Jump-Diffusion Models in Foreign Exchange Markets

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Abstract

Jump-diffusion models to price FX options will be considered. Several distributions for the jump size will be stated and analysed. Features and limitations of the obtained models will be presented. Besides theoretical results and notes on implementation, the question will be discussed whether this models can be used to price options taking the volatility smile in todays FX markets into account.

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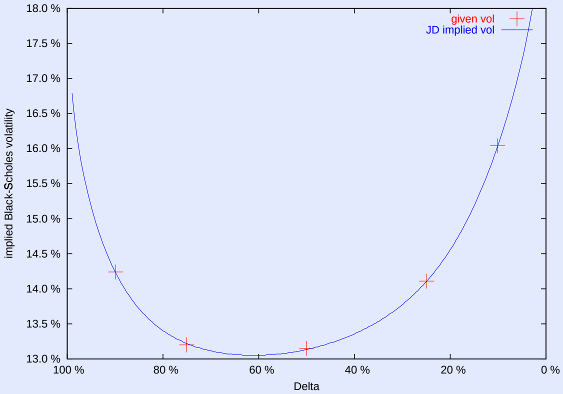
1. Motivation



- Foreign exchange rates do not change continuously (`USD/JYP` spot rate from April 1998 to April 1999)

- Observation of volatility smile in FX markets (implied volatilities for `6m` `USD/JPY` options)

2. The model

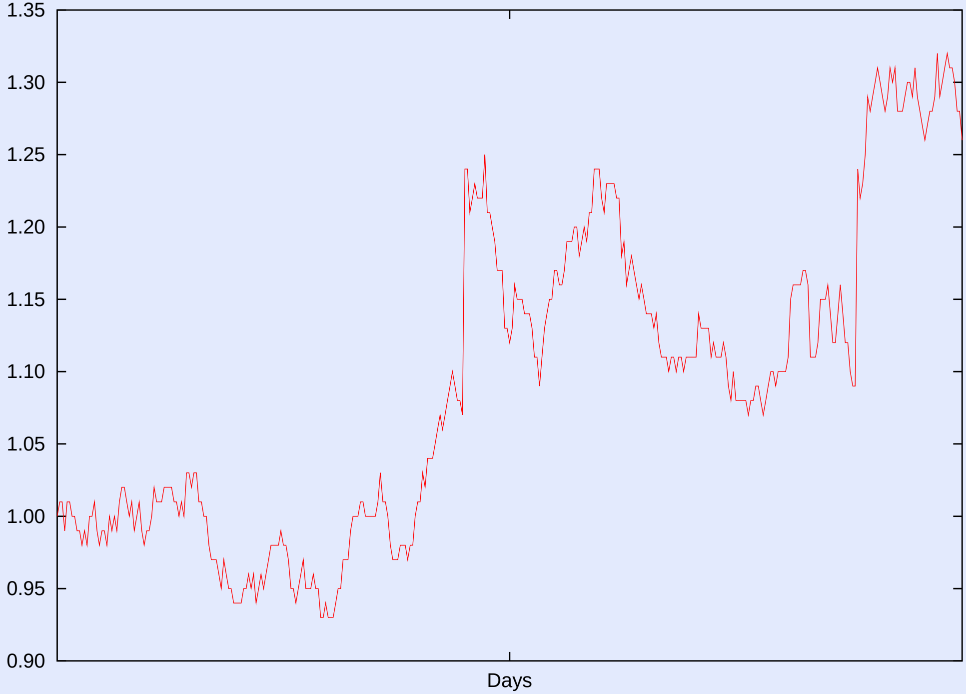
Describing the underlying by

$$\frac{dS_t}{S_{t^-}} = \mu dt + \sigma dW_t + \gamma dN_t$$

resolves into a process with

- small spot movements are modeled using Brownian Motion,
- larger spot movements described by discrete jump events,
- relative jump size $\gamma \in (-1, \infty)$.

Simulated path of exchange rate in jump-diffusion model



In particular we use

$$\frac{dS_t}{S_{t^-}} = (r_d - r_f - \lambda \mathbb{E}[\gamma_0])\, dt + \sigma dW_t + \gamma_t dN_t,$$

where

(W_t) is a standard Brownian motion,

(N_t) is a Poisson process with intensity $\lambda > 0$,

(γ_t) are i.i.d. random variables representing the relative jump size, with values in $(-1, \infty)$.

