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Expanding Further the Universe of Exotic Options Closed Pricing Formulas in the Black and Scholes Framework

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Abstract. A pricing method resulting in a closed formula is proposed for a large class of options such as *Best Of* and *Rainbow* based on the analysis of the return profile of the option. We suppose that returns follow a brownian motion and the usual hypotheses of the Black-Scholes model extended for the multi-underlying/multi-currency case.
The result builds on a previous result, and expands the set of exercise conditions, to allow the use of different reference maturities to define an adapted payoff. The result states that, if the return of an option is a linear combination of the prices at maturity of traded assets multiplied by an indicator function generated by an exercise condition, then, the pricing formula is also a linear combination of the current market prices of the traded assets multiplied by a probability expressed in the risk-neutral measure where the asset is the risk-free asset.
The proof the of the result uses the *change of numeraire* technique based on Girsonov's theorem for the multi-underlying/ multi-currency case.
We apply the method to develop closed formulas for Target Redemption options. Comparison results of simulation are also presented.

1. Introduction.

The aim of this paper is to expand further the universe of exotic options closed formulas, developed in a previous work by the author[22]. The closed formulas are valid in the context of the Black Scholes model but the usual generalizations can also be applied. Namely, to allow volatility to be time dependent.
Instead of trying to find a pricing formula for one or more types of exotic options, a method is developed that always results in a closed formula pricing function, given that some *a priori* conditions are met. This paper relaxes the original *a priori* conditions allowing comparisons of asset prices at different moments in time.
The general form of the return profile of the option is a linear combination of traded asset prices. Each term of the combination is associated with a set that determines whether the term is included at maturity in the return or not. The set is defined in respect to the option's underlying asset prices as a conjunction of n subsets. Each subset is generated by a condition of the form $S_x^{T_\alpha} > S_y^{T_\beta}$, where S_x, S_y are price of traded assets, and T_α, T_β are moments in time earlier (or equal) than the option's maturity date, so that the return profile is adapted.

2. The Pricing Formula.

Let the payoff profile be of the following form:

$$\Phi^T(S_0, ..., S_n) = \sum_{i=1}^{n} k_i \cdot S_i^T \cdot \mathbb{I}_i, \ n \in \mathbb{N} < \infty, \tag{1}$$

where,

- $k_i \in \mathbb{R}$,
- S_i^T is the market value of the traded asset S_i at time T,
- $\mathbb{I}_i$ is the indicator function of the set C_i defined below.

The payoff profile is a linear combination of asset prices at maturity, where each term is multiplied by an indicator function, that determines whether the term is realized or not.

Theorem 2.1. *The pricing formula for any exotic option Φ^T is of the form:*

$$\Pi(t,\Phi^T) = e^{-q_i(T-t)} \sum_{i=0}^{n} k_i S_i^t \mathbb{P}^{\mathcal{Q}_i}(C_i),\ n \in \mathbb{N} < \infty \tag{2}$$

where,

- S_i^t, *is the market value of* $S_i(t)$ *at time* $t < T$,
- $\mathbb{P}^{\mathcal{Q}_i}(C_i)$, *is the risk-neutral probability of the set* C_i *when the asset* S_i *is chosen to be the* numeraire,
- $q_i = (r - (d_{fx_i} - d_{S_i} - \sigma_{S_i}\sigma_{fx_i}\rho_{S_i fx_i}))$,
- r *is the risk-free interest rate of the currency on which* Φ^T *is expressed,*
- d_{fx_i} *is the risk-free interest rate of the currency on which* S_i *is expressed,*
- d_{S_i} *is the dividend yield of the asset* S_i,
- σ_{S_i} *is the volatility of the returns of* S_i,
- σ_{fx_i} *is the volatility of the returns of the foreign exchange rate between the currency of* Φ^T *and the currency of* S_i, *and*
- $\rho_{S_i fx_i}$ *is the correlation between the returns of* S_i *and the returns of* fx_i.

Proof. Using as *numerarie* the risk-free asset expressed in the currency of Φ^T, with $\mathcal{Q}$ being its risk-neutral measure, the diffusion model for the underlying asset prices in $\mathcal{Q}$, is the following:

$$\frac{dS_i^t}{S_i^t} = (r - q_i)\,dt + \sigma_{S_i} dW_{S_i} \tag{3}$$

$$d\left\langle W_{S_i}, W_{S_j}\right\rangle_t = \rho_{i,j} dt \tag{4}$$

with

$$q_i := r - (d_{fx_i} - d_{S_i} - \sigma_{S_i}\sigma_{fx_i}\rho_{S_i,fx_i}) \qquad \text{and} \qquad i,j = 1,\dots,n.$$

The pricing formula for $\Pi(t,\Phi^T)$ is the well known result

$$F = e^{-r(T-t)} E^{\mathcal{Q}}\left[\Phi^T\right],$$

$$F = e^{-r(T-t)} E^{\mathcal{Q}}\left[\sum_{i=1}^{n} k_i S_i^T \mathbb{I}_i\right].$$

Or equivalently, using the fact that $E^{\mathcal{Q}}$ is a linear operator,

$$F = e^{-r(T-t)} \sum_{i=1}^{n} k_i E^{\mathcal{Q}}\left[S_i^T \mathbb{I}_i\right].$$

Let $\mathcal{Q}_i$ be the risk-neutral measure when the asset $S_i^t e^{q_i t}$ is the *numeraire*, defined making use of the Radon-Nikodym derivative:

$$\left.\frac{d\mathcal{Q}_i}{d\mathcal{Q}}\right| \mathcal{F}_T = \frac{S_i^T e^{q_i T}}{S_i^t e^{q_i t}} \frac{e^{rt}}{e^{rT}}. \tag{5}$$

