calculate moments of the distribution of δV .
- For this we differentiate the *Moment Generating Function (MGF)* defined in 2.1 (see 2.4).

– The MGF of the $N(0, \sum_{i \notin J} L_i^2)$ -distributed random variable u_0 is

$$G_{u_0}(s) = \exp\left(\frac{1}{2} s^2 \sum_{i \notin J} L_i^2\right) \quad (3.22)$$

– The MGF of a $\chi^2(1, (L_j/\lambda_j)^2)$ -distributed random variables u_j are

$$G_{u_j}(s) = \frac{1}{\sqrt{1-2s}} \exp\left\{\frac{s}{1-2s} \frac{L_j^2}{\lambda_j^2}\right\}, \quad j \in J \quad (3.23)$$

* well-defined for $s < 1/2$, which is sufficient for us since we need it for s close to zero.

- The MGF of δV in 3.21 follows directly from the properties 2.2 and 2.3:

$$G_{\delta V}(s) = \exp\left\{-s \sum_{j \in J} \frac{L_j^2}{2\lambda_j}\right\} G_{u_0}(s) \prod_{j \in J} G_{u_j}\left(\frac{1}{2}\lambda_j s\right)$$

- Inserting the above expressions for G_{u_0} and G_{u_j} yields (after simple rearranging):

$$G_{\delta V}(s) = \exp\left(\frac{1}{2}s^2 \sum_{i \notin J} L_i^2\right) \prod_{j \in J} \frac{1}{\sqrt{1 - \lambda_j s}} \exp\left\{\frac{1}{2}L_j^2 \frac{s^2}{1 - \lambda_j s}\right\} \quad (3.24)$$

- Trick: Since $\lambda_i = 0$ for all $i \notin J$, we can re-write the first exp-function:

$$\exp\left(\frac{1}{2}s^2 \sum_{i \notin J} L_i^2\right) = \prod_{i \notin J} \exp\left(\frac{1}{2}s^2 L_i^2\right) = \prod_{i \notin J} \frac{1}{\sqrt{1 - \lambda_i s}} \exp\left\{\frac{1}{2}L_i^2 \frac{s^2}{1 - \lambda_i s}\right\}$$

- Using this form in Equation 3.24 allows us to write δV very compactly as a product over *all* indexes $j = 1, \dots, n$

$$G_{\delta V}(s) = \prod_{j=1}^n \frac{1}{\sqrt{1 - \lambda_j s}} \exp\left\{\frac{1}{2}L_j^2 \frac{s^2}{1 - \lambda_j s}\right\} \quad (3.25)$$

- This is well-defined for all $s < \min_{j \in J} \left(\frac{1}{2|\lambda_i|}\right)$, which is sufficient for our needs since we are only interested in values of s close to zero.

- Now, using 2.4, arbitrary moments of δV can be computed by differentiating 3.25.
- We first abbreviate the argument of the exp-function in 3.25 as

$$a_j := \frac{1}{2} L_j^2 \frac{s^2}{1 - \lambda_j s}$$

- For the first moment we need the first derivative.
 - Application of the *product rule* yields

$$\begin{aligned} \mathbb{E}[\delta V] &= \left. \frac{\partial G_{\delta V}(s)}{\partial s} \right|_{s=0} \\ &= \left. \frac{\partial}{\partial s} \prod_{j=1}^n \frac{e^{a_j}}{\sqrt{1 - \lambda_j s}} \right|_{s=0} \\ &= \sum_{j=1}^n \left(\frac{\partial}{\partial s} \frac{e^{a_j}}{\sqrt{1 - \lambda_j s}} \right) \prod_{k=1, k \neq j}^n \frac{e^{a_k}}{\sqrt{1 - \lambda_k s}} \Bigg|_{s=0} \end{aligned}$$

