ON STRONG APPROXIMATION OF SCALAR STOCHASTIC DIFFERENTIAL EQUATIONS

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Stochastic Differential Equations

$$dX(t) = a(t, X(t))\, dt + b(t, X(t))\, dW(t) \qquad (1)$$

where

$a : [0,1] \times \mathbb{R} \to \mathbb{R}$ drift coefficient

$b : [0,1] \times \mathbb{R} \to \mathbb{R}$ diffusion coefficient

$W(t)$, $t \in [0,1]$ Brownian motion

$X(0) = x_0$ initial value

Example : geometric Brownian motion

$$dX(t) = \mu X(t)\, dt + \sigma X(t)\, dW(t)$$

(stock price in the Black-Scholes model)

Two problems

1. Approximation of $X(t)$, $t \in [0,1]$ on the basis of $W(t), t = t_1, \ldots, t_n$

   a and b fulfill certain regularity conditions.

2. Approximation of the **square root process**

   $$dX(t) = (\alpha - \beta X(t))\, dt + \sigma \sqrt{X(t)}\, dW(t),$$

   where α, β, $\sigma > 0$.

Classical Methods

Choose a discretization

$$0 = t_0 < t_1 < \ldots < t_n = 1$$

of $[0, 1]$.
Put

$$\Delta_k = t_{k+1} - t_k, \quad \Delta_k W = W(t_{k+1}) - W(t_k).$$

- Most theoretical results deal with **mean square errors**

  $$E(X(t_k) - \widehat{X}(t_k))^2$$

  at the discretization points t_k.

Euler Method is defined by

$$\widehat{X}^E(0) \; = \; x_0 \qquad \text{and}$$

$$\widehat{X}^E(t_{k+1}) \; = \; \widehat{X}^E(t_k) + a(y_k) \cdot \Delta_k + b(y_k) \cdot \Delta_k W,$$

where $y_k = (t_k, \widehat{X}^E(t_k))$.

- $\widehat{X}^E$ contains the single stochastic integrals

  $$\int_{t_k}^{t_{k+1}} dW(s) = \Delta_k W.$$

Error bound

$$\max_{1\le k\le n}\left(E(X(t_k)-\widehat{X}^E(t_k))^2\right)^{1/2}\le K\Delta_{\max}^{1/2},$$

where

$$\Delta_{\max}:=\max_{0\le j\le n-1}\Delta_j$$

Milstein Method is defined by

$$\widehat{X}^M(0) = x_0 \qquad \text{and}$$

$$\begin{aligned}\widehat{X}^M(t_{k+1}) &= \widehat{X}^M(t_k)+a(z_k)\cdot\Delta_k+b(z_k)\cdot\Delta_k W,\\ &+\frac{1}{2}\left(b\cdot b^{(0,1)}\right)(z_k)\cdot\left((\Delta_k W)^2-\Delta_k\right),\end{aligned}$$

where $z_k=(t_k,\widehat{X}^M(t_k))$.

- $\widehat{X}^M$ contains the double stochastic integrals

$$\int_{t_k}^{t_{k+1}}\int_{t_k}^{s}dW(u)dW(s)=\frac{1}{2}\left((\Delta_k W)^2-\Delta_k\right).$$

Error bound Milstein (1974)

$$\max_{1\le k\le n}\left(E(X(t_k)-\widehat{X}^M(t_k))^2\right)^{1/2}\le K\Delta_{\max}$$

Stochastic Taylor Methods

Wagner, Platen (1978), Kloeden, Platen (1992)

- take into account additional multiple stochastic integrals
- upper error bounds

  $$K \cdot \Delta_{\max}^{\gamma}$$

  for arbitrary $\gamma \in \{\frac{1}{2}, 1, \frac{3}{2}, 2, \ldots\}$ can be achieved

Questions

- Asymptotic constants ?
- Error in $t \notin \{t_1, \dots, t_n\}$?
- Adaptive step-size control ?
- Optimal methods ?