Following Geman, Karoui and Rochet [6],

$$E^{\mathcal{Q}}\left[S_i^T \mathbb{I}_i\right] = E^{\mathcal{Q}_i}\left[S_i^T \mathbb{I}_i \frac{d\mathcal{Q}}{d\mathcal{Q}_i}\right].$$

Thus,

$$\begin{aligned}F &= e^{-r(T-t)} \sum_{i=1}^{n} k_i E^{\mathcal{Q}_i} \left[S_i^T \mathbb{I}_i \frac{d\mathcal{Q}}{d\mathcal{Q}_i} \right] \\ &= e^{-r(T-t)} \sum_{i=1}^{n} k_i E^{\mathcal{Q}_i} \left[S_i^T \mathbb{I}_i \frac{S_i^t e^{q_i t}}{S_i^T e^{q_i T}} \frac{e^{rT}}{e^{rt}} \right] \\ &= e^{-q_i(T-t)} \sum_{i=1}^{n} k_i S_i^t E^{\mathcal{Q}_i} \left[\mathbb{I}_i \right].\end{aligned}$$

$$F = e^{-q_i(T-t)} \sum_{i=1}^{n} k_i S_i^t \mathbb{P}^{\mathcal{Q}_i} \left(C_i \right) \tag{6}$$

■

This result is valid even if no restrictions are made on the form of the sets C_i. However, in general, this result is not a closed formula. The critical term is the probability factor $\mathbb{P}^{\mathcal{Q}_i}(C_i)$, which analytical form depends of the form of the set C_i. Now we will define C_i such that F is a closed formula while allowing Φ^T to be as general as possible.

Proposition 2.2. *If C_i is of the form below, then $\mathbb{P}^{\mathcal{Q}_i}(C_i)$ can be evaluated using the cumulative function of the Normal Multivariate distribution.*

$$C_i = \left\{ S_0, ..., S_n : \cap_{q=1}^{Q} \left\{ \frac{S_{u(q)}^{T_{u(q)}}}{S_{v(q)}^{T_{v(q)}}} < h_q \right\} \right\},$$

with $u_{(q)}, v_{(q)} \in \{1, ..., n\}, h_q \in \mathbb{R}, Q \in \mathbb{N}$ *and* $T_{u(q)}, T_{v(q)} \leq T$.

Proof. Let $Z_1^{\mathcal{Q}}(t), \ldots, Z_n^{\mathcal{Q}}(t)$ be n independent brownian motions. The model defined above by (3) and (4) can be restated equivalently by orthogonalizing the Wiener increments, thus

$$\frac{dS_i^t}{S_i^t} = (r - q_i)dt + \sum_{k=1}^{n} \theta_{S_i,k} dZ_k^{\mathcal{Q}}(t) \tag{7}$$

with

$$\sum_{k=1}^{n} \theta_{S_i,k}\theta_{S_j,k} = \rho_{S_i,S_j}\sigma_{S_i}\sigma_{S_j}, \qquad i,j = 1, \ldots, n. \tag{8}$$

Solving the stochastic diferencial equation (7), we have

$$\frac{S_i^{T_i}}{S_i^t} = \exp\left\{ \left(r - q_i - \frac{1}{2}\sum_{k=1}^{n} \theta_{S_i,k}^2 \right)(T_i - t) + \int_t^{T_i} \sum_{k=1}^{n} \theta_{S_i,k} dZ_k^{\mathcal{Q}}(u) \right\}. \tag{9}$$

However, to calculate the $\mathbb{P}^{\mathcal{Q}_i}(C_i)$ in (6) we need to know $S_i^{T_i}$ expressed in any risk-neutral measure $\mathcal{Q}_M$, i.e., taking $S_M^t e^{q_m t}$ as *numeraire*, with $M \in \{1, \ldots, n\}$.
Using equations (9) and (5),

$$\begin{aligned}\left.\frac{d\mathcal{Q}_M}{d\mathcal{Q}}\right| \mathcal{F}_T &= \frac{S_M^T}{S_M^t} e^{-(r-q_M)(T-t)} \\ &= \exp\left\{ -\frac{1}{2}\sum_{k=1}^{n} \theta_{S_M,k}^2 (T-t) + \int_t^T \sum_{k=1}^{n} \theta_{S_M,k}^2 dZ_k^{\mathcal{Q}}(u) \right\},\end{aligned}$$

and applying Girsanov's theorem:

$$dZ_k^{\mathcal{Q}_M}(t) = -\theta_{S_M,k}dt + dZ_k^{\mathcal{Q}}(t). \tag{10}$$

Gathering equations (7) and (10) we have the diffusion of S_i^t written in the measure $\mathcal{Q}_M$,

$$\frac{dS_i^t}{S_i^t} = \left(r - q_i + \sum_{k=1}^{n} \theta_{S_i,k}\theta_{S_M,k}\right) dt + \sum_{k=1}^{n} \theta_{S_i,k} dZ_k^{\mathcal{Q}_M}(t).$$

Finally, applying Itô's Lemma to the relative price $\frac{S_i^t}{S_M^t}$, we have:

$$d\left(\frac{S_i^t}{S_M^t}\right) = d\left(\frac{S_i^t}{S_M^t}\right)\left[(q_M - q_i)dt + \sum_{k=1}^{n}(\theta_{S_i,k} - \theta_{S_M,k})dZ_k^{\mathcal{Q}_M}(t)\right], \tag{11}$$