(W_t), (N_t), and (γ_t) are assumed to be mutually independent and defined on some probability space $(\Omega, \mathbb{F}, \mathbb{P})$.

With some regularity conditions on the coefficients there exists a unique strong solution of the form

$$\begin{aligned} S_T &= S_t e^{\left(r_d - r_f - \lambda \boldsymbol{E}[\gamma] - \frac{\sigma^2}{2}\right)\tau + \sigma W_\tau + \int_t^T \ln(1+\gamma_s)\,dN_s} \\ &= S_t e^{\left(r_d - r_f - \lambda \boldsymbol{E}[\gamma] - \frac{\sigma^2}{2}\right)\tau + \sigma W_\tau + \sum_{j=1}^{N_\tau} \ln(1+\gamma_{\tau_j})} \\ &= S_t e^{\left[\left(r_d - r_f - \lambda \boldsymbol{E}[\gamma] - \frac{\sigma^2}{2}\right)\tau + \sigma W_\tau\right]} \prod_{j=1}^{N_\tau} (1+\gamma_{\tau_j}). \end{aligned}$$

(S_t) is a right-continuous process with left-hand limits.
In any given finite interval it has only finitely many jumps.

3. The relative jump size

(γ_t) are i.i.d. random variables representing the relative jump size with $\gamma_t \in (-1, \infty), \ \forall t$.
Describing its distribution in terms of

$$Y_t := \ln \left\{ \gamma_t + 1 \right\},$$

we get $Y_t \in (-\infty, \infty)$ and $\gamma_t = e^{Y_t} - 1$.
We consider two distributions

- $Y_t \sim \mathcal{N}\left(\mu_Y, \sigma_Y^2\right)$,
- $Y_t \sim$ Double Exponential with $p, \ \eta_1, \ \eta_2$ $(\eta_1 > 1, \ \eta_2 > 0)$.

Normal distribution

The pdf for $Y = \ln\{\gamma + 1\}$ then satisfies

$$f_Y(x) = \frac{1}{\sqrt{2\pi}\sigma_Y} \exp\left[-\frac{(x-\mu_Y)^2}{2\sigma_Y^2}\right],$$

and

$$\mathbb{E}[Y] = \mu_Y,$$

$$\mathbb{V}[Y] = \sigma_Y^2.$$

For $\gamma = e^Y - 1$ it follows

$$\mathbb{E}[\gamma] = \mathbb{E}[e^Y] - 1 = e^{\mu_Y + \frac{\sigma_Y^2}{2}} - 1.$$

Double Exponential distribution

The pdf for $Y = \ln\{\gamma + 1\}$ then satisfies

$$f_Y(x) = p \cdot \eta_1 e^{-\eta_1 x} 1\!\!1_{\{x \geq 0\}} + q \cdot \eta_2 e^{\eta_2 x} 1\!\!1_{\{x < 0\}},$$

where $q = 1 - p$, and

$$\begin{aligned} I\!\!E[Y] &= \frac{p}{\eta_1} - \frac{q}{\eta_2}, \\ I\!\!V[Y] &= 2\left(\frac{p}{\eta_1^2} + \frac{q}{\eta_2^2}\right) - \left(I\!\!E[Y]\right)^2. \end{aligned}$$

For $\gamma = e^Y - 1$ it follows

$$I\!\!E[\gamma] = I\!\!E[e^Y] - 1 = p\frac{\eta_1}{\eta_1 - 1} + q\frac{\eta_2}{\eta_2 + 1} - 1.$$

