

# Unifying Exotic Option Closed Formulas

Carlos Veiga<sup>13</sup>, Uwe Wystup<sup>1</sup> and Manuel L. Esquivel<sup>2</sup>

<sup>1</sup>Frankfurt School of Finance & Management, Centre for Practical Quantitative Finance

<sup>2</sup>Universidade Nova de Lisboa, Faculty of Science and Technology

<sup>3</sup>Millennium bcp, S.A.

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## Motivation

- Pricing and partial derivatives (greeks) formulas for exotic options are scattered over many papers and books.
- We set out to write the most general pricing formula to cover as much formulas as we could, including formulas for the greeks.
- By doing we aimed also at being able to mix the features of each classical option type in one single option and still get a formula.
- Different option types would then handled by proper parameterization creating and implicit payoff language.
- We use the Black-Scholes because it allows to consider the widest class of exotics pricing by closed formulas and the only one available to address multivariate problems effectively.

## Multivariate Black-Scholes Model

- Assume there are  $n$  assets  $A_i$ , and the respective bank accounts  $B_i$ .
- Each  $B_i$  yields a return at a continuous rate of  $r_i$  with the dynamics

$$dB_i(t)/B_i(t) = r_i(t)dt. \quad (1)$$

- Assume also the existence of price processes  $S_{ij}$ , i.e., the price of one unit of  $A_i$  expressed in units of  $A_j$  with the dynamics

$$dS_{ij}(t)/S_{ij}(t) = \mu_{ij}(t)dt + \sigma_{ij}(t)dW_{ij}(t). \quad (2)$$

- Furthermore,  $W_{ij}(t)$  is correlated with the other  $W_{kl}(t)$  that drive the other asset prices. Let  $\rho_{ij,kl}(t)$  be its correlation and  $\varsigma_{ij,kl}(t) = \rho_{ij,kl}(t)\sigma_{ij}(t)\sigma_{kl}(t)$  the respective covariance.

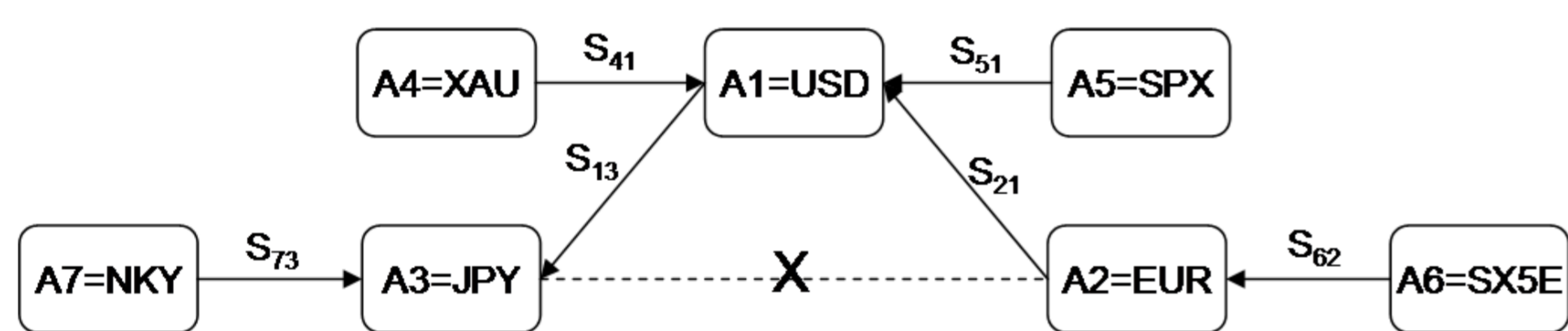


Figure: Example of market setup.