Considering that $d\left(\frac{S_{u(q)}^t}{S_M^t}\right)$ and $d\left(\frac{S_{v(q)}^t}{S_M^t}\right)$ are geometric brownian motions, the distributions of $\frac{S_{u(q)}^t}{S_M^t}$, $\frac{S_{v(q)}^t}{S_M^t}$ and $\frac{S_{u(q)}^t}{S_{v(q)}^t}$ are lognormals. Consequently, $\mathbb{P}^{\mathcal{Q}_i}\left(\left\{\frac{S_{u(q)}^{T_{u(q)}}}{S_{v(q)}^{T_{v(q)}}} < h_q\right\}\right)$ can be evaluated with the cumulative function of the Lognormal distribution. However the logarithm of $\frac{S_{u(q)}^{T_{u(q)}}}{S_{v(q)}^{T_{v(q)}}}$, follows a Normal distribution, thus $\mathbb{P}^{\mathcal{Q}_i}\left(\left\{\log\left(\frac{S_{u(q)}^{T_{u(q)}}}{S_{v(q)}^{T_{v(q)}}}\right) < \log(h_q)\right\}\right)$ can be evaluated with the cumulative function of the Normal distribution. Accordingly $\mathbb{P}^{\mathcal{Q}_i}(C_i)$ can be evaluated using the cumulative function of the Normal Multivariate distribution. ∎

3. The Probability Factor $\mathbb{P}^{\mathcal{Q}_i}(C_i)$. It is well known that the cumulative function of the Multivariate Normal distribution does not have an explicit expression, and that it has to be evaluated numerically. In fact, that is also the case of the Univariate Normal. However, several algorithms have been developed that enable the evaluation of the cumulative function efficiently for problems with up to 10 dimensions, considering the current computational capabilities. In the results presented in the section below, it was used the transformation of the function proposed by Alan Genz [5] in conjunction with Sobol Low Discrepancy Sequences for the generation of *Quasi-Random* samples.
From section 2 we have,

$$\mathbb{P}^{\mathcal{Q}_i}(C_i) = N\left(\log\left(\frac{S_{u(1)}^{T_{u(1)}}}{S_{v(1)}^{T_{v(1)}}}\right) < \log(h_1), \ldots, \log\left(\frac{S_{u(Q)}^{T_{u(Q)}}}{S_{v(Q)}^{T_{v(Q)}}}\right) < \log(h_Q)\right) \tag{12}$$

To evaluate (12) we have to solve the variable terms $\log\left(\frac{S_{u(q)}^{T_{u(q)}}}{S_{v(q)}^{T_{v(q)}}}\right)$ and to determine the correlations between each of the dimensions of the Multivariate Normal distribution.

3.1. Variables. Solving the stochastic differential equation (11), we have

$$\frac{S_i^{T_i}}{S_M^{T_i}} = \frac{S_i^t}{S_M^t} \exp\left\{\left(q_M - q_i - \frac{1}{2}\sum_{k=1}^{n}(\theta_{S_i,k} - \theta_{S_M,k})^2\right)(T_i - t) + \int_t^{T_i} \sum_{k=1}^{n}(\theta_{S_i,k} - \theta_{S_M,k})dZ_k^{\mathcal{Q}_M}(u)\right\}. \tag{13}$$

with $i = 1, \ldots, n$.

Consequently, and assuming $T_{u(q)} \geq T_{v(q)}$ without loss of generality, since h_q can be positive or negative,

$$\begin{aligned}\frac{S_{u(q)}^{T_{u(q)}}}{S_{v(q)}^{T_{v(q)}}} =& \frac{S_{u(q)}^{t}}{S_{v(q)}^{t}} \exp\left\{ \left(q_M - q_{u(q)} - \frac{1}{2}\sum_{k=1}^{n}(\theta_{S_{u(q)},k} - \theta_{S_M,k})^2 \right)(T_{u(q)} - T_{v(q)}) + \right. \\ &+ \left(q_{v(q)} - q_{u(q)} - \frac{1}{2}\sum_{k=1}^{n}(\theta_{S_{u(q)},k} - \theta_{S_M,k})^2 + \frac{1}{2}\sum_{k=1}^{n}(\theta_{S_{v(q)},k} - \theta_{S_M,k})^2 \right)(T_{v(q)} - t) + \\ &\left. + \int_{T_{v(q)}}^{T_{u(q)}} \sum_{k=1}^{n}(\theta_{S_{u(q)},k} - \theta_{S_M,k}) dZ_k^{\mathcal{Q}_M}(u) + \int_{t}^{T_{v(q)}} \sum_{k=1}^{n}(\theta_{S_{u(q)},k} - \theta_{S_{v(q)},k}) dZ_k^{\mathcal{Q}_M}(u) \right\} \end{aligned} \tag{14}$$