Adaptive Methods

- W may be observed at adaptively chosen points
$$t_1, \dots, t_\nu \in (0, 1],$$
i.e., the choice of t_{k+1} may depend on $W(t_1), \dots, W(t_k)$
- total number ν of observations may be determined by any termination criterion
- $W(t_1), \dots, W(t_\nu)$ may then be used to produce a pathwise approximation

Remark An adaptive discretization should take into account

(D) the drift and diffusion coefficient

(T) the particular trajectory

(E) the error criterion

In the following : global characterization of the quality of $\widehat{X}$ on $[0,1]$

- **mean-squared L_2-error :**

$$e(\widehat{X}) = \left(E\left(\|X - \widehat{X}\|_2^2\right)\right)^{1/2},$$

  where

$$\|X - \widehat{X}\|_2 = \left(\int_0^1 (X(t) - \widehat{X}(t))^2 dt\right)^{1/2}$$

- **computational cost :**

  expected number $n(\widehat{X})$ of evaluations of W

Adaptive Discretization

that reflects the local smoothness of the solution

- We have

$$\left(E\left((X(t+\delta) - X(t))^2 | X(t) = x\right)\right)^{1/2}$$

$$= |b(t,x)| \cdot \delta^{1/2} + o(\delta^{1/2}),$$

  i.e., X is Hölder continuous with exponent $1/2$ and conditional Hölder constant $|b(t,x)|$.

Reasonable : Decreasing the step-size $\Delta_k = t_{k+1} - t_k$ with increasing value of $|b(t_k, X(t_k))|$. (Hereby we deal with (D) and (T).)

Rule of thumb (for L_2-error) :

$$\Delta_k \sim \frac{1}{|b(t_k, X(t_k))|}$$

Asymptotically Optimal Method

Adaptive step-size control : Choose parameter

$$h > 0,$$

put $t_0 = 0$ and define

$$t_{k+1} = t_k + \frac{h}{|b(t_k, \widehat{X}^M(t_k))|} \tag{2}$$

as long as the right-hand side ≤ 1.

Use the **Milstein scheme with (2) plus piecewise linear interpolation.**

$$\Longrightarrow \quad \text{method} \quad \widehat{X}_h$$

Error Analysis for $\widehat{X}_h$

Assumptions

(A) a, b differentiable with respect to the state variable.

Moreover, there exists $K > 0$ such that $f = a$ and $f = b$ satisfy

$$|f(t,x) - f(t,y)| \le K \cdot |x-y|,$$

(Lipschitz condition)

$$|f(s,x) - f(t,x)| \le K \cdot (1+|x|) \cdot |s-t|,$$

$$|f^{(0,1)}(t,x) - f^{(0,1)}(t,y)| \le K \cdot |x-y|$$

for all $s, t \in [0,1]$ and $x, y \in \mathbb{R}$. $\left(f^{(0,1)} := \frac{\partial f}{\partial x}\right)$

(B) $X(0)$ is independent of W and

$$E(X(0))^4 < \infty.$$

Remark :

$$\text{(A)} \Longrightarrow |f(t,x)| \le c \cdot (1+|x|) \quad \text{(linear growth)}$$

Given the above properties a pathwise unique strong solution of SDE (1) with initial value $X(0)$ exists.

THEOREM H., Müller-Gronbach, Ritter (2001)

$$\lim_{h \to 0} \left(n(\widehat{X}_h) \right)^{1/2} \cdot e(\widehat{X}_h) = C/\sqrt{6},$$

where

$$C = E\left(\int_0^1 |b(t, X(t))| dt \right).$$

Consider

$\widehat{X}_n^{\text{equi}}$ Milstein scheme with step-size $1/n$

Proposition $\widehat{X}_n^{\text{equi}}$ satisfies

$$\lim_{n \to \infty} n^{1/2} \cdot e(\widehat{X}_n^{\text{equi}}) = C^{\text{equi}}/\sqrt{6},$$

where

$$C^{\text{equi}} = \left(\int_0^1 E(b^2(t, X(t)))\, dt \right)^{1/2}$$

Note :