## Risk-Neutral Dynamics

As there is one and only one martingale measure for a given *numéraire*  $B_k(t)$ , the market of this model is arbitrage free and complete. The new dynamics of  $S_{ij}$  under that measure  $Q_k$  are, with  $B_k(t)$  as the *numéraire* portfolio

$$dS_{ij}(t) = S_{ij}(t) \left( \left( r_j(t) - r_i(t) - \sum_{h=1}^n \lambda_{jh} \varsigma_{ij,ih}(t) \right) dt + \sigma_{ij}(t)dW_{ij,k}(t) \right). \quad (3)$$

with

- $n$  is the number of price conversions needed to express  $A_j$  in units of  $A_k$ .
- $i_h$  are the indexes of the assets that stand between assets  $A_j$  and  $A_k$ .
- $W_{ij,k}(t)$  are Brownian motions under the measure  $Q_k$ .

## Abstract Assets

Consider now a new set of abstract assets  $A_i$  that fulfill the following conditions:

- their price  $S_i$ , expressed in terms of  $A_k$ , is a function of the prices  $S_{ij}$ ;
- their price has, under the measure  $Q_k$ , dynamics of the type

$$dS_i(t) = S_i(t) (d_{S_i}(t)dt + \sigma_{S_i}(t)dW_{S_i,k}(t)), \quad (4)$$

$d_{S_i}(t)$  and  $\sigma_{S_i}(t)$  are deterministic functions of time.

The most general case we can conceive is

$$S_i(t) = \prod_{j=1}^m (S_{i_j,t_j}(t))^{\alpha_j}, \quad (5)$$

where  $S_{i_j,t_j}(t)$  is the process  $S_{ij}(t)$  frozen at time  $t_j$ , i.e.,  $S_{ij}(t \wedge t_j)$  and  $i_j$  an index of an asset.

$d_{S_i}(t)$  and  $\sigma_{S_i}(t)$  are obtained using the Itô formula. The covariance of  $S_i$  with any other price  $S_j$  follows from  $\zeta_{S_i S_j}(t)dt = \sigma_{S_i}(t)\sigma_{S_j}(t)dW_{S_i,k}(t)dW_{S_j,k}(t)$

$$\zeta_{S_i S_j}(t) = \sum_{a=1}^{m_i} \sum_{b=1}^{m_j} \alpha_a \alpha_b \zeta_{i_a i_b}(t) = \sum_{a=1}^{m_i} \sum_{b=1}^{m_j} \alpha_a \alpha_b \varsigma_{a,b}(t) \mathbb{I}_{\{t < t_a, t < t_b\}}. \quad (6)$$

## Abstract Assets' Bank Account

The return rate  $r_{S_i}$  of the bank account  $B_i$  associated with asset  $A_i$  can be easily calculated using the fact that the dynamics of  $B_i$  expressed in terms of  $B_k$ , under the measure  $Q_k$ , is a martingale. Thus, by the Itô formula

$$r_{S_i}(t) = r_k(t) - d_{S_i}(t). \quad (7)$$

- Clearly these abstract assets are not traded *per se*, and they cannot be replicated by any self-financing portfolio.
- However, we were able to find the  $Q_{S_i}$  measure in which all the portfolios  $\vartheta$  of the above assets and abstract assets, expressed in units of  $B_k$ , are martingales when translated to units of  $B_i$ , by  $\frac{\vartheta(t)B_k(t)}{B_i(t)S_i(t)}$ .
- Therefore, we can use the martingale measure  $Q_{S_i}$  but we may not call  $S_i$  a *numéraire*.

## General Contract

Consider the following payoff definition expressed in terms of asset  $A_k$

$$\Phi_k = \sum_{i=1}^n c_i S_{i,t_i}(T_i) \mathbb{I}_{C_i}, \quad (8)$$

with

- $c_i \in \mathbb{R}$ ,  $S_{i,t_i}(T_i)$  the price of an abstract asset  $A_i$  observed at time  $t_i$ , to be settled at time  $T_i$ , and  $\mathbb{I}_{C_i}$  the indicator function of the exercise set  $C_i$ .