Simplifying this equation using the facts

- all θ are constant,
- $\sum_{k=1}^{n}(\theta_{x,k} - \theta_{y,k})^2 = \sigma_x^2 + \sigma_y^2 - 2\sigma_x\sigma_y\rho_{x,y} \equiv \sigma_{x,y}^2$,
- $\sum_{k=1}^{n}(\theta_{x,k} - \theta_{y,k}) = \sigma_{x,y}$ and

we have

$$\begin{aligned}\frac{S_{u(q)}^{T_{u(q)}}}{S_{v(q)}^{T_{v(q)}}} =& \frac{S_{u(q)}^{t}}{S_{v(q)}^{t}} \exp\left\{ \left(q_M - q_{u(q)} - \frac{1}{2}\sigma^2_{S_{u(q)},S_M} \right)(T_{u(q)} - T_{v(q)}) + \right. \\ &+ \left(q_{v(q)} - q_{u(q)} - \frac{1}{2}\sigma^2_{S_{u(q)},S_M} + \frac{1}{2}\sigma^2_{S_{v(q)},S_M} \right)(T_{v(q)} - t) + \\ &\left. + \sqrt{T_{u(q)} - T_{v(q)}}\sigma_{S_{u(q)},S_M} Z_1 + \sqrt{T_{v(q)} - t}\sigma_{S_{u(q)},S_{v(q)}} Z_2 \right\}\end{aligned}$$

where $Z_1, Z_2 \sim N(0,1)$ are independent.
Furthermore, knowing that the variance of the sum of two independent random variables is the sum of the variances, a final simplification can be applied.

$$\begin{aligned}\frac{S_{u(q)}^{T_{u(q)}}}{S_{v(q)}^{T_{v(q)}}} =& \frac{S_{u(q)}^{t}}{S_{v(q)}^{t}} \exp\left\{ \left(q_M - q_{u(q)} - \frac{1}{2}\sigma^2_{S_{u(q)},S_M} \right)(T_{u(q)} - T_{v(q)}) + \right. \\ &+ \left(q_{v(q)} - q_{u(q)} - \frac{1}{2}\sigma^2_{S_{u(q)},S_M} + \frac{1}{2}\sigma^2_{S_{v(q)},S_M} \right)(T_{v(q)} - t) + \\ &\left. + \sqrt{(T_{u(q)} - T_{v(q)})\sigma^2_{S_{u(q)},S_M} + (T_{v(q)} - t)\sigma^2_{S_{u(q)},S_{v(q)}}} Z \right\}, \text{ with } Z \sim N(0,1).\end{aligned}$$

Solving $\log\left(\frac{S_{u(q)}^{T_{u(q)}}}{S_{v(q)}^{T_{v(q)}}}\right) < \log(h_q)$ for Z, we have

$$Z < \frac{\log\left(\frac{S_{v(q)}^{t} h_q}{S_{u(q)}^{t}}\right) - \left(q_M - q_{u(q)} - \frac{1}{2}\sigma^2_{S_{u(q)},S_M}\right)(T_{u(q)} - T_{v(q)}) - \left(q_{v(q)} - q_{u(q)} - \frac{1}{2}\sigma^2_{S_{u(q)},S_M} + \frac{1}{2}\sigma^2_{S_{v(q)},S_M}\right)(T_{v(q)} - t)}{\sqrt{(T_{u(q)} - T_{v(q)})\sigma^2_{S_{u(q)},S_M} + (T_{v(q)} - t)\sigma^2_{S_{u(q)},S_{v(q)}}}} \tag{15}$$

3.2. Correlations. To determine the correlations between each of the dimensions of the Multivariate Normal distribution, we will a generic pair q_1 and q_2. Thus, we will develop the correlation between the random variables $\log\left(\frac{S_{u(q_1)}^{T_{u(q_1)}}}{S_{v(q_1)}^{T_{v(q_1)}}}\right)$ and $\log\left(\frac{S_{u(q_2)}^{T_{u(q_2)}}}{S_{v(q_2)}^{T_{v(q_2)}}}\right)$. Without loss of generality we will consider $T_{u(q_1)} \geq T_{v(q_1)}$, $T_{u(q_2)} \geq T_{v(q_2)}$ and $T_{u(q_1)} \geq T_{u(q_2)}$. Though three possible combinations are still left, namely

- $T_{u(q_1)} \geq T_{u(q_2)} \geq T_{v(q_1)} \geq T_{v(q_2)}$
- $T_{u(q_1)} \geq T_{u(q_2)} \geq T_{v(q_2)} \geq T_{v(q_1)}$
- $T_{u(q_1)} \geq T_{v(q_1)} \geq T_{u(q_2)} \geq T_{v(q_2)}$

The analysis will be developed for the first case, having the remaining possibilities an analogous reasoning. By definition, the covariance between the two random variables is

$$\text{Covar}\left(\log \frac{S_{u(q_1)}^{T_{u(q_1)}}}{S_{v(q_1)}^{T_{v(q_1)}}}, \log \frac{S_{u(q_2)}^{T_{u(q_2)}}}{S_{v(q_2)}^{T_{v(q_2)}}}\right) = E\left[\left(\log \frac{S_{u(q_1)}^{T_{u(q_1)}}}{S_{v(q_1)}^{T_{v(q_1)}}} - \mu_{q_1}\right)\left(\log \frac{S_{u(q_2)}^{T_{u(q_2)}}}{S_{v(q_2)}^{T_{v(q_2)}}} - \mu_{q_2}\right)\right]$$

from equation (14),

$$\begin{aligned}\mu_{q_1} = \log\left(\frac{S_{u(q_1)}^t}{S_{v(q_1)}^t}\right) &+ \left(q_M - q_{u(q_1)} - \frac{1}{2}\sigma^2_{S_{u(q_1)},S_M}\right)(T_{u(q_1)} - T_{v(q_1)}) \\ &+ \left(q_{v(q_1)} - q_{u(q_1)} - \frac{1}{2}\sigma^2_{S_{u(q_1)},S_M} + \frac{1}{2}\sigma^2_{S_{v(q_1)},S_M}\right)(T_{v(q_1)} - t)\end{aligned}$$

$$\begin{aligned}\mu_{q_2} = \log\left(\frac{S_{u(q_2)}^t}{S_{v(q_2)}^t}\right) &+ \left(q_M - q_{u(q_2)} - \frac{1}{2}\sigma^2_{S_{u(q_2)},S_M}\right)(T_{u(q_2)} - T_{v(q_2)}) \\ &+ \left(q_{v(q_2)} - q_{u(q_2)} - \frac{1}{2}\sigma^2_{S_{u(q_2)},S_M} + \frac{1}{2}\sigma^2_{S_{v(q_2)},S_M}\right)(T_{v(q_2)} - t)\end{aligned}$$