Memoryless property

If $\forall t,\ s > 0$

$$I\!\!P[x > s + t | x > t] = I\!\!P[x > s].$$

Applying definition of $I\!\!P[\cdot|\cdot]$,

$$\frac{I\!\!P[x > s + t, x > t]}{I\!\!P[x > t]} = I\!\!P[x > s],$$

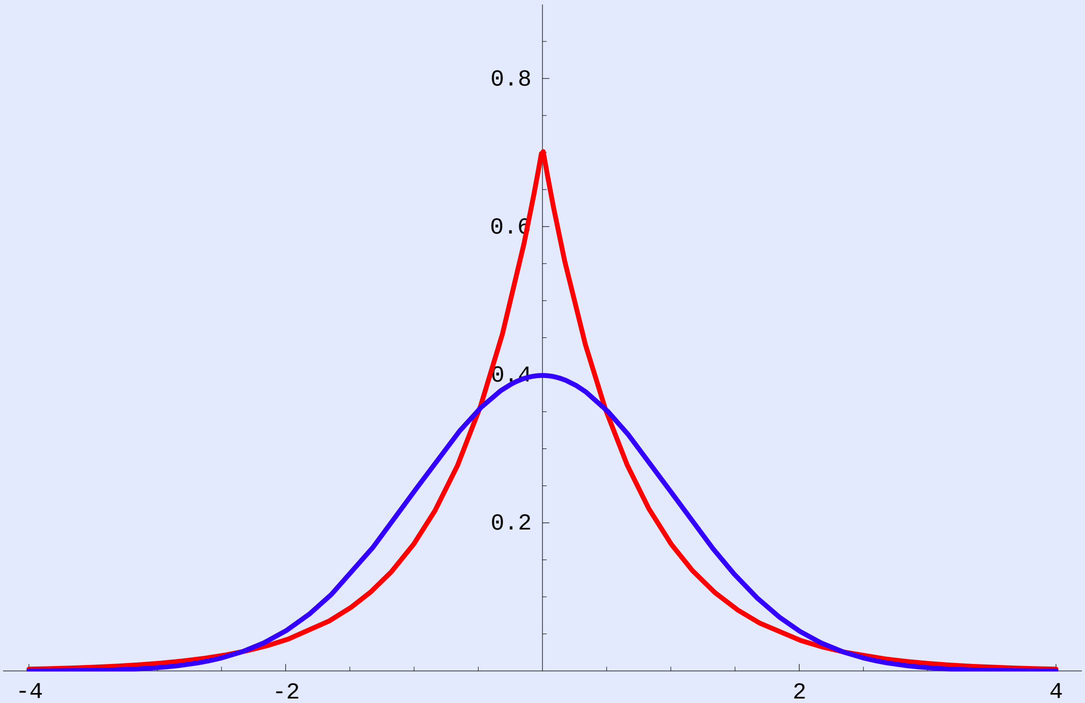
using $s > 0$,

$$I\!\!P[x > s + t] = I\!\!P[x > s] I\!\!P[x > t].$$

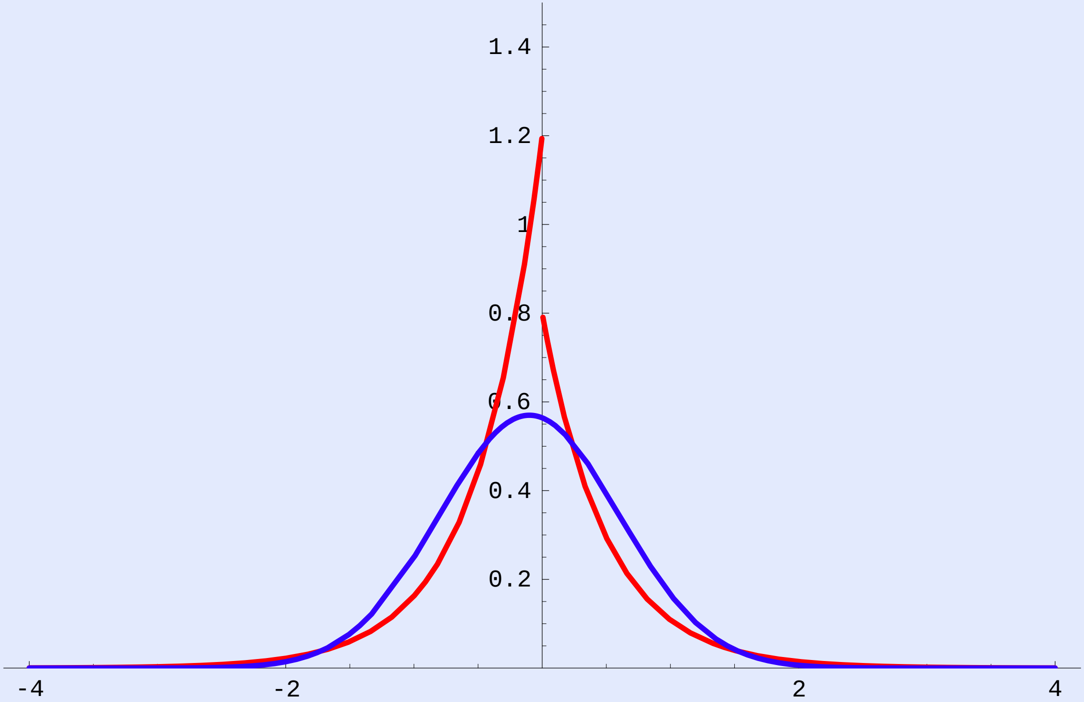
Exponential distribution has the memoryless property