– The derivative we need to calculate is

$$\begin{aligned} \frac{\partial}{\partial s} \frac{e^{a_j}}{\sqrt{1 - \lambda_j s}} &= e^{a_j} \frac{\partial}{\partial s} \frac{1}{\sqrt{1 - \lambda_j s}} + \frac{1}{\sqrt{1 - \lambda_j s}} \frac{\partial}{\partial s} e^{a_j} \\ &= \frac{\frac{1}{2} \lambda_j e^{a_j}}{(1 - \lambda_j s)^{3/2}} + \frac{\frac{1}{2} L_j^2 e^{a_j}}{\sqrt{1 - \lambda_j s}} \left(\frac{2s}{1 - \lambda_j s} + \frac{\lambda_j s^2}{(1 - \lambda_j s)^2} \right) \\ &= \frac{1}{2} \frac{e^{a_j}}{(1 - \lambda_j s)^{3/2}} \left(\lambda_j + 2L_j^2 s + \lambda_j L_j^2 \frac{s^2}{1 - \lambda_j s} \right) \end{aligned}$$

– For $s = 0$ almost all terms vanish and we are left with $\lambda_j/2$. Thus $E[\delta V]$ is simply

$$E[\delta V] = \sum_{j=1}^n \frac{1}{2} \lambda_j \prod_{k \in J, k \neq j} \frac{e^{a_k}}{\sqrt{1 - \lambda_k s}} \Bigg|_{s=0} = \frac{1}{2} \sum_{j=1}^n \lambda_j$$

- This is by definition one half times the *trace* of the eigenvalue matrix λ . With

Equations 3.17 and 3.18 this becomes:

$$\mathbb{E}[\delta V] = \frac{1}{2} \text{tr}(\lambda) = \frac{1}{2} \text{tr}(\mathbf{D}^T \tilde{\Gamma} \mathbf{D}) = \frac{1}{2} \text{tr}(\tilde{\Gamma} \mathbf{D} \mathbf{D}^T) = \frac{1}{2} \text{tr}(\tilde{\Gamma} \delta \Sigma) \quad (3.26)$$

- Note that the drifts of all risk factors have been neglected.
 - But still the expectation of the *portfolio* changes (the drift of the portfolio) is *not* zero because *non-linear* effects were taken into consideration.
 - The Gamma matrix gives rise to the drift of δV .
- To find out more about the distribution of δV , we proceed by computing its variance.
 - The variance is the second *central* moment which can be calculated via Equation 2.6:

$$\begin{aligned} \text{var}[\delta V] &= \mathbb{E}[(\delta V - \mathbb{E}[\delta V])^2] \\ &= \left. \frac{\partial^2}{\partial s^2} \exp(-s\mathbb{E}[\delta V]) G_{\delta V}(s) \right|_{s=0} \end{aligned}$$

– Doing the (tedious!) differentiations finally yields

$$\text{var}[\delta V] = \sum_{j=1}^n \left(L_j^2 + \frac{1}{2} \lambda_j^2 \right)$$

- The sum $\sum L_j^2$ is just the square of the transformed sensitivity *vector* and $\sum \lambda_j^2$ is the trace of the square of the the *matrix* of eigenvalues, i.e.

$$\text{var}[\delta V] = \mathbf{L}^T \mathbf{L} + \frac{1}{2} \text{tr}(\lambda^2)$$

- With the transformations 3.17 and the property 3.18, the variance of the portfolio's value becomes

$$\begin{aligned} \text{var}[\delta V] &= \tilde{\Delta}^T \mathbf{D} \mathbf{D}^T \tilde{\Delta} + \frac{1}{2} \text{tr} \left(\mathbf{D}^T \tilde{\Gamma} \mathbf{D} \mathbf{D}^T \tilde{\Gamma} \mathbf{D} \right) \\ &= \tilde{\Delta}^T \delta \Sigma \tilde{\Delta} + \frac{1}{2} \text{tr} \left(\tilde{\Gamma} \delta \Sigma \tilde{\Gamma} \delta \Sigma \right) \end{aligned} \quad (3.27)$$

– The first term resulting from the linear portfolio sensitivities $\tilde{\Delta}$ is identical to the portfolio variance in the Delta-Normal method.

- The non-linear sensitivities $\tilde{\Gamma}$ effect a correction of the linear portfolio variance.