Thus, referring to equation (14), we have

$$\begin{aligned}&E\left[\left(\log \frac{S_{u(q_1)}^{T_{u(q_1)}}}{S_{v(q_1)}^{T_{v(q_1)}}} - \mu_{q_1}\right)\left(\log \frac{S_{u(q_2)}^{T_{u(q_2)}}}{S_{v(q_2)}^{T_{v(q_2)}}} - \mu_{q_2}\right)\right] = \\ &= E\left[\left(\int_{T_{v(q_1)}}^{T_{u(q_1)}} \sum_{k=1}^n (\theta_{S_{u(q_1)},k} - \theta_{S_M,k}) dZ_k^{\mathcal{Q}_M}(u) + \int_t^{T_{v(q_1)}} \sum_{k=1}^n (\theta_{S_{u(q_1)},k} - \theta_{S_{v(q_1)},k}) dZ_k^{\mathcal{Q}_M}(u)\right)\right. \\ &\left.\left(\int_{T_{v(q_2)}}^{T_{u(q_2)}} \sum_{k=1}^n (\theta_{S_{u(q_2)},k} - \theta_{S_M,k}) dZ_k^{\mathcal{Q}_M}(u) + \int_t^{T_{v(q_2)}} \sum_{k=1}^n (\theta_{S_{u(q_2)},k} - \theta_{S_{v(q_2)},k}) dZ_k^{\mathcal{Q}_M}(u)\right)\right]\end{aligned}$$

Taking in consideration the initial assumption $T_{u(q_1)} \geq T_{u(q_2)} \geq T_{v(q_1)} \geq T_{v(q_2)}$ we can rewrite the covariance as

$$\begin{aligned}E\Bigg[\Bigg(&\int_{T_{u(q_2)}}^{T_{u(q_1)}} \sum_{k=1}^n (\theta_{S_{u(q_1)},k} - \theta_{S_M,k}) dZ_k^{\mathcal{Q}_M}(u) + \int_{T_{v(q_1)}}^{T_{u(q_2)}} \sum_{k=1}^n (\theta_{S_{u(q_1)},k} - \theta_{S_M,k}) dZ_k^{\mathcal{Q}_M}(u) + \\ &+ \int_{T_{v(q_2)}}^{T_{v(q_1)}} \sum_{k=1}^n (\theta_{S_{u(q_1)},k} - \theta_{S_{v(q_1)},k}) dZ_k^{\mathcal{Q}_M}(u) + \int_t^{T_{v(q_2)}} \sum_{k=1}^n (\theta_{S_{u(q_1)},k} - \theta_{S_{v(q_1)},k}) dZ_k^{\mathcal{Q}_M}(u)\Bigg) \\ \Bigg(&\int_{T_{v(q_1)}}^{T_{u(q_2)}} \sum_{k=1}^n (\theta_{S_{u(q_2)},k} - \theta_{S_M,k}) dZ_k^{\mathcal{Q}_M}(u) + \int_{T_{v(q_2)}}^{T_{v(q_1)}} \sum_{k=1}^n (\theta_{S_{u(q_2)},k} - \theta_{S_M,k}) dZ_k^{\mathcal{Q}_M}(u) + \\ &+ \int_t^{T_{v(q_2)}} \sum_{k=1}^n (\theta_{S_{u(q_2)},k} - \theta_{S_{v(q_2)},k}) dZ_k^{\mathcal{Q}_M}(u)\Bigg)\Bigg]\end{aligned}$$

Knowing that the increments $Z_k^{Q_M}(u)$ are independent, the covariance simplifies to

$$\text{Covar}\left(\log \frac{S_{u(q_1)}^{T_{u(q_1)}}}{S_{v(q_1)}^{T_{v(q_1)}}}, \log \frac{S_{u(q_2)}^{T_{u(q_2)}}}{S_{v(q_2)}^{T_{v(q_2)}}}\right) = (T_{u(q_2)} - T_{v(q_1)})\sigma_{S_{u_{q_1}},S_M,S_{u_{q_2}},S_M} +$$
$$+ (T_{v(q_1)} - T_{v(q_2)})\sigma_{S_{u_{q_1}},S_{v_{q_1}},S_{u_{q_2}},S_M} + (T_{v(q_2)} - t)\sigma_{S_{u_{q_1}},S_{v_{q_1}},S_{u_{q_2}},S_{v_{q_2}}}$$

where $\sigma_{S_\alpha,S_\beta,S_\gamma,S_\delta} = \rho_{S_\alpha,S_\gamma}\sigma_{S_\alpha}\sigma_{S_\gamma} - \rho_{S_\alpha,S_\delta}\sigma_{S_\alpha}\sigma_{S_\delta} - \rho_{S_\beta,S_\gamma}\sigma_{S_\beta}\sigma_{S_\gamma} + \rho_{S_\beta,S_\delta}\sigma_{S_\beta}\sigma_{S_\delta}$
and $\alpha, \beta, \gamma, \delta = S_{u_{q_1}}, S_{v_{q_1}}, S_{u_{q_2}}, S_{v_{q_2}}, S_M$.
To find the correlation, we need to divide de covariance by the standard deviation of each random variable that can be found on the denominator of equation (15).