$$\begin{aligned} \mathbb{P}[x > s + t] &= e^{-\eta(s+t)} \\ &= e^{-\eta s} \cdot e^{-\eta t} \\ &= \mathbb{P}[x > s] \cdot \mathbb{P}[x > t]. \end{aligned}$$

Normal distribution does not have memoryless property.



Comparison of the probability density function for normal distribution and double exponential distribution with $\mathbb{E} = 0.0$ and $\mathbb{V} = 1.0$ $(p = 0.5,\ \eta_1 = \eta_2 = \sqrt{2.0})$.



Comparison of the probability density function for normal distribution and double exponential distribution with $\mathbb{E} = -0.10$ and $\mathbb{V} = 0.49$ ($p = 0.4,\ \eta_1 = \eta_2 = 2.0$).

Recalling the spot process

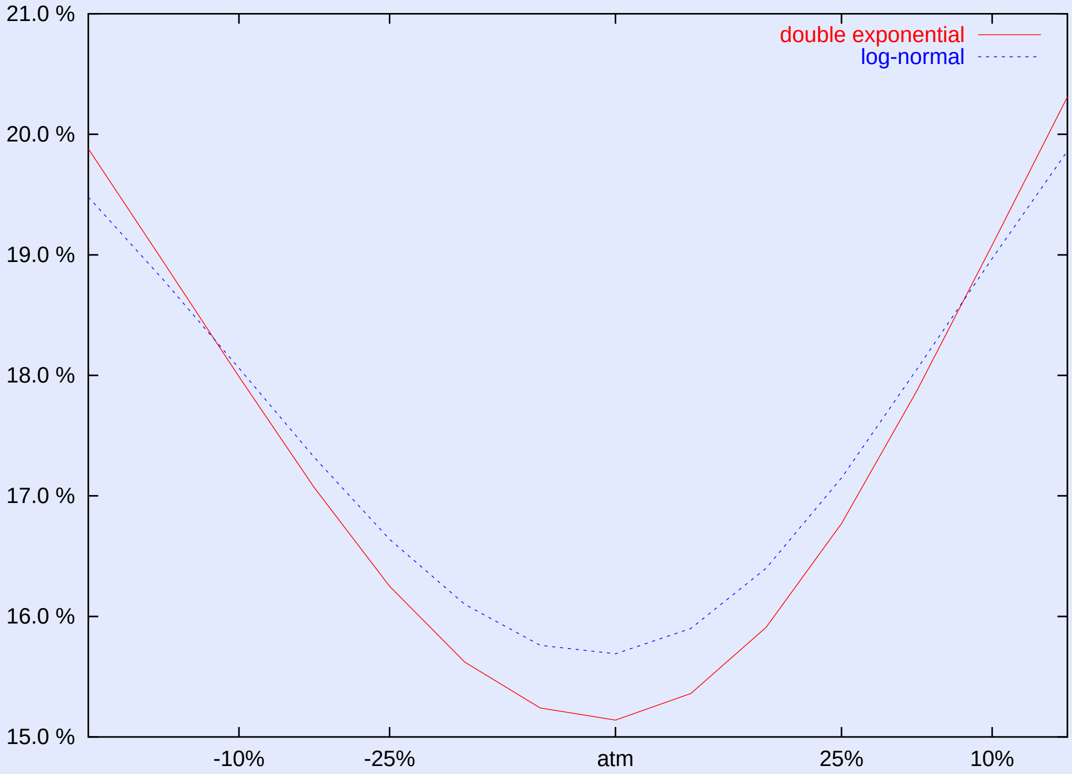
$$S_T = S_t e^{\left(r_d - r_f - \lambda \boldsymbol{E}[\gamma] - \frac{\sigma^2}{2}\right)\tau + \sigma W_\tau + \sum_{j=1}^{N_\tau} \ln(1+\gamma_{\tau_j})}$$

we get in terms of $Y_t = \ln\left\{\gamma_t + 1\right\}$

$$= S_t e^{\left(r_d - r_f - \lambda \boldsymbol{E}[\gamma] - \frac{\sigma^2}{2}\right)\tau + \sigma W_\tau + \sum_{j=1}^{N_\tau} Y_{\tau_j}}$$