- Analogously, one can continue to calculate further central moments of δV :

$$\begin{aligned}
\mu &= \mathbb{E}[\delta V] = \frac{1}{2} \text{tr} \left(\tilde{\Gamma} \delta \Sigma \right) \\
\mu_2 &= \mathbb{E}[(\delta V - \mathbb{E}[\delta V])^2] = \tilde{\Delta}^T \delta \Sigma \tilde{\Delta} + \frac{1}{2} \text{tr} \left((\tilde{\Gamma} \delta \Sigma)^2 \right) \\
\mu_3 &= \mathbb{E}[(\delta V - \mathbb{E}[\delta V])^3] = 3 \tilde{\Delta}^T \delta \Sigma \tilde{\Gamma} \delta \Sigma \tilde{\Delta} + \text{tr} \left((\tilde{\Gamma} \delta \Sigma)^3 \right) \\
\mu_4 &= \mathbb{E}[(\delta V - \mathbb{E}[\delta V])^4] = 12 \tilde{\Delta}^T \delta \Sigma (\tilde{\Gamma} \delta \Sigma)^2 \tilde{\Delta} + 3 \text{tr} \left((\tilde{\Gamma} \delta \Sigma)^4 \right) + 3 \mu_2^2
\end{aligned} \tag{3.28}$$

- In this way, a great deal of additional information about the distribution of δV can be generated.

– For instance *skewness* and *kurtosis* of the distribution of δV are¹

$$\text{Skewness} \equiv \frac{\mu_3}{\mu_2^{3/2}} = \frac{3\tilde{\Delta}^T \delta \Sigma \tilde{\Gamma} \delta \Sigma \tilde{\Delta} + \text{tr}(\tilde{\Gamma} \delta \Sigma)^3}{\left(\tilde{\Delta}^T \delta \Sigma \tilde{\Delta} + \frac{1}{2} \text{tr}(\tilde{\Gamma} \delta \Sigma)^2\right)^{3/2}}$$

$$\text{Curtosis} \equiv \frac{\mu_4}{\mu_2^2} = \frac{12\tilde{\Delta}^T \delta \Sigma (\tilde{\Gamma} \delta \Sigma)^2 \tilde{\Delta} + 3 \text{tr}(\tilde{\Gamma} \delta \Sigma)^4 + 3\mu_2^2}{\left(\tilde{\Delta}^T \delta \Sigma \tilde{\Delta} + \frac{1}{2} \text{tr}(\tilde{\Gamma} \delta \Sigma)^2\right)^2}$$

- A *percentile*, however, is needed for the computation of the value at risk as given in Equation 1.3.

3.3.2 Johnson Transformation

- Computation of a percentile necessitates knowledge of the distribution *directly* and not of its moments.
- One way to proceed is to assume a particular functional form of the distribution.

¹Recall that a normal distribution has skewness 0 and kurtosis 3, see Equation 2.10.

- Then establish a relation between the parameters of this functional form and the moments of the random variable via *moment matching*.
- Since the moments can be explicitly computed (using the MGF), the parameters of the assumed distribution can thus be determined.
- For example, one could *assume* that δV normally or lognormally distributed.
 - Then Equations 3.26 and 3.27 would determine the parameter values of the assumed distribution.
- Additional functional forms for approximating the distribution of δV were suggested by Johnson [32].
 - These *Johnson transformations* have four parameters which can be determined from the first four moments in Equation 3.28.

3.3.3 Cornish-Fisher Expansion

- Approximate the *percentiles* of a distribution from its *moments* and the (well known) percentiles $Q^{N(0,1)}$ of the standard normal distribution.

- First transform δV into a centered and normalized random variable $\widetilde{\delta V}$:

$$\widetilde{\delta V} := \frac{\delta V - \mathbb{E}[\delta V]}{\sqrt{\text{var}[\delta V]}} = \frac{\delta V - \mu}{\sqrt{\mu_2}}$$