- order of convergence is $1/2$ for $\widehat{X}_h$ and $\widehat{X}_n^{\text{equi}}$
- $C \leq C^{\text{equi}}$

In most cases :

$$C << C^{\text{equi}}$$

Example : geometric Brownian motion with $\mu = 0$ and initial value $X(0) = 1$,
i.e.

$$dX(t) = \sigma X(t) dW(t)$$

solution :

$$X(t) = \exp(-\sigma^2/2 \cdot t + \sigma \cdot W(t))$$

Asymptotic constants

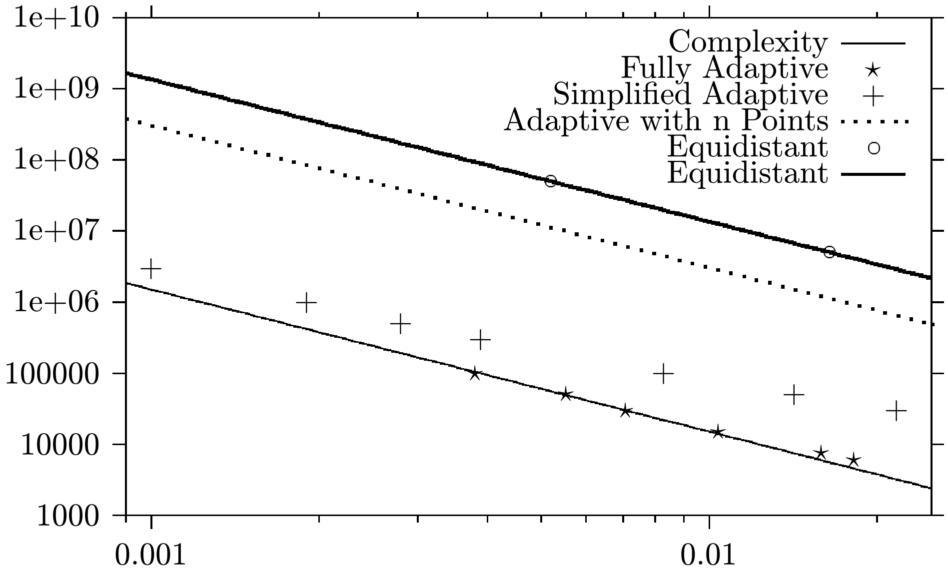
- for the method $\widehat{X}_n^{\,\mathrm{equi}}$:

  $$C^{\,\mathrm{equi}} = (\exp(\sigma^2) - 1)^{1/2},$$

  i.e., $C^{\,\mathrm{equi}}$ depends exponentially on the volatility.
- for the method $\widehat{X}_h$:

  $$C = \sigma,$$

  i.e., C depends only linearly on the volatility.



Expected number of evaluations vs. error for the geometric Brownian Motion with
$\mu = 0,\ \sigma = 3,\ X(0) = 1.$

We have even more :

THEOREM H., Müller-Gronbach, Ritter (2001)

$$\lim_{N\to\infty} \inf\{e(\widehat{X}) : n(\widehat{X}) \le N\} \cdot N^{1/2} = C/\sqrt{6}$$

Conclusions :

- $\widehat{X}_h$ is asymptotically optimal among all methods.
- The best order of convergence is 1/2 in terms of $n(\widehat{X})$.
- The best asymptotic constant is given by the mean of the conditional Hölder constant in space and time.