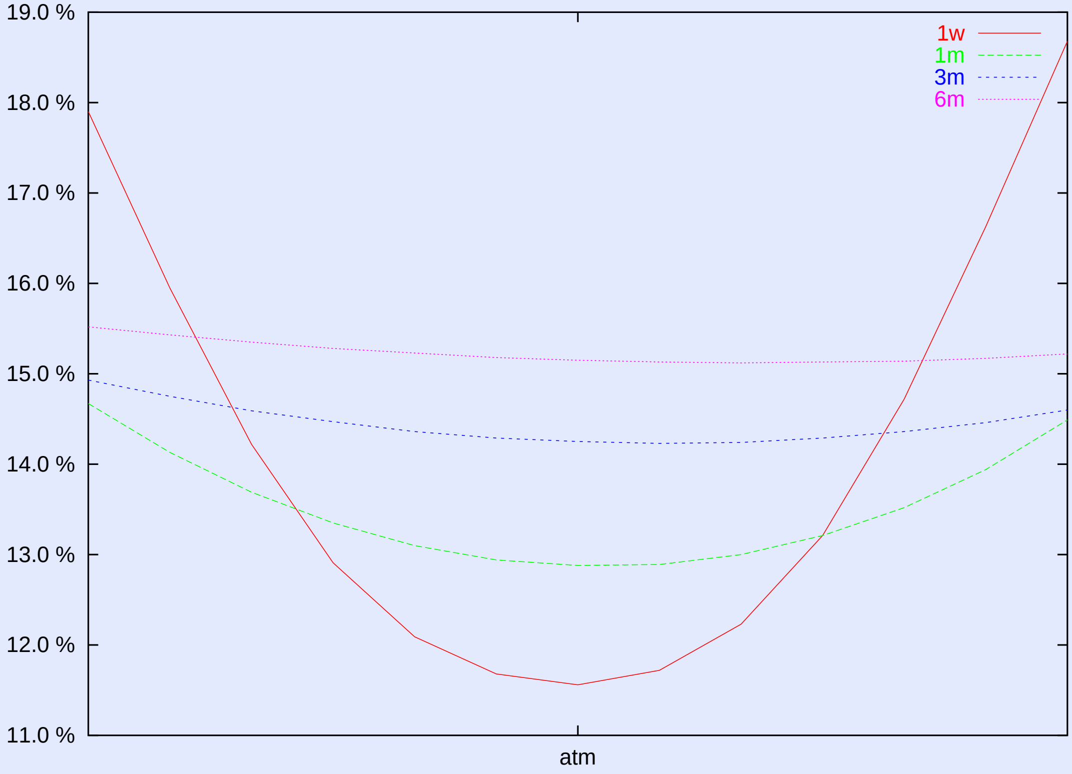
$$= S_t e^{\left[\left(r_d - r_f - \lambda \boldsymbol{E}[\gamma] - \frac{\sigma^2}{2}\right)\tau + \sigma W_\tau\right]} \prod_{j=1}^{N_\tau} e^{Y_{\tau_j}},$$

4. The model and the volatility smile



Implied Black-Scholes volatilities for prices obtained from jump-diffusion model with $\mathbb{E}[Y] = 0,\ \mathbb{V}[Y] = 0.02$.
$(r_d = 0,\ r_f = 0,\ \sigma = 10.0\%,\ \lambda = 1.0;\ p = 0.50,\ \eta_1 = \eta_2 = 10.0)$.

Can one set of parameters be used for different maturities?