- Cornish-Fisher expansion (see [10], [51]) for the percentiles of the distribution of $\widetilde{\delta V}$ up to the order involving the first four moments in Equation 3.28:

$$\begin{aligned} Q^{\text{cpf}_{\widetilde{\delta V}}} &\approx Q^{\text{N}(0,1)} + \frac{1}{6} \left[(Q^{\text{N}(0,1)})^2 - 1 \right] \frac{\mu_3}{\mu_2^{3/2}} \\ &+ \frac{1}{24} \left[(Q^{\text{N}(0,1)})^3 - 3Q^{\text{N}(0,1)} \right] \left(\frac{\mu_4}{\mu_2^2} - 3 \right) \\ &+ \frac{1}{36} \left[2(Q^{\text{N}(0,1)})^3 - 5Q^{\text{N}(0,1)} \right] \left(\frac{\mu_3}{\mu_2^{3/2}} \right)^2 \end{aligned} \quad (3.29)$$

- The probability that $\widetilde{\delta V}$ is less than a number a is, naturally, the same as the probability that δV is less than $\mu + \sqrt{\mu_2} a$:

$$Q^{\text{cpf}_{\delta V}} \approx \mu + \sqrt{\mu_2} Q^{\text{cpf}_{\widetilde{\delta V}}}$$

- From Equation 1.3, the Value at Risk is thus

$$\text{VaR}(c) = -Q_{1-c}^{\text{cpf}_{\delta v}} \approx -\mu - \sqrt{\mu_2} Q_{1-c}^{\text{cpf}_{\delta \bar{v}}}$$

- for $Q^{\text{cpf}_{\delta \bar{v}}}$ approximation 3.29 is used with the $(1 - c)$ percentile of the standard normal distribution.

3.3.4 Fourier-Transformation of the Portfolio-Value Distribution

- Up to now the distribution itself has not been calculated directly.
- For this, characteristic functions (CFs) are necessary.
- As in 2.11, the *characteristic function*, Φ_x , of a random variable x with density function $\text{pdf}(x)$ is defined as the *Fourier transformation* of the density function:

$$\Phi_x(s) \equiv \mathbb{E}[e^{isx}] = \int_{-\infty}^{\infty} e^{isx} \text{pdf}(x) dx \quad (3.30)$$

- The CF of the $N(0, \sum_{i \notin J} L_i^2)$ -distributed random variable u_0 is:

$$\Phi_{u_0}(s) = \exp\left(-\frac{1}{2}s^2 \sum_{i \notin J} L_i^2\right) \quad (3.31)$$

- The CF of the $\chi^2(1, (L_j/\lambda_j)^2)$ -distributed random variable u_j is:

$$\Phi_{u_j}(s) = \frac{1}{\sqrt{1-2is}} \exp\left\{\frac{is}{1-2is} \frac{L_j^2}{\lambda_j^2}\right\}, \quad j \in J, \quad i \equiv \sqrt{-1} \quad (3.32)$$

- The CF of the distribution of δV is of course the same as Equation 3.25 for the MGF with the obvious substitution $s \rightarrow is$:

$$\Phi_{\delta V}(s) = \prod_{j=1}^n \frac{1}{\sqrt{1-i\lambda_j s}} \exp\left\{-\frac{1}{2}L_j^2 \frac{s^2}{1-i\lambda_j s}\right\} \quad \text{with } i \equiv \sqrt{-1} \quad (3.33)$$

- There exists an inverse transformation for the characteristic function, namely the *inverse Fourier Transformation*, see 2.12.

- Thus the density function $\text{pdf}(\delta V)$ can be computed explicitly (at least numerically):

$$\begin{aligned} \text{pdf}_{\delta V}(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-isx} \Phi_{\delta V}(s) ds \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \prod_{j=1}^n \frac{1}{\sqrt{1-i\lambda_j s}} \exp \left\{ -\frac{1}{2} L_j^2 \frac{s^2}{1-i\lambda_j s} - isx \right\} ds \end{aligned} \quad (3.34)$$

- The *cumulative* distribution function of δV is obtained through the (numerical) integration of this probability density

$$\begin{aligned} \text{cpf}_{\delta V}(c) &\equiv \int_{-\infty}^c \text{pdf}_{\delta V}(x) dx \\ &= \frac{1}{2\pi} \int_{-\infty}^c \int_{-\infty}^{\infty} \prod_{j=1}^n \frac{\exp \left\{ -\frac{1}{2} L_j^2 \frac{s^2}{1-i\lambda_j s} - isx \right\}}{\sqrt{1-i\lambda_j s}} ds dx \end{aligned}$$

- The recommended method for numerically performing the Fourier transformation and inverse Fourier transformation is the *fast Fourier transformation* (FFT).