Problem 2

Aim: Methods $\widehat{X}$ for the strong approximation of the **square root process** (or Cox-Ingersoll-Ross process)

$$dX(t) = (\alpha - \beta X(t))\, dt + \sigma \sqrt{X(t)}\, dW(t),$$

α, β, $\sigma \in \mathbb{R}$, $X(0) = x_0$, W Brownian motion

Properties of the square root process

- $X(t) \geq 0$ for $t \geq 0$

Put

$$d = \frac{4\alpha}{\sigma^2} > 0$$

$$\Longrightarrow dX(t) = \left(\frac{d\sigma^2}{4} - \beta X(t) \right) dt + \sigma \sqrt{X(t)} dW(t) \tag{3}$$

- If d is an integer :

$$X(t) := \sum_{i=1}^{d} Y_i^2(t)$$

solves (3) with Brownian motion

$$W(t) = \sum_{j=1}^{d} \int_0^t \frac{Y_j(u)}{\sqrt{X(u)}} dW_j(u),$$

where $Y_1, \ldots, Y_d$ are given by

$$dY_j(t) = -\frac{1}{2}\beta Y_j(t)\,dt + \frac{1}{2}\sigma dW_j(t),\ \ j = 1, \ldots, d$$

(independent Ornstein-Uhlenbeck processes),

$(W_1, \ldots, W_d)$ d-dimensional Brownian motion

- $d < 2 \Longrightarrow \text{Prob}\,(X(t) = 0 \text{ infinitely often}) = 1$
- $d \geq 2 \Longrightarrow \text{Prob}\,(X(t) = 0 \text{ at least once}) = 0$

Remark : A numerical solution should be always positive.

Consider a discretization $0 = t_0 < t_1 < \ldots < t_n = T$ of $[0, T]$.

1. **Modified Euler scheme**

$$\begin{aligned} \widetilde{X}_0^E &= X(0), \\ \widetilde{X}_{k+1}^E &= \left| \widetilde{X}_k^E + (\alpha - \beta \widetilde{X}_k^E)\Delta_k + \sigma \sqrt{\widetilde{X}_k^E}\Delta_k W \right|, \\ &\qquad\qquad k = 0, \ldots, n-1, \end{aligned}$$

where $\widetilde{X}_k^E = \widetilde{X}^E(t_k)$.

- no result on the order of convergence

2. **Balanced implicit method**

$$\begin{aligned} \widehat{X}_0^B &= X(0), \\ \widehat{X}_{k+1}^B &= \widehat{X}_k^B \\ &+ \frac{(\alpha - \beta \widehat{X}_k^B)\Delta_k + \sigma\sqrt{\widehat{X}_k^B}\Delta_k W}{1 + c^0(t_k, \widehat{X}_k^B)\Delta_k + c^1(t_k, \widehat{X}_k^B)|\Delta_k W|}, \\ &\qquad \text{for } k = 0, \ldots, n-1, \end{aligned}$$

where $\widehat{X}_k^B = \widehat{X}^B(t_k)$,

$c^0, c^1 : [0,T] \times (0,\infty) \to [0,\infty)$.

THEOREM

If

$$c^0(t,x) = \frac{|\alpha - \beta x|}{x} \quad \text{and} \quad c^1(t,x) = \frac{\sigma}{\sqrt{x}}$$

then, with probability 1,

$$\widehat{X}^B(t_k) > 0 \Longrightarrow \widehat{X}^B(t_{k+1}) > 0$$

for $k = 0, \ldots, n-1$.

Numerical Experiments

We consider

$$dX(t) = \left(\frac{d\sigma^2}{4} - \beta X(t)\right) dt + \sigma\sqrt{X(t)}dW(t)$$

and study the dependence on the dimension d for

- $\widetilde{X}^E$
- $\widehat{X}^B$ with weights

$$c^0(t,x) = \frac{|d\sigma^2/4 - \beta x|}{x},\ c^1(t,x) = \frac{\sigma}{\sqrt{x}}$$