Implied Black-Scholes volatilities for prices of `1w, 1m, 3m,` and `6m` European Plain Vanilla options obtaines from jump-diffusion model with double exponential distributed jump size with parameters $r_d = 0,\ r_f = 0,\ \sigma = 10.0\%,\ \lambda = 1.0;\ p = 0.50,\ \eta_1 = \eta_2 = 10.0.$

5. Option pricing

- Monte Carlo methods
- Closed-form solutions for special cases (including European Plain Vanilla options)
- Finite-Difference
    - combined with fast-fourier transforms (log-normal)
    - boundary conditions take overshoot into account
- Inversion of Laplace transforms (double exponential)

The process (S_t) is decribed by a combination of Brownian motion and Poisson random sum of random variables Y

$$\sigma W_\tau + \sum_{j=1}^{N_\tau} Y_{\tau_j}.$$

- for $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$ it can be calculated analytically
- for $Y \sim$ Double Exponential
  - $\sum$ *Double Exponential* $\sim$ *Double Gamma*
  - $+$ *Brownian:* can be described using so called Hh functions, where for $n = 1,\ 2,\ \ldots,$

$$Hh_n(x) = \int_x^\infty Hh_{n-1}(s)ds = \frac{1}{n!}\int_x^\infty (s-x)^n e^{-s^2/2}ds,$$
$$Hh_{-1}(x) = \sqrt{2\pi}\phi(x),\ Hh_0(x) = \sqrt{2\pi}\Phi(x).$$

Price of European Plain Vanilla

Following the general framework of option pricing

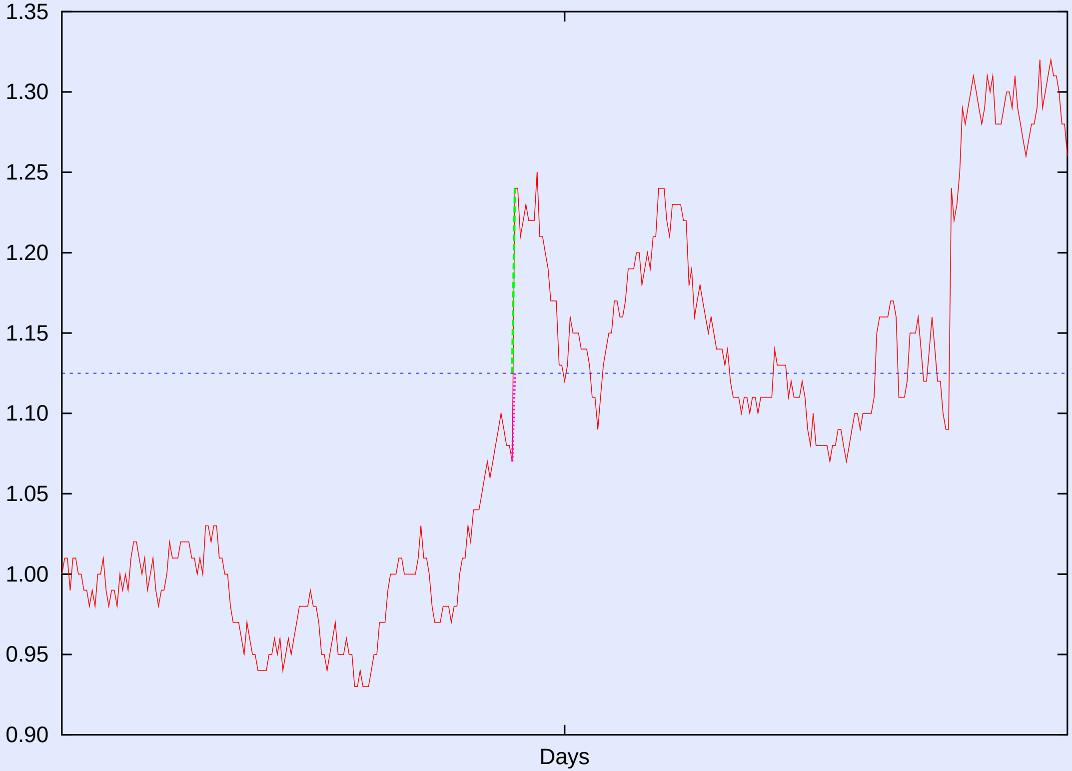
$$
\begin{aligned}
P_t &= \mathbb{E}\left[e^{-r_d\tau}\left(S_T - K\right)^+ \middle| \mathcal{F}_t\right] \\
&= e^{-r_d\tau}\,\mathbb{E}\left[S_T \mathbb{1}_{\{S_T > K\}} \middle| S_t\right] - e^{-r_d\tau} K\,\mathbb{P}\left[\{S_T > K\} \middle| S_t\right] \\
&= S_t e^{(-r_f - \lambda \boldsymbol{E}[\gamma])\tau} \cdot \Psi^* - e^{-r_d\tau} K \cdot \Psi
\end{aligned}
$$

where $\Psi^*,\ \Psi$ are of the form

$$\Psi^{(*)} = \sum_{n=0}^{\infty} e^{-\lambda\tau} \frac{(\lambda\tau)^n}{n!} \psi^{(*)},$$

with $\psi^{(*)}$ being cumulative distribution functions of the combination of the Brownian motion and the relative jump size.