4. Applications.

4.1. Exchange Option - William Magrabe [11]. The contract function

$$\Phi^T(S_1, S_2) = Max\,(S_2 - S_1; 0)$$

may be rewritten as

$$\Pi\left(t, \Phi^T\left(S_1, S_2\right)\right) = \sum_{i=1}^{2} k_i \cdot S_i\left(t\right) \cdot e^{-\alpha_i(T-t)} \cdot N\left(d_{S_i}^1\right)$$

Term $i = 1$

- $k_1 = -1$, negative because this term corresponds to the payment of the value of S_1;
- S_1, underlying asset delivered if the option is exercised;
- $C_1 = \left\{S_1^T, S_2^T : \left\{\frac{S_1^T}{S_2^T} < 1\right\}\right\}$

Term $i = 2$

- $k_2 = 1$, positive because this term corresponds to the value of the underlying asset received;
- S_2, underlying asset received if the option is exercised;
- $C_2 = \left\{S_1^T, S_2^T : \left\{\frac{S_1^T}{S_2^T} < 1\right\}\right\}$

Consequently,

$$\Pi\left(t, \Phi^T(S_1, S_2)\right) = S_2\left(t\right) \cdot e^{-\alpha_2(T-t)} \cdot N\left(d_{S_2}^1\right) - S_1\left(t\right) \cdot e^{-\alpha_1(T-t)} \cdot N\left(d_{S_1}^1\right)$$

4.2. Option on the Maximum of Several Assets - Herb Johnson [9]. The contract function

$$\Phi^T\left(S_0, S_1, \ldots, S_n\right) = Max\left(S_1 - S_0, \ldots, S_n - S_0\right)$$

may be rewritten as

$$\Pi\left(t, \Phi^T\left(S_0, S_1, \ldots, S_n\right)\right) = \sum_{i=0}^{1} k_i \cdot S_i\left(t\right) \cdot e^{-\alpha_i(T-t)} \cdot \mathbb{P}^{S_i}(C_i)$$

Term $i = 0$

- $k_0 = -1$, negative because this term corresponds to the payment of the strike price;
- S_0 is the strike price;

- $C_0 = \left\{ S_0^T, ..., S_n^T : \cup_{i=1}^{n} \left\{ \frac{S_0^T}{S_i^T} < 1 \right\} \right\} \Leftrightarrow C_0 = \left\{ S_0^T, ..., S_n^T : \cap_{i=1}^{n} \left\{ \frac{S_0^T}{S_i^T} < 1 \right\}^c \right\}^c$

Term $i = 1, ..., n$

- $k_i = 1$, positive because this term corresponds to the value of the underlying asset received;
- S_i the value of the underlying asset;
- $C_i = \left\{ S_0^T, ..., S_n^T : \cap_{j=0, j\neq i}^{n} \left\{ \frac{S_j^T}{S_i^T} < 1 \right\} \right\}$.

Consequently,

$$\Pi^t \left(t, \Phi^T(S_0, \dots, S_n)\right) = \sum_{i=1}^{n} S_i(t) \cdot e^{-\alpha_i (T-t)} \cdot N\left(d_{S_i}^1, \dots, d_{S_i}^n\right) -$$

$$- S_0(t) \cdot e^{-\alpha 0(T-t)} \cdot \left[1 - N\left(-d_{S_0}^1, \dots, -d_{S_0}^n\right)\right]$$

4.3. Option on the 2^{nd} Best Asset. This option gives the right to buy the second most valuable asset at maturity in exchange for the strike price.
The following analysis can be easily generalized to develop a closed formula for a *Best M of N* option, see [22].
Let $BOf_{2^{nd}}(S_1, ..., S_4; S_0)$ be an option on the 2^{nd}*Best of* 4 with a strike price of S_0.

$$BOf_{2^{nd}}(S_1, ..., S_4; S_0) = \sum_{i=0}^{4} S_i \cdot e^{-\alpha_i (T-t)} \mathbb{P}^{S_i}(C_{S_i})$$

Term $i = 1, ..., 4$

- $k_i = 1$, positive because this term corresponds to the value of the underlying asset received;
- S_i the value of the underlying asset;
- $C_{S_i} = \left\{ S_1^T, ..., S_4^T : \cup_{j=1 \wedge j \neq i}^{4} C_{S_i, j} \right\}$

  Each $C_{S_i,j}$ describes a result state where S_i^T is greater than every other asset except one. For example, on the set $C_{S_i,1}$, S_i^T is greater than all other asset except S_1^T.

  $C_{S_i,1} = \left\{ \frac{S_1^T}{S_i^T} > 1, \frac{S_2^T}{S_i^T} < 1, \frac{S_3^T}{S_i^T} < 1, \frac{S_4^T}{S_i^T} < 1, \frac{S_0^T}{S_i^T} < 1 \right\}$

  $C_{S_i,2} = \left\{ \frac{S_1^T}{S_i^T} < 1, \frac{S_2^T}{S_i^T} > 1, \frac{S_3^T}{S_i^T} < 1, \frac{S_4^T}{S_i^T} < 1, \frac{S_0^T}{S_i^T} < 1 \right\}$

  $C_{S_i,3} = \left\{ \frac{S_1^T}{S_i^T} < 1, \frac{S_2^T}{S_i^T} < 1, \frac{S_3^T}{S_i^T} > 1, \frac{S_4^T}{S_i^T} < 1, \frac{S_0^T}{S_i^T} < 1 \right\}$

  $C_{S_i,4} = \left\{ \frac{S_1^T}{S_i^T} < 1, \frac{S_2^T}{S_i^T} < 1, \frac{S_3^T}{S_i^T} < 1, \frac{S_4^T}{S_i^T} > 1, \frac{S_0^T}{S_i^T} < 1 \right\}$

Term $i = 0$

- $k_0 = -1$, negative because this term corresponds to the payment of the strike price;
- S_0, is the strike price;
- $C_{S_0} = \left\{ S_1^T, ..., S_4^T : \cup_{j=1}^{11} C_{S_0^T, j} \right\}$

  Conversely, each $C_{S_0^T,j}$ describes a result state where S_0^T is smaller than all S_i^T; or smaller than all S_i^T except one; ou smaller than all S_i^T except two.

S_0^T smaller than all S_i:
$C_{S_0^T,1} = \left\{ \frac{S_0^T}{S_1^T} < 1, \frac{S_0^T}{S_2^T} < 1, \frac{S_0^T}{S_3^T} < 1, \frac{S_0^T}{S_4^T} < 1 \right\}$
S_0^T smaller than all S_i except one:
$C_{S_0^T,2} = \left\{ \frac{S_1^T}{S_0^T} < 1, \frac{S_0^T}{S_2^T} < 1, \frac{S_0^T}{S_3^T} < 1, \frac{S_0^T}{S_4^T} < 1 \right\}$
$C_{S_0^T,3} = \left\{ \frac{S_0^T}{S_1^T} < 1, \frac{S_2^T}{S_0^T} < 1, \frac{S_0^T}{S_3^T} < 1, \frac{S_0^T}{S_4^T} < 1 \right\}$
$C_{S_0^T,4} = \left\{ \frac{S_0^T}{S_1^T} < 1, \frac{S_0^T}{S_2^T} < 1, \frac{S_3^T}{S_0^T} < 1, \frac{S_0^T}{S_4^T} < 1 \right\}$
$C_{S_0^T,5} = \left\{ \frac{S_0^T}{S_1^T} < 1, \frac{S_0^T}{S_2^T} < 1, \frac{S_0^T}{S_3^T} < 1, \frac{S_4^T}{S_0^T} < 1 \right\}$
S_0^T smaller than all S_i except two:
$C_{S_0^T,6} = \left\{ \frac{S_1^T}{S_0^T} < 1, \frac{S_2^T}{S_0^T} < 1, \frac{S_0^T}{S_3^T} < 1, \frac{S_0^T}{S_4^T} < 1 \right\}$
$C_{S_0^T,7} = \left\{ \frac{S_1^T}{S_0^T} < 1, \frac{S_0^T}{S_2^T} < 1, \frac{S_3^T}{S_0^T} < 1, \frac{S_0^T}{S_4^T} < 1 \right\}$
$C_{S_0^T,8} = \left\{ \frac{S_1^T}{S_0^T} < 1, \frac{S_0^T}{S_2^T} < 1, \frac{S_0^T}{S_3^T} < 1, \frac{S_4^T}{S_0^T} < 1 \right\}$
$C_{S_0^T,9} = \left\{ \frac{S_0^T}{S_1^T} < 1, \frac{S_2^T}{S_0^T} < 1, \frac{S_3^T}{S_0^T} < 1, \frac{S_0^T}{S_4^T} < 1 \right\}$
$C_{S_0^T,10} = \left\{ \frac{S_0^T}{S_1^T} < 1, \frac{S_2^T}{S_0^T} < 1, \frac{S_0^T}{S_3^T} < 1, \frac{S_4^T}{S_0^T} < 1 \right\}$
$C_{S_0^T,11} = \left\{ \frac{S_0^T}{S_1^T} < 1, \frac{S_0^T}{S_2^T} < 1, \frac{S_3^T}{S_0^T} < 1, \frac{S_4^T}{S_0^T} < 1 \right\}$

$$BOf_{2^{nd}}\left(S_1, ..., S_4; S_0\right) = \sum_{i=1}^{4} S_i \cdot e^{-\alpha_i (T-t)} \sum_{j=1 \wedge j \neq i}^{4} N\left(d_{S_i,j}^1, \ldots, d_{S_i,j}^4, d_{S_i,j}^0\right) -$$
$$- S_0 \cdot e^{-\alpha_0 (T-t)} \cdot \sum_{i=0}^{11} N\left(d_{S_0,i}^1, \ldots, d_{S_0,i}^4\right)$$

4.4. Target Redemption Options - TAR. This option gives the right to receive a stream of payments, up to maximum total amount. For example, an option with four years maturity that pays every 5% of the initial price of the underlying, up to a total maximum of 10%, if the underlying is above its initial value in each of the years. This option would only have two payments maximum, after the second payment the option would be worthless.

Term $i = 1$

- $k_1 = 5\%$, quantity of the underlying asset received;
- S^{T_0} initial value of the underlying asset;
- $C_1 = \left\{ S^{T_1} : \frac{K}{S^{T_1}} < 1 \right\}$

Term $i = 2$

- $k_2 = 5\%$, quantity of the underlying asset received;
- S^{T_0} initial value of the underlying asset;
- $C_2 = \left\{ S^{T_2} : \frac{K}{S^{T_2}} < 1 \right\}$