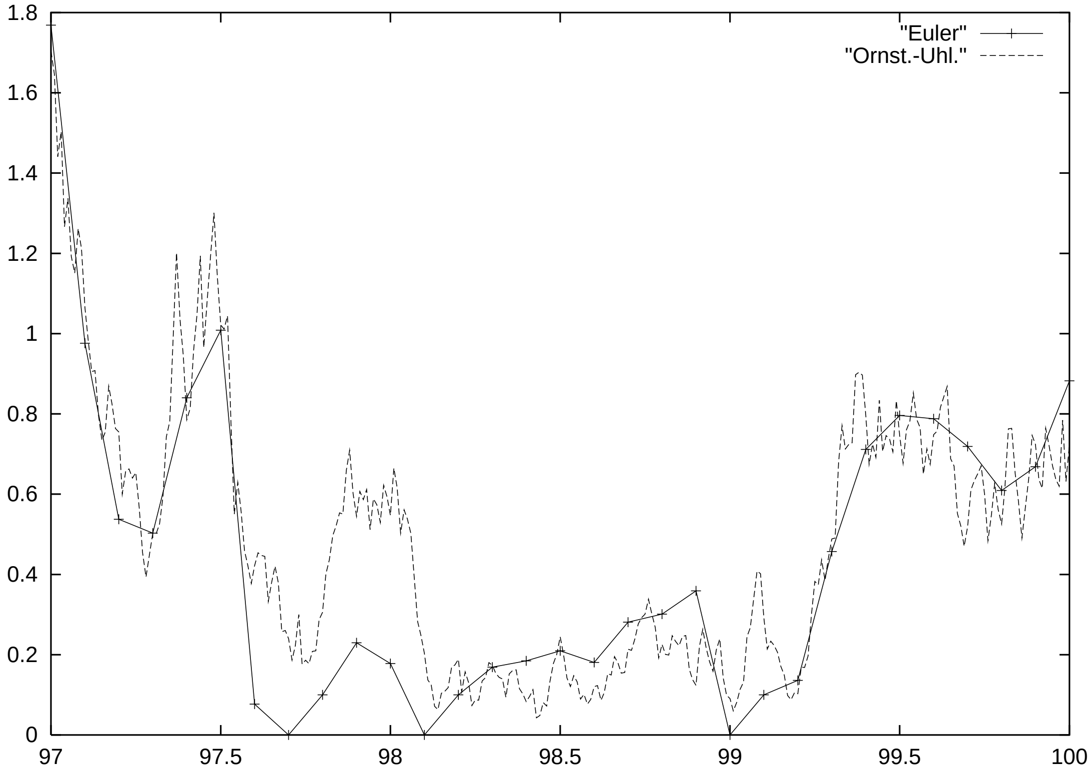
We choose

- $\sigma = 1,\ \beta = 1:$ mean reversion with reference level $d/4$ and speed of adjustment 1
- $0 \leq t \leq 100, \quad X(0) = 0.1$
- $d =$ integer

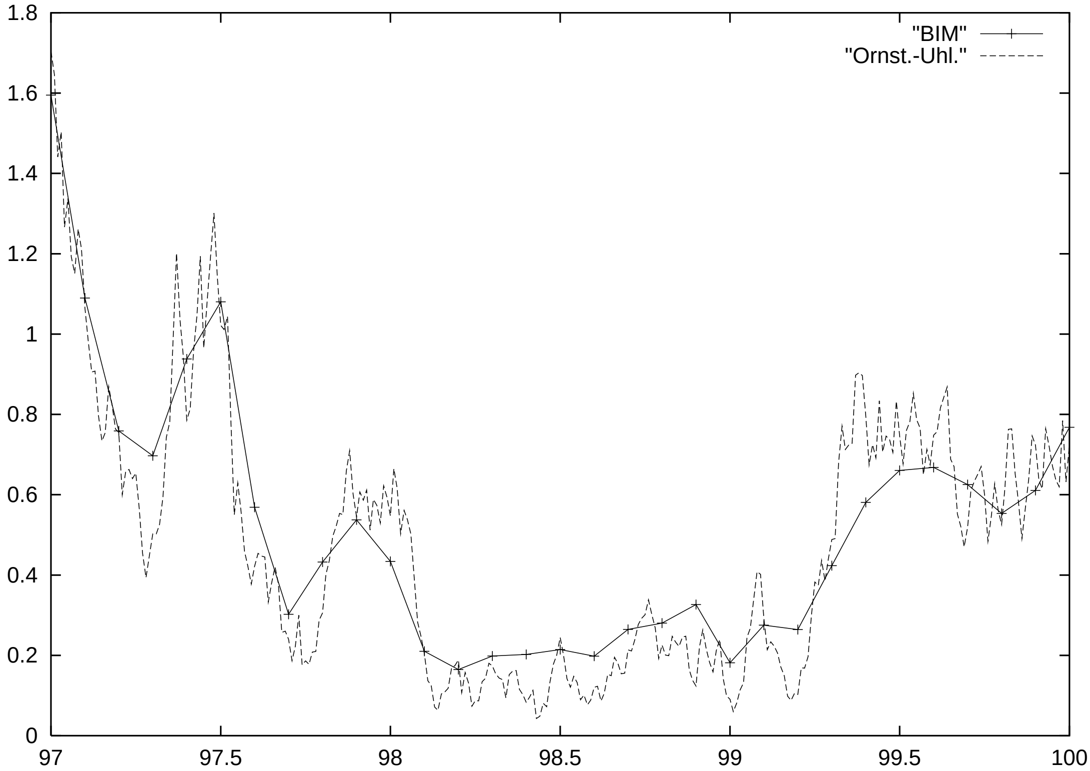
Comparison with exact solution :
As a representative of the exact solution we take