- The FFT reduces the number of multiplications from order $O(N^2)$ to $O(N \ln(N))$.
- See, for example [4] oder [44].

3.3.5 Monte Carlo Simulations of the Portfolio-value Distribution

- All methods above offer sufficient possibilities for error.
 - Calculating the cumulative distribution of δV with characteristic functions involves complicated numerical procedures.
 - Using moment generating functions, one needs additional assumptions and approximations to establish a relation between the moments and the distribution or the percentiles.
 - The Delta-Gamma method itself is only the second order Taylor approximation of the portfolio's value.
 - Significant difficulties and assumptions and approximations are often involved in calculating the Gamma and Covariance matrices.

- Hence a simple Monte Carlo simulation may not even be less accurate if a sufficient number of simulations are run.
 - Draw n standard normally distributed iid random numbers and compute the simulated change in the portfolio's value immediately from Equation 3.15:

$$\delta V = \sum_{i=1}^n \left[L_i \delta X_i + \frac{1}{2} \lambda_i \delta X_i^2 \right]$$

- Repeat this procedure N times (several thousand times) to obtain N simulated changes δV .
- The percentiles of the distribution can be approximated by simply sorting the simulated values of δV in increasing order.
- In this method a (lengthy) full valuation of the portfolio is replaced by the 2nd order proxy for the simulated value change generated *directly* with Equation 3.15.
- However, before the simulation can be performed,
 - * the eigenvalues of the transformed Gamma matrix 3.4

- * and the transformed sensitivities L_i first need to be determined.
- * Because of Equations 3.17 and 3.16,
 - the Cholesky decomposition of the Covariance matrix as well as
 - the eigenvectors of the Gamma matrix must be computed.

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Index

- Brownian Motion
 - Geometric, 3
- Characteristic Function, 39, 75
 - of the Chi-Squared Distribution, 45
 - Non-Central, 47
- Chi-Squared Distribution, 42, 60
 - Non-Central, 45, 63
- Cholesky- Decomposition, 19
- Cornish-Fisher Expansion, 73
- Correlation, 14
- Covariance, 13
- Covariance Matrix, 14
- cpf, 1
- Cumulative Probability Function, 1
- Current Level, 22
- Curtosis, 72
 - of the Normal Distribution, 39
- Delta Approximation, 22
- Delta-Gamma Approximation, 22

- Delta-Normal Method, 30
- Determinant, 53
- Dirac Delta Function, 40

- Eigenvalues, 53
- Eigenvector, 53
- Expectation
 - of the Chi-Squared Distribution, 44

- Fast Fourier Transformation, 77
- Fourier Transformation, 39, 75
 - Inverse, 40, 76

- Gamma-Matrix, 50
- GBM, 3

- Hessian Matrix, 22

- Identity Matrix, 16
- iid, 52

- Imaginary Number, 39
- independent identically distributed, 52

- Johnson Transformationen, 73

- Linear Algebra, 52
- Liquidation Period, 7

- Matrix
 - singular, 53
- MGF, 33
- Moment Generating Function, 33, 64
 - of the Chi-Squared Distribution, 43
 - Non-Central, 46
 - of the Normal Distribution, 38
- Moment Matching, 73
- Moments
 - of the Chi-Squared Distribution, 43

- of the Normal Distribution, 38
- monotonous function, 2
- Non-Central Parameter, 45
- pdf, 2
- probability density function, 2
- Product Rule, 67
- Quantil
 - der Standardnormalverteilung, 10
- Scalar, 53
- Sensitivity
 - of a Portfolio, 22
- Skewness, 72
 - of the Normal Distribution, 39
- Square Root of Time Law, 13
- Straddle, 23
- Taylor Series, 20
- Trace
 - of a Matrix, 68
- Transformation
 - Orthonormal, 55
- Transpose
 - of a Matrix, 17
- VaR, 1
- Variance
 - of the Chi-Squared Distribution, 44
- Variance-Covariance Method, 20
- Vector
 - Orthogonal, 54
 - Orthonormal, 54
- Volatility, 14