Barrier options and the overshoot problem



If memoryless property holds

$$I\!P(x > s + t \,|\, x > t) = I\!P(x > s).$$

Remark on first passage time

From memoryless property of double exponential distribution the Laplace transform of the first passage time hitting a fixed barrier can be obtained.
Numerical inversion of Laplace transform can give prices for Touch and Barrier options. (using Gaver-Stehfest algorithm)

Caution: Numerical problems

- algorithm involves sum with alternating signs,
- terms involve factorials

Calculation with high precision is needed!

6. Calibration

Using jump-diffusion model requires to find parameters (Θ).

σ is the volatility of the Brownian motion.

λ is the intensity of the Poisson process.
It is assumed to be deterministic and $\lambda > 0$.

log-normal

μ_Y is the mean of normal distribution.
σ_Y^2 is the variance of normal distribution.

double exponential

p probability of upward jump.
$\eta_1 = \eta_2$ parameter of exponential distribution.

Mean-square minimization of market implied volatilities and model implied volatilities.

- using implied volatilities $\tilde{\sigma}_i$ for `atm`, 10%- and 25%-Delta Call and Put options,
- calibrations for one fixed maturity.

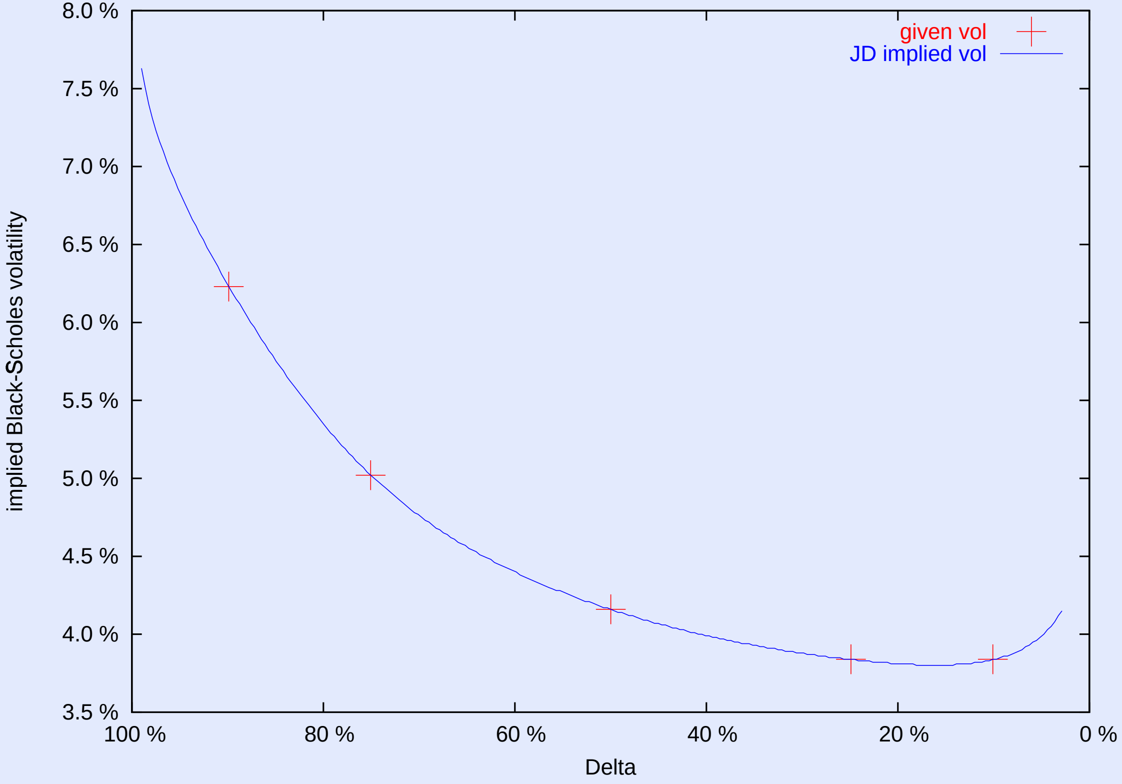
Procedure

1. retrieving the strikes corresponding to given volatilities and deltas using Black-Scholes,
2. computing call prices in jump diffusion model,
3. retrieving implied Black-Scholes volatilities from obtained jump diffusion prices.