$$\sum_{j=1}^{d} \overline{Y}_j(t),$$

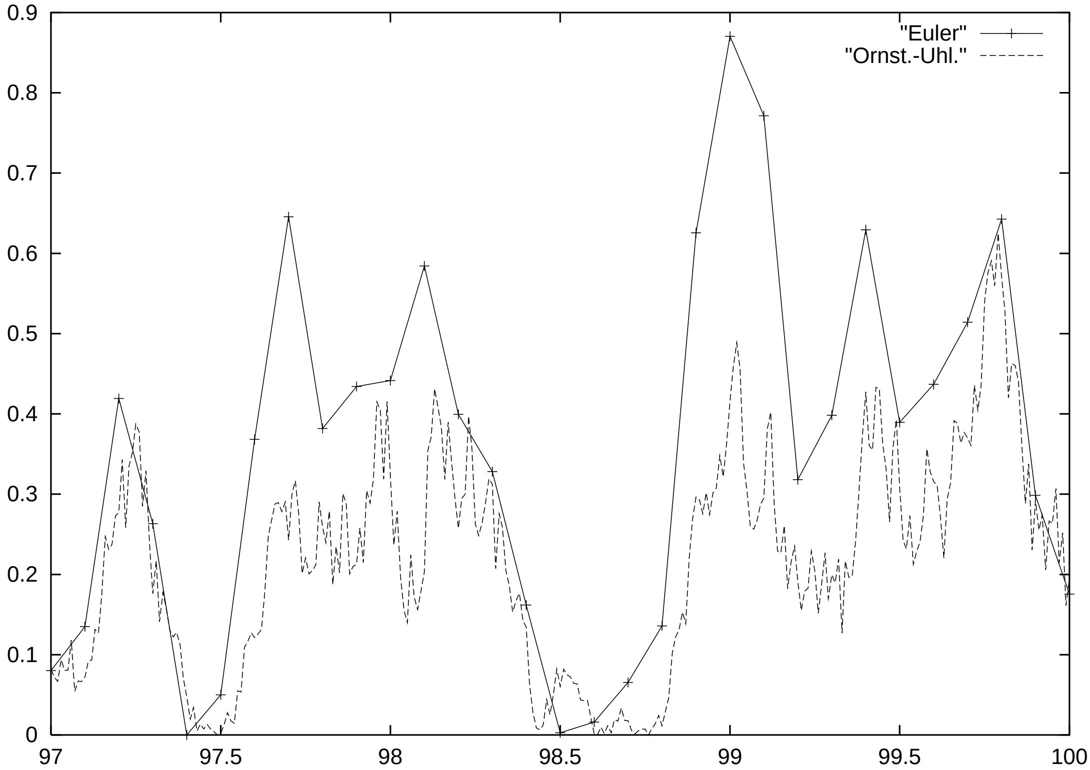
where $\overline{Y}_1, \ldots, \overline{Y}_d$ are approximate independent Ornstein-Uhlenbeck processes