Term $i = 3$

- $k_3 = 5\%$, quantity of the underlying asset received;
- S^{T_0} initial value of the underlying asset;
- $C_{3,1} = \left\{ S^{T_1}, S^{T_2}, S^{T_3} : \frac{K}{S^{T_1}} < 1; \frac{S^{T_2}}{K} < 1; \frac{K}{S^{T_3}} < 1 \right\}$
- $C_{3,2} = \left\{ S^{T_1}, S^{T_2}, S^{T_3} : \frac{S^{T_1}}{K} < 1; \frac{K}{S^{T_2}} < 1; \frac{K}{S^{T_3}} < 1 \right\}$
- $C_{3,3} = \left\{ S^{T_1}, S^{T_2}, S^{T_3} : \frac{S^{T_1}}{K} < 1; \frac{S^{T_2}}{K} < 1; \frac{K}{S^{T_3}} < 1 \right\}$

Term $i = 4$

- $k_4 = 5\%$, quantity of the underlying asset received;
- S^{T_0} initial value of the underlying asset;
- $C_{4,1} = \left\{ S^{T_1}, S^{T_2}, S^{T_3}, S^{T_4} : \frac{K}{S^{T_1}} < 1; \frac{S^{T_2}}{K} < 1; \frac{S^{T_3}}{K} < 1; \frac{K}{S^{T_4}} < 1 \right\}$
- $C_{4,2} = \left\{ S^{T_1}, S^{T_2}, S^{T_3}, S^{T_4} : \frac{S^{T_1}}{K} < 1; \frac{K}{S^{T_2}} < 1; \frac{S^{T_3}}{K} < 1; \frac{K}{S^{T_4}} < 1 \right\}$
- $C_{4,3} = \left\{ S^{T_1}, S^{T_2}, S^{T_3}, S^{T_4} : \frac{S^{T_1}}{K} < 1; \frac{S^{T_2}}{K} < 1; \frac{K}{S^{T_3}} < 1; \frac{K}{S^{T_4}} < 1 \right\}$
- $C_{4,4} = \left\{ S^{T_1}, S^{T_2}, S^{T_3}, S^{T_4} : \frac{S^{T_1}}{K} < 1; \frac{S^{T_2}}{K} < 1; \frac{S^{T_3}}{K} < 1; \frac{K}{S^{T_4}} < 1 \right\}$

$$
\begin{aligned}
\mathrm{TAR}\left(S^{T_1}, S^{T_2}, S^{T_3}, S^{T_4}\right) = & 5\% \cdot S^{T_0} \cdot e^{-\alpha_0 (T_i - t)} \cdot N\left(d_{S^{T_1}}\right) + \\
& + 5\% \cdot S^{T_0} \cdot e^{-\alpha_0 (T_i - t)} \cdot N\left(d_{S^{T_2}}\right) + \\
& + 5\% \cdot S^{T_0} \cdot e^{-\alpha_0 (T_i - t)} \cdot \sum_{i=1}^{3} N_i\left(d_{S^{T_3},1}, d_{S^{T_3},2}, d_{S^{T_3},3}\right) + \\
& + 5\% \cdot S^{T_0} \cdot e^{-\alpha_0 (T_i - t)} \cdot \sum_{i=1}^{4} N_i\left(d_{S^{T_4},1}, d_{S^{T_4},2}, d_{S^{T_4},3}, d_{S^{T_4},4}\right)
\end{aligned}
$$

5. Results - Method Performance. To evaluate the performance of the method, the pricing function was implemented in Microsoft Excel Visual Basic and compared with a regular Monte Carlo experiment. The cumulative function of the Multivariate Normal distribution was implemented using the algorithm suggested by Alan Genz [5] and with Sobol Low Discrepancy Sequences for the generation of *Quasi-Random* samples, also used in the Monte Carlo Experiment.
Three test were performed on an exotic option with an increasing number of exercise conditions.

Test 1

For a given precision of 0.005, with a confidence interval of 99%, how many seconds were need to calculate the price of the option?

Test 2

For a given precision of 0.002, with a confidence interval of 99%, how many seconds were need to calculate the price of the option?

Test 3

For a given time of 300 seconds, what precision, with a confidence interval 99%, is the proposed method able to reach?

The results of the experiment are in the table 1 on page 11.

Test	Precision*=0.005		Precision*=0.002		300seconds
Algorithm	MC	**New Method**	MC	**New Method**	**New Method**
Dimensions	(sec.)	(sec.)	(sec.)	(sec.)	(precision*)
2	57	1	356	1	5.20E-5
3	57	1	362	2	1.51E-4
4	68	1	438	3	1.80E-4
5	50	2	316	12	3.94E-4
6	50	2	319	15	3.54E-4
7	54	2	340	9	3.37E-4
8	58	4	382	27	2.71E-4
9	58	4	371	26	2.29E-4

Table 1: Test Results.

6. Conclusions.

The exotic options pricing method proposed in this paper does more than creating a general setting that covers several classical results, it sets a framework that can be applied to any exotic option that has the predefined properties, namely,

- extended *european* style contracts, allowing comparisons of underlying asset prices at different moments in time;
- with tradable underlying assets (foreign or domestic);
- with a return profile that is a linear combination of the prices of the underlyings at each maturity; and
- that the conditions of exercise are defined as a relationship of the underlying assets defined in (2.2.) on page 3.

Based on the analysis developed, the results of Black e Scholes [2], William Magrabe [11] and Herb Johnson [9] where recovered and placed under the same general framework. Further more, this analysis enables the inclusion of the *Quanto* property on all this contracts.
This paper also presents original pricing closed formulas for (i) options on the Best M of N risky assets[1] and on (ii) TAR - Target Redemption options, both widely used in the industry.
Future research can be focused on generalizing further the method here developed to include several other properties present in exotic options, namely, multiple exercise or compound features. Alternatively, this analysis could be transposed to other asset classes, for example, interest rates.

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