The function to be minimized can be chosen to be

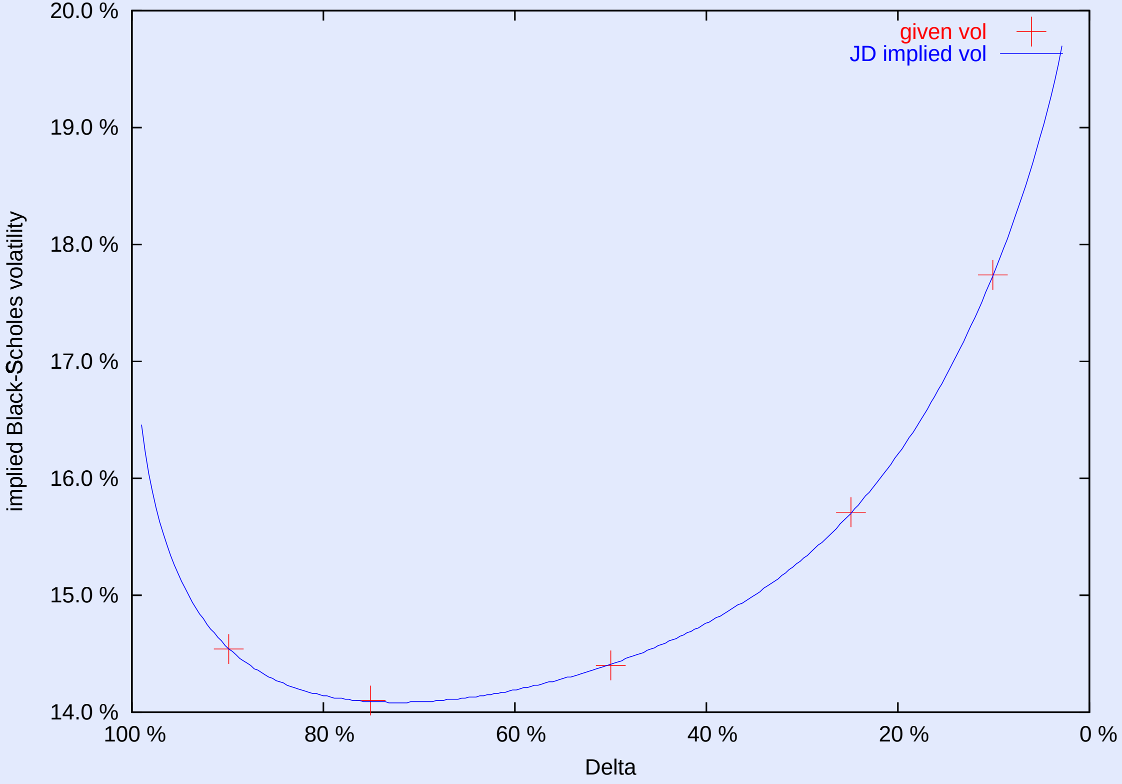
$$M(\Theta) := \sum_{i=1}^{5} \Big(\tilde{\sigma}_i - \sigma_i(\Theta)\Big)^2.$$

Calibrating `6m EUR/CHF` options (log-normal)



$$(\sigma = 2.73\%,\ \lambda = 0.6,\ \mu_Y = -0.03,\ \sigma_Y^2 = 4.33\%)$$

Calibrating `1m` `USD/JPY` options (log-normal)



$$(\sigma = 9.21\%,\ \lambda = 5.7,\ \mu_Y = 0.01,\ \sigma_Y^2 = 4.47\%)$$

7. Concluding remarks

- jump-diffusion models can explain volatility smile for short and medium term options
- very good quality of fit for a given maturity
- model cannot explain *term structure* of volatility smile
- option prices for European Plain Vanilla options can be obtained for many variations of the model
- for barrier options problem of overshoot can be solved using double exponential distributed jump sizes
- efficient methods for pricing exotic options need to be developed

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