The sets  $C_i$  are of the form

$$\bigcap_{l=1}^{m_i} \left\{ \frac{S_{I_u, t_{I_u}}(T_i)}{S_{I_d, t_{I_d}}(T_i)} < h_l \right\}, \quad (9)$$

with  $I_u, I_d$  denoting the indexes of abstract assets,  $h_l \geq 0$  and  $t_{I_u}, t_{I_d} \leq T_i$ .

## Formulas for Price and Greeks

The arbitrage free price of the claim with payoff  $\Phi_k$  as in (8) and the sets  $C_i$  as in (9) can be calculated using the formula

$$V(t_0) = \sum_{i=1}^n c_i S_{i,t_i}(t_0) \exp \left\{ - \int_{t_0}^{T_i} r_{S_{i,t_i}}(u) du \right\} \mathcal{N}_{m_i}^{S_{i,t_i}}(v; \phi, \Sigma). \quad (10)$$

with  $\mathcal{N}_{m_i}^{S_{i,t_i}}$  denoting the  $m_i$ -dimensional multivariate normal cumulative distribution function, with covariance matrix  $\Sigma$  and mean vector  $\phi$ , evaluated at vector  $v$ .

Formulas for spatial derivatives of the pricing function are obtained using the results of Carr [1]. Results from Reiß and Wystup [2] enable the calculation of the derivatives with respect to all other model parameters.

## Examples

- A one year ATM cliquet option with five periods of equal length.
- A best of five that pays off the difference, if positive, between the maximum of the five index values at maturity and 100.
- A discrete lookback that pays off the difference, if positive, of the highest stored value of the first index and 100. The index values are stored 12 times during the year at evenly spaced times, starting at 1/12.
- A Himalaya option that pays off, at the end of each period of 1/3 units of time, the best return, if positive, over the period of the three first indexes times 100. The indexes that pay out are not considered for any of the subsequent periods.

## Payoff Language – Parameterization of Best of 5 (parts of)

This option payoff is composed of 6 terms, five for the reception of each of the five possible maximum index prices at maturity, and one payment of the exercise price 100.

The terms for each of the five index payments follow the following rule. Let

$a_i = 1, \dots, 5$  and let  $b_{i,1}, \dots, b_{i,4}$  be the elements of the set  $\{1, 2, 3, 4, 5\} \setminus a_i$ .

The terms for reception of each of the five index prices is parameterized by

Term  $i = 1, \dots, 5$  (index price)

$c_i$	$I_i$	$t_i$	$T_i$	Set Complement	Flag
1	$a_i$	1	1		false

Set  $C_i$

$I_u$	$t_{I_u}$	$I_d$	$t_{I_d}$	$h_l$
0	0	$a_i$	1	1/100
$b_{i,1}$	1	$a_i$	1	1
$b_{i,2}$	1	$a_i$	1	1
$b_{i,3}$	1	$a_i$	1	1
$b_{i,4}$	1	$a_i$	1	1

## Results

	Cliquet		Best of 5		Lookback		Himalaya	
calculation time = 10''								
Monte Carlo	18.27	0.53%	19.16	0.43%	13.50	1.03%	174.46	0.57%
Closed Formula	18.33	—	19.16	0.29%	13.47	1.24%	173.90	0.05%
calculation time = 5'								
Monte Carlo	18.33	0.10%	19.16	0.08%	13.50	0.19%	173.97	0.10%
Closed Formula	18.33	—	19.15	0.05%	13.51	0.23%	173.93	0.01%

<sup>13</sup> Carr, P. (2001) Deriving Derivatives of Derivative Securities. *Journal of Computational Finance*, Winter 2000/2001, 4(2), 5–29.

<sup>14</sup> Reiß, O. and Wystup, U. (2001). Efficient Computation of Option Price Sensitivities Using Homogeneity and Other Tricks. *The Journal of Derivatives*, 9(2), Winter.

<sup>15</sup> Veiga, C., Wystup, W. and Esquivel, M. (2010). Unifying Exotic Option Closed Formulas. *CPQF Working Papers*, Frankfurt School of Finance & Management, Nr. 23.