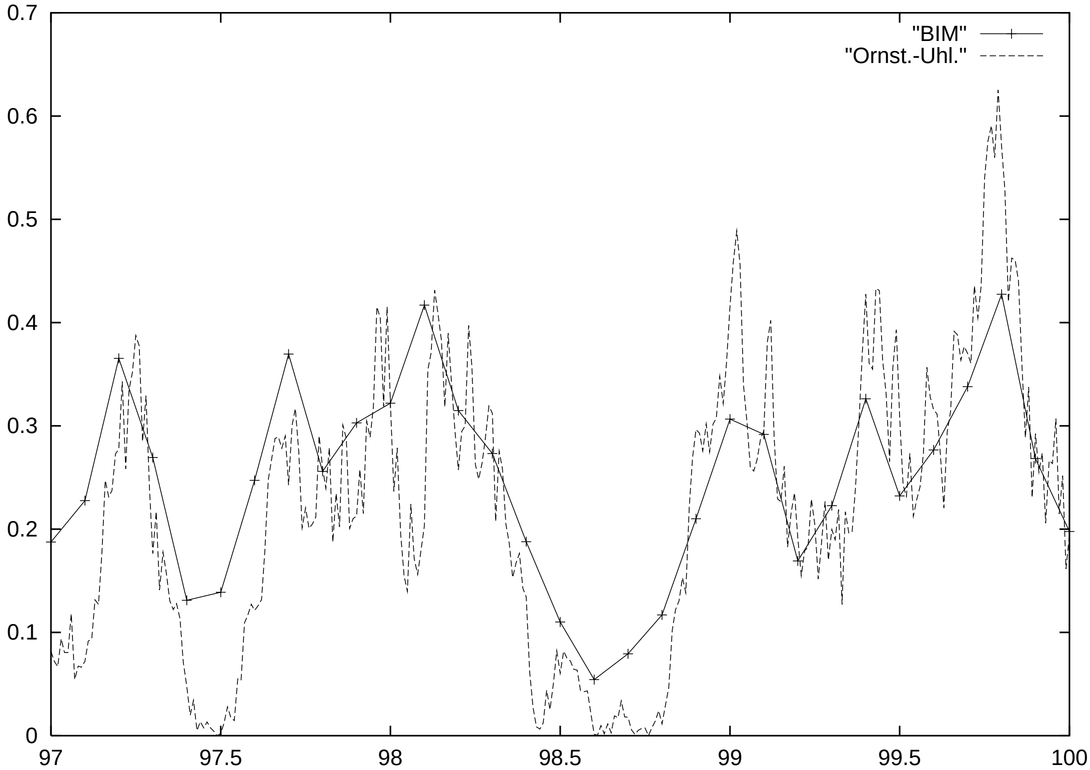
Mod. Euler vs. exact solution, $d = 4$, $\Delta = 1/10$.



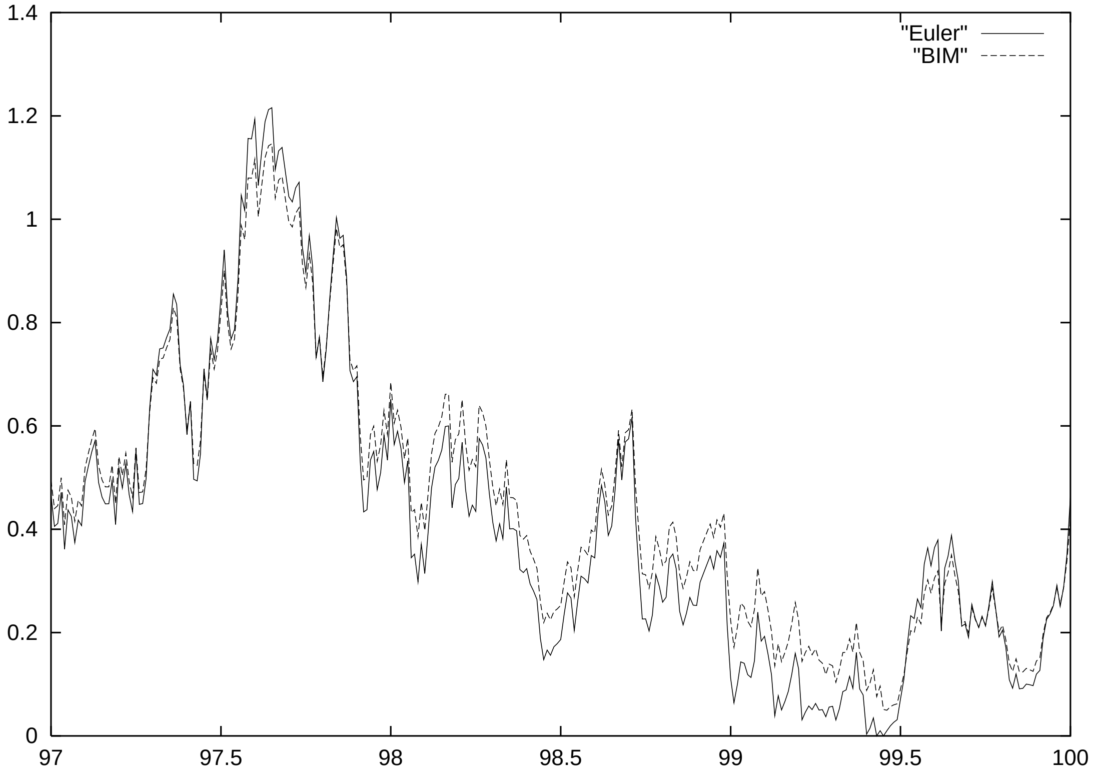
BIM vs. exact solution, $d = 4$, $\Delta = 1/10$.



Mod. Euler vs. exact solution, $d = 2$, $\Delta = 1/10$.



BIM vs. exact solution, $d = 2$, $\Delta = 1/10$.



Mod. Euler vs. BIM, $d = 4$, $\Delta = 1/100$.

Convergence of Balanced Implicit Methods

d–dimensional SDE

$$dX(t) = a(t, X(t))\, dt + \sum_{j=1}^{m} b^j(t, X(t))\, dW^j(t),$$

$a, b^j : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$

$(W_1, \ldots, W_m)$ m-dimensional Brownian motion

General Form of Balanced Implicit Methods

$$\begin{aligned} \widehat{X}^B(t_0) &= X(0), \\ \widehat{X}^B(t_{k+1}) &= \widehat{X}^B_k + a(t_k, \widehat{X}^B_k)\, \Delta_k \\ &\quad + \sum_{j=1}^{m} b^j(t_k, \widehat{X}^B_k)\, \Delta_k W^j \\ &\quad + C_k\, (\widehat{X}^B_k - \widehat{X}^B_{k+1}), \end{aligned}$$

for $k = 0, \ldots, n-1$,

where

$$C_k = c^0(t_k, \widehat{X}^B_k)\, \Delta_k + \sum_{j=1}^{m} c^j(t_k, \widehat{X}^B_k) |\, \Delta_k W^j|$$

- $\widehat{X}^B_k = \widehat{X}^B(t_k)$

- $c^0, \ldots, c^m$ are $d \times d$–matrix valued functions with certain properties (e.g. positive definite)
- $\Delta_k W^j = W^j(t_{k+1}) - W^j(t_k)$

Error Bound Milstein, Platen, Schurz (1998)

$$\max_{1 \le k \le n} \left(E \left| X(t_k) - \widehat{X}^B(t_k) \right|^2 \right)^{1/2} \le K \Delta_{\max}^{1/